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HAMILTONIAN ANALYSIS FOR TOPOLOGICAL AND YANG-MILLS THEORIES EXPRESSED AS A CONSTRAINED BF-LIKE THEORY

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Abstract: The Hamiltonian analysis for the Euler and second-Chern classes is performed. We show that, in spite of the fact that the Second-Chern and Euler invariants give rise to the same equations of motion, their corresponding symplectic structures on the phase space are different, therefore, one can expect different quantum formulations. In addition, the symmetries of actions written as a BF-like theory that lead to Yang-Mills equations of motion are studied. A close relationship with the results obtained in previous works for the Second-Chern and Euler classes is found.

1. Introduction

Nowadays BF theories are a topic of great interest in physics [1, 2] due to the close relation with General Relativity, since they are background independent, diffeomorphisms covariant although are devoid of local physical degrees of freedom, [3, 4, 5]. In the literature there exist several examples where BF theories come to be relevant models, for instance in alternative formulations of gravity using the MacDowell-Maunsouri approach or of Yang-Mills [YM] theory using Martellini's model. MacDowell-Maunsouri formulation of gravity consists in

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breaking down the $SO(5)$ symmetry of a BF-theory from the $SO(5)$ group to the $SO(4)$, in order to obtain the Palatini action plus the sum of the Second Chern and the Euler topological invariants. Since those topological classes have trivial local variations that do not contribute classically to the dynamics, one obtains essentially general relativity [6]. Furthermore, the study of those invariants has been the subject of recent works since they have a close relation with general relativity as well [4]. In addition, they are expected to be related to physical observables, as for instance in the case of anomalies [7, 8, 9, 10, 11, 12]. On the other hand, Martellini's model consists in expressing YM theory as a BF-like theory [13], thus, in this so-called first-order formulation, YM theory can be viewed as a perturbative expansion in the coupling constant g around the pure topological BF theory; additionally the BF first-order formulation is *on shell* equivalent to the usual (second-order) YM theory. In this context, both formulations of the theory possess the same perturbative quantum properties [14]. On the other side, there also exists an alternative way to express YM as a BF-like theory [16], and it is possible to show again a close relation among topological theories and physical theories. In particular, in [16] has been analyzed the quantum aspects of the BF-like theory with the abelian group $U(1)$, and it has been concluded that the model gives the same physical description than Maxwell theory at both classical and quantum level.

At the light of these facts, in this paper we analyze the Hamiltonian formulation for the theories discussed above. First we perform the Hamiltonian analysis for the Second-Chern and the Euler invariants. We show that, in spite of the fact that both theories give rise to the same equations of motion, their corresponding symplectic structures are different from each other, fact that becomes important at both classical and quantum levels. It is important to mention that there exist studies including those invariants; in particular they play an important role at quantum level in modified versions of General Relativity [17]. On the other hand, in [18], the canonical covariant formalism has been developed for those invariants showing that the topological classes share the Chern-Simons state within the self-dual scenario. Nevertheless, in these works, the study of the symmetries was not performed in detail, and the present work attempts to go on along these lines. Furthermore, we perform the Hamiltonian analysis for Martellini's model [13, 14, 15] and for the action worked out in [16]. We show that both theories yield YM equations of motion, however our Hamiltonian study shows that those theories are not topological because there exist physical degrees of freedom. In addition, their symplectic structures are different from each other, fact that becomes to be relevant in the quantum treatment of both theories just as it happens for the Second-Chern and Euler classes. At the end, we show a close relation among these physical actions and the topological invariants discussed at the beginning of the paper.

2. Canonical Analysis for Actions Written as BF-Type Ones: Euler and Second-Chern Class

In the following two sections we develop Dirac's canonical analysis for the Euler and Second-Chern topological invariants on a smaller phase space context. This means that we shall consider as dynamical variables those whose time derivatives occur in the Lagrangian. The analysis performed in this part will be relevant for later sections, because we will learn that in spite of two actions give rise to same equations of motion, their corresponding symplectic structures are quite different from each other and this fact could yield different quantum descriptions. Partial results were studied in [18], however we develop our analysis in this work by a different way extending those results.

The theories that we will study in this part are described by the following actions [4, 18]

$$S_{SC}[A,B] = \Xi \int_M F^{IJ} \wedge B_{IJ} - \frac{1}{2} B^{IJ} \wedge B_{IJ}, \tag{1}$$

and

$$S_E[A,B] = \Omega \int_M \star F^{IJ} \wedge B_{IJ} - \frac{1}{2} \star B^{IJ} \wedge B_{IJ}, \tag{2}$$

where the former corresponds to the Second-Chern invariant and the later to the Euler class. Here Ξ, Ω are constants, $I, J, .. = 0, 1, 2, 3$ are $SO(3,1)$ index that are raised and lowered with the Minkowski metric $\eta^{IJ} = (-1, 1, 1, 1)$, the $\star$ product acts on internal indices namely, $\star B^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL} B^{KL}$, and we will assume that M is a manifold with topology $\Sigma \times R$.

The equations of motion for the action (1) are given by

$$DB^{IJ} = 0, \qquad F^{IJ}[\omega] = B^{IJ}, \tag{3}$$

whereas for the action (2) are given by

$$D \star B^{IJ} = 0, \qquad \star F^{IJ}[\omega] = \star B^{IJ}, \tag{4}$$

after the application of the $\star$ operation to the equations (4), those are reduced to (3); thus, we would expect at classical level the same physical description. However, it has been remarked in [18, 19] that two theories sharing the same equations of motion, in general do not yield the same physical description, in particular within the quantum context. In this way, we need to be more careful by carrying out a deep analysis of the symmetries of the systems under study. In this respect, we will develop the Hamiltonian framework for the actions given in (1) and (2) allowing us to know the principal symmetries of the actions and the relation among them. It is important to remark that the analysis that we develop in this section is performed by following ideas presented in [19] and the study carry out here has been not reported in the literature.

For our aim, it is convenient to make the following change of variables: $B^i = -\frac{1}{2}\epsilon^i{}_{jk}B^{jk}$, $\Upsilon^i = -\frac{1}{2}\epsilon^i{}_{jk}\omega^{jk}$, $\pi^a{}_i = \Xi\widetilde{\eta}^{abc}B_{ibc}$, $P^a_i = \Xi\widetilde{\eta}^{abc}B_{0ibc}$, $\tau^i = -\Upsilon^i_0$, $\Lambda^i = -{\omega_0}^{0i}$, $\chi_{i0a} = -2B_{ioa}$, $\varsigma_{ia} = -2B_{0i0a}$, $B^i{}_{bc} = -\frac{1}{2}\epsilon^i{}_{jk}B^{jk}{}_{bc}$, where latin indices are raised and lowered with the metric $\delta^{ij} = (1,1,1)$ and the totally anty-symmetric Levi-Civita density of weight $+1$, $\widetilde{\eta}^{\alpha\beta\mu\nu}$, is such that $\widetilde{\eta}^{0123} = 1$ with $\widetilde{\eta}^{0abc} \equiv \widetilde{\eta}^{abc}$. The two-form curvature F^{IJ} in terms of these variables is given by

$$
\begin{aligned}
F_{ibc} &= \left[\partial_b\Upsilon_{ic} - \partial_c\Upsilon_{ib} - \epsilon_{ijk}{\omega_b}^{j0}{\omega_{c0}}^k + \epsilon_{ijk}\Upsilon^j_b\Upsilon^k_c\right], \\
F_{0ibc} &= \left[\partial_b\omega_{c0i} - \partial_c\omega_{b0i} + \epsilon_{ijk}{\omega_b}^k{}_0\Upsilon^j_c - \epsilon_{ijk}{\omega_c}^k{}_0\Upsilon^j_b\right].
\end{aligned} \tag{5}
$$

In this manner, using the new variables, the Hamiltonian analysis for the Second-Chern and Euler invariants leads to

$$
\begin{aligned}
S_{SC} \quad & [\Upsilon^i{}_a, \pi^a_i, {\omega_a}^{0i}, P^a_i, \tau^i, \Lambda^i, \varsigma^i{}_a, \chi^i_a] \\
&= \int dx^0 \int_\Sigma dx^3 \Big\{ \dot{\Upsilon}^i{}_a\pi^a_i + \dot{\omega}_a{}^{0i}P^a_i - \tau^i\left(\partial_a\pi^a_i + \epsilon_{ij}{}^kP^a_k{\omega_a}^{0j} + \epsilon_{ij}{}^k\Upsilon^j_a\pi^a_k\right) \\
&-\Lambda^i(\partial_aP^a_i + \epsilon_{ij}{}^kP^a_k\Upsilon^j_a - \epsilon_{ij}{}^k\pi^a_k{\omega_a}^{0j}) + \frac{\varsigma^i{}_a}{2}\left(\pi^a_i - \Xi\widetilde{\eta}^{abc}F_{ibc}\right) \\
&+\frac{\chi^i_a}{2}\left(P^a_i - \Xi\widetilde{\eta}^{abc}F_{0ibc}\right)\Big\},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
S_E \quad & [\Upsilon^i{}_a, \pi^a_i, {\omega_a}^{0i}, P^a_i, \tau^i, \Lambda^i, \varsigma^i{}_a, \chi^i_a] \\
&= \int dx^0 \int_\Sigma dx^3 \Big\{ -\frac{\Omega}{\Xi}\dot{\Upsilon}^i{}_aP^a_i - \frac{\Omega}{\Xi}\dot{\omega}_a{}^{0i}\pi^a_i + \frac{\Omega}{\Xi}\Lambda^i(\partial_a\pi^a_i + \epsilon_{ij}{}^kP^a_k{\omega_a}^{0j} \\
&+\epsilon_{ij}{}^k\Upsilon^j_a\pi^a_k) + \frac{\Omega}{\Xi}\tau^i(\partial_aP^a_i + \epsilon_{ij}{}^kP^a_k\Upsilon^j_a - \epsilon_{ij}{}^k\pi^a_k{\omega_a}^{0j}) \\
&+\frac{\Omega\chi^i_a}{2\Xi}\left(\pi^a_i - \Xi\widetilde{\eta}^{abc}F_{ibc}\right) + \frac{\Omega}{2\Xi}\varsigma^i{}_a\left(P^a_i - \Xi\widetilde{\eta}^{abc}F_{0ibc}\right)\Big\}.
\end{aligned} \tag{7}
$$

From the extended actions (6) and (7) we are able to identify the corresponding symplectic structures for Second Chern and Euler classes given by

Second Chern class

$$
\begin{aligned}
\{\Upsilon^i_a(x), \pi^b_j(y)\}_{sc} &= \delta^b_a \delta^i_j \delta^3(x-y), \\
\{\omega_a{}^{0i}(x), P^b_j(y)\}_{sc} &= \delta^b_a \delta^i_j \delta^3(x-y).
\end{aligned} \tag{8}
$$

Euler class

$$
\begin{aligned}
\{\Upsilon^i_a(x), \frac{\Omega}{\Xi} P^b_j(y)\}_E &= \delta^b_a \delta^i_j \delta^3(x-y), \\
\{\omega_a{}^{0i}(x), -\frac{\Omega}{\Xi} \pi^b_j(y)\}_E &= \delta^b_a \delta^i_j \delta^3(x-y).
\end{aligned} \tag{9}
$$

We observe that the two actions share the same dynamical variables, however, the corresponding symplectic structures are quite different from each other. In this manner, in spite of the actions (1) and (2) giving rise to the same equations of motion, in virtue of (8) and (9) we expect a different quantum description of the systems; these results confirm those reported in [18] where it was found that the Second-Chern and the Euler classes have different quantum states. In this way, the action principle presents a double role [19]; on one hand, the action provides the equations of motion and on the other ones fixes the symplectic structure, thus that the role of the action is very important as matter of fact beyond the equations of motion. Of course, there are approaches where the equations of motion are used to determine the symplectic geometry on the phase space [5], however in that approach the phase space is not endowed with a natural or preferred symplectic structure, this fact becomes to be important because the freedom in the choice of symplectic structures will be relevant at the classical and quantum level [19, 18].

From (1) and (2) we are able to identify that the actions share the following 24 constraints

$$
\begin{aligned}
\phi_i &:= \partial_a \pi^a_i + \epsilon_{ij}{}^k P^a_k \omega_a{}^{0j} + \epsilon_{ij}{}^k \Upsilon^j_a \pi^a_k &\approx& \ 0, \\
\psi_i &:= \partial_a P^a_i + \epsilon_{ij}{}^k P^a_k \Upsilon^j_a - \epsilon_{ij}{}^k \pi^a_k \omega_a{}^{0j} &\approx& \ 0, \\
\Phi^a_i &:= \left(\pi^a_i - \Xi \widetilde{\eta}^{abc} \left[2\partial_b \Upsilon_{ic} - \epsilon_{ijk} \omega_b{}^{j0} \omega_{c0}{}^k + \epsilon_{ijk} \Upsilon^j_b \Upsilon^k_c\right]\right) &\approx& \ 0, \\
\Psi^a_i &:= \left(P^a_i - \Xi \widetilde{\eta}^{abc} \left[2\partial_b \omega_{c0i} + 2\epsilon_{ijk} \omega_b{}^k{}_0 \Upsilon^j_c\right]\right) &\approx& \ 0,
\end{aligned} \tag{10}
$$

this fact will be relevant, because by using the new variables defined above we observe that the actions under study share the same constraints, however, the symplectic structures are different, this crucial part was not considered in [18]. On the other side, from (1) and (2) we are able to identify the corresponding extended Hamiltonians given by

$$
\begin{aligned}
H_{SC_E} &= \int_\Sigma \Big\{ \tau^i \left(\partial_a \pi^a_i + \epsilon_{ij}{}^k P^a_k \omega_a{}^{0j} + \epsilon_{ij}{}^k \Upsilon^j_a \pi^a_k \right) \\
&+ \Lambda^i (\partial_a P^a_i + \epsilon_{ij}{}^k P^a_k \Upsilon^j_a - \epsilon_{ij}{}^k \pi^a_k \omega_a{}^{0j}) \\
&- \frac{\varsigma^i{}_a}{2} \left(\pi^a_i - \alpha \widetilde{\eta}^{abc} \left[2\partial_b \Upsilon_{ic} - \epsilon_{ijk} \omega_b{}^{j0} \omega_{c0}{}^k + \epsilon_{ijk} \Upsilon^j_b \Upsilon^k_c \right] \right) \\
&- \frac{\chi^i_a}{2} \left(P^a_i - \alpha \widetilde{\eta}^{abc} \left[2\partial_b \omega_{c0i} + 2\epsilon_{ijk} \omega_b{}^k{}_0 \Upsilon^j_c \right] \right) \Big\}, \\
H_{E_E} &= \int_\Sigma \Big\{ - \frac{\Omega}{\Xi} \Lambda^i \left(\partial_a \pi^a_i + \epsilon_{ij}{}^k P^a_k \omega_a{}^{0j} - \epsilon_{ij}{}^k \Upsilon^j_a \pi^a_k \right) \\
&- \frac{\Omega}{\Xi} \tau^i (\partial_a P^a_i - \epsilon_{ij}{}^k P^a_k \Upsilon^j_a - \epsilon_{ij}{}^k \pi^a_k \omega_a{}^{0j}) \\
&- \frac{\Omega \chi^i_a}{2\Xi} \left(\pi^a_i - \Xi \widetilde{\eta}^{abc} \left[2\partial_b \Upsilon_{ic} - \epsilon_{ijk} \omega_b{}^{j0} \omega_{c0}{}^k + \epsilon_{ijk} \Upsilon^j_b \Upsilon^k_c \right] \right) \\
&- \frac{\Omega}{2\Xi} \varsigma^i{}_a \left(P^a_i - \Xi \widetilde{\eta}^{abc} \left[2\partial_b \omega_{c0i} + 2\epsilon_{ijk} \omega_b{}^k{}_0 \Upsilon^j_c \right] \right) \Big\}. \qquad (11)
\end{aligned}
$$

these Hamiltonians are linear combination of constraints. The constraints (10) are first class, whose algebra is given by

Second Chern class

$$
\begin{aligned}
\{\phi_i(x), \psi_j(y)\} &= \epsilon_{ij}{}^k \psi_k \delta^3(x-y), \\
\{\phi_i(x), \phi_j(y)\} &= \epsilon_{ij}{}^k \phi_k \delta^3(x-y), \\
\{\psi_i(x), \psi_j(y)\} &= -\epsilon_{ij}{}^k \phi_k \delta^3(x-y), \\
\{\phi_i(x), \Psi^a_j(y)\} &= \epsilon_{ij}{}^k \Psi^a_k \delta^3(x-y), \\
\{\psi_i(x), \Psi^a_j(y)\} &= -\epsilon_{ij}{}^k \Phi^a_k \delta^3(x-y), \\
\{\Phi^a_i(x), \Phi^b_j(y)\} &= 0, \\
\{\phi_i(x), \Phi^a_j(y)\} &= \epsilon_{ij}{}^k \Phi^a_k \delta^3(x-y), \\
\{\psi_i(x), \Phi^a_j(y)\} &= \epsilon_{ij}{}^k \Psi^a_k \delta^3(x-y), \\
\{\Psi^a_i(x), \Phi^b_j(y)\} &= 0, \\
\{\Psi^a_i(x), \Psi^b_j(y)\} &= 0. \qquad (12)
\end{aligned}
$$

Euler class

$$
\begin{aligned}
\{\phi_i(x), \psi_j(y)\} &= \frac{\Xi}{\Omega} \epsilon_{ij}{}^k \phi_k \delta^3(x-y), \\
\{\phi_i(x), \phi_j(y)\} &= -\frac{\Xi}{\Omega} \epsilon_{ij}{}^k \psi_k \delta^3(x-y),
\end{aligned}
$$

$$
\begin{aligned}
\{\psi_i(x), \psi_j(y)\} &= \frac{\Xi}{\Omega}\epsilon_{ij}{}^{k}\psi_k\delta^3(x-y), \\
\{\phi_i(x), \Psi^a_j(y)\} &= \frac{\Xi}{\Omega}\epsilon_{ij}{}^{k}\Phi^a_k\delta^3(x-y), \\
\{\psi_i(x), \Psi^a_j(y)\} &= \frac{\Xi}{\Omega}\epsilon_{ij}{}^{k}\Psi^a_k\delta^3(x-y), \\
\{\phi_i(x), \Phi^a_j(y)\} &= -\frac{\Xi}{\Omega}\epsilon_{ij}{}^{k}\Psi^a_k\delta^3(x-y), \\
\{\Phi^a_i(x), \Phi^b_j(y)\} &= 0, \\
\{\psi_i(x), \Phi^a_j(y)\} &= \frac{\Xi}{\Omega}\epsilon_{ij}{}^{k}\Phi^a_k\delta^3(x-y), \\
\{\Psi^a_i(x), \Phi^b_j(y)\} &= 0, \\
\{\Psi^a_i(x), \Psi^b_j(y)\} &= 0.
\end{aligned}
$$

It is important to observe that the algebra of the constraints for Second-Chern and Euler class is closed, however, because of the symplectic structures are different, both theories has an algebra with different structure, namely; for Second-Chern class we see that $\{\phi_i(x), \psi_j(y)\}$ is proportional to ψ_k, while for Euler class is proportional to ϕ_k, and so on with the rest of the Poisson brackets among the constraints. On the other hand, the extended Hamiltonians (11) are a linear combination of first class constraints as expected because of the background independence of the theories [4, 16, 18]. Furthermore, the constraints (10) are not independent because do exist 6 reducibility conditions given by

$$
\begin{aligned}
\partial_a\Phi^a_i &= \phi_i, \\
\partial_a\Psi^a_i &= \psi_i.
\end{aligned}
$$

Thus, with all constraints identified we are able to carry out the counting of degrees of freedom as follows: There are 36 canonical variables, [24-6]=18 independents first class constraints, and there are not second class constraints. Therefore the theories under study are devoid of physical degrees of freedom and correspond to topological theories. Therefore, these results complete and extend those ones reported in [18] in the sense of the present analysis was performed using a different approach.

3. Dirac's Canonical Analysis for Yang-Mills Theory Written as a BF-Like Theory

It has been commented above that there exist models where YM theory can be written as a BF-like theory [16]. In the later two sections, we perform

Dirac's canonical analysis for two different actions, leading to YM theory but with different symplectic structures, so we will find a similar situation as it was found for the topological invariants studied above.

First let us start with the following action [16]

$$S[A,B] = \int_M *B_a \wedge B^a - 2B_a \wedge *F^a, \tag{13}$$

where $a, b, c..$ are $SU(N)$ index, $B^a = \frac{1}{2}B^a{}_{\mu\nu}dx^\mu \wedge dx^\nu$ is a set of (N^2-1) $SU(N)$ valued 2-forms, $F^a = \frac{1}{2}F^a{}_{\mu\nu}dx^\mu \wedge dx^\nu$, with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a{}_{bc}A^b_\mu A^c_\nu$ is the curvature of the connection 1-form $A^a = A^a_\mu dx^\mu$. Here, $\mu, \nu = 0, 1, .., 3$ are spacetime indices, x^μ are the coordinates that label the points for the 4-dimensional Minkowski manifold M, and $*B_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}B^{\mu\nu}$ is the usual Hodge-duality operation. It is important to remark that the action (13) is the coupling of two topological theories in the sense that if we split (13), namely; in a term $S_1[A,B] = \int_M B_a \wedge *F^a$ and $S_2[A,B] = \int_M *B_a \wedge B^a$, S_1 and S_2 are topological ones [16]. It its important to observe that for the Euler class the star product acts on internal indices, while for the action (13) the star product acts on space-time indices; this fact will be very important because Euler class is a topological theory as it has been showed above, however (13) will not be topological anymore as it will be showed below.

So, the action on a Minkowski background takes the following form

$$S[B,A] = \int_M \left[\frac{1}{4}B^a{}_{\mu\nu}B_a{}^{\mu\nu} - \frac{1}{2}B^{\mu\nu}_a\left(\partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f^a_{\ bc}A^b{}_\mu A^c{}_\nu\right)\right] dx^4, \tag{14}$$

the equations of motion obtained from (14) read

$$\begin{aligned} B^a_{\mu\nu} &= \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + f^a_{\ bc}A^b{}_\mu A^c{}_\nu, \\ D_\mu B^{\mu\nu a} &= 0. \end{aligned} \tag{15}$$

We can observe that by substituting the first equation of motion in the second one, (15) is reduced to usual YM equations of motion. We can appreciate at this level the double role of the action (14), as we have already commented above; the first one is that the action give us the equations of motion (15), and the second one the action will fix the symplectic structure as we shall see below performing the Hamiltonian framework.

Thus, by performing the Hamiltonian analysis of (14) we obtain

$$S_E \quad [A^a_\mu, \Pi^\mu_a, B^a_{\mu\nu}, \Pi^{\mu\nu}_a, \lambda^a_0, \lambda^a, u^a_i, u^a_{0i}, u^a_{ij}, v^a_{ij}]$$

$$
\begin{aligned}
= &\quad \int d^4x(\dot{A}^a_\mu \Pi^\mu_a + \dot{B}^a_{\mu\nu}\Pi^{\mu\nu}_a - \frac{1}{2}\Pi^i_a \Pi^a_i + \frac{1}{4}B^a_{ij}B^{ij}_a \\
+ &\quad A^a_0 D_i \Pi^i_a - \frac{1}{2}B^{ij}_a F^a_{ij} - \lambda^a_0 \gamma^0_a - \lambda^a \gamma_a - u^a_i \chi^i_a - u^a_{0i}\chi^{0i}_a \\
- &\quad u^a_{ij}\chi^{ij}_a - v^a_{ij}\phi^{ij}_a),
\end{aligned} \tag{16}
$$

here $(\Pi^\alpha_a, \Pi^{\alpha\beta}_a)$ are canonically conjugate to $(A^a_\alpha, B^a_{\alpha\beta})$; $i, j, k.. = 1, 2, 3$, and $D_i \pi^i_a = \partial_i \pi^i_a + f_{abc}A^b_i \pi^{ic}$.

From (16) can be identified the following symplectic structure

$$
\begin{aligned}
\{A^a_\alpha(x), \Pi^\mu_b(y)\} &= \delta^\mu_\alpha \delta^a_b \delta^3(x-y), \\
\{B^a_{\alpha\beta}(x), \Pi^{\mu\nu}_b(y)\} &= \frac{1}{2}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)\delta^a_b \delta^3(x-y),
\end{aligned} \tag{17}
$$

and λ^a_0, λ^a are Lagrange multipliers enforcing the following $2(N^2-1)$ first class constraints

$$
\begin{aligned}
\gamma^0_a &= \Pi^0_a \approx 0, \\
\gamma_a &= D_i \Pi^i_a + 2f_{abc}B^b_{0i}\Pi^{0ic} + f_{abc}B^b_{ij}\Pi^{ijc} \approx 0,
\end{aligned} \tag{18}
$$

u^a_i, u^a_{0i}, u^a_{ij}, v^a_{ij} are Lagrange multipliers enforcing the following $12(N^2-1)$ second class constraints

$$
\begin{aligned}
\chi^i_a &= \Pi^i_a + B^{0i}_a \approx 0, \\
\chi^{0i}_a &= \Pi^{0i}_a \approx 0, \\
\chi^{ij}_a &= \Pi^{ij}_a \approx 0, \\
\phi^{ij}_a &= B^{ij}_a - F^{ij}_a \approx 0.
\end{aligned} \tag{19}
$$

Therefore, the counting of degrees of freedom is carry out as follows; there are $20(N^2-1)$ phase space variables, $2(N^2-1)$ independent first class constraints and $12(N^2-1)$ second class constraints, thus the theory given in (16) has $2(N^2-1)$ degrees of freedom, corresponding to the number of degrees of freedom for YM theory.

From the action (16) we also identify the extended Hamiltonian for the theory

$$
H_E = \frac{1}{2}\Pi^i_a \Pi^a_i - \frac{1}{4}B^a_{ij}B^{ij}_a - A^a_0 D_i \Pi^i_a + \frac{1}{2}B^{ij}_a F^a_{ij} - \lambda^a_0 \gamma^0_a - \lambda^a \gamma_a. \tag{20}
$$

Of course, the difference between the Hamiltonian (20) and the Hamiltonians (11), is that (20) is not linear combination of constraints; this fact is due to the action (13) is not background independent. Furthermore, if the second class constraints (19) are considered as strong equations, we recover the standard YM Hamiltonian.

With the constraints identified, we are able to calculate the gauge transformations on the phase space. For this aim we define the gauge generator in the following form

$$G = \int_{\Sigma} \left[D_0 \epsilon_0^a \gamma_a^0 + \epsilon^a \gamma_a \right] d^3x, \tag{21}$$

so, the gauge transformations are given by

$$\begin{aligned} \{A_\alpha^a, G\} &= -D_\alpha \epsilon^a, \\ \{\Pi_a^\alpha, G\} &= -f_{ba}{}^c \Pi_c^\alpha \epsilon^b, \end{aligned} \tag{22}$$

or

$$\begin{aligned} A_\alpha^a &\rightarrow A_\alpha^a - D_\alpha \epsilon^a, \\ \Pi_a^\alpha &\rightarrow \Pi_a^\alpha - \epsilon^b f_{ba}{}^c \Pi_c^\alpha. \end{aligned} \tag{23}$$

We finish this section with some remarks. On one hand, it is important to perform the quantum treatment of the action (13) and to find the differences respect to the standard YM theory; this subject of study is already in progress [16]. On the other, we can also introduce a constant g^2 in the second term of the action (13) namely, $S[A, B] = \int_M g^2 * B_a \wedge B^a - 2B_a \wedge *F^a$, thus, we are able to analyze the perturbative behavior in the constant g around the second term $\int_M B_a \wedge *F^a$ which corresponds to be topological one, and then comparing this behavior with the results obtained in [15] for Martellini's model. On the other side, in the following section we will analyze the relation between the action (13) and Martellini's model because both actions yield *on shell* to YM theory, however we shall see that their corresponding symplectic structures are different jus as in the case for topological invariants studied above.

4. Dirac's Canonical Analysis for Martellini's Model

It is well-know in the literature that there exists a different model to (13), with the particularity that its equations of motion also yield YM equations; that one is called Martellini's model [13, 14, 15]. As it has been comment above, Martellini's model is a deformation of a BF topological field theory where it is possible show that gives the first order formulation of YM theory, and it has been showed that the standard uv-behaviour is recovered [15], and new non local observables can be defined, which are related to the phase structure of the theory.

For these reasons, in this section we shall perform the Hamiltonian analysis for Martellini's model which is absent the literature, and we will compare the results obtained in this part with those found in Sec. III, all this part will be developed with in the aim to observe if there exists a similar situation just as for Second-Chern and Euler invariants.

Our starting point is the following action [13, 14, 15]

$$S[A,B] = \int_M \mathbf{i} B_a \wedge F^a + \frac{g^2}{4} B_a \wedge *B^a, \tag{24}$$

where $\mathbf{i}$ is the complex number, B^a, F^a and $*B$ are defined as in Sec. III, g is a coupling constant. We can observe that the action (13) and (24) share the same dynamical variables, however they are different, in (13) the star product acts in both terms of the action, while in (24) it does not; something similar occurs for the topological invariants. Furthermore, (24) has an imaginary factor $\mathbf{i}$ which is relevant in the path integral formulation.

Hence, on a Minkowski space-time the action takes the form

$$S[A,B] = \int_M \left[\frac{\mathbf{i}}{4} \epsilon^{\mu\nu\alpha\beta} F_{a\mu\nu} B^a{}_{\alpha\beta} + \frac{g^2}{4} B_{a\mu\nu} B^{a\mu\nu} \right] dx^4. \tag{25}$$

The equations of motion obtained from (25) read

$$\begin{aligned} \frac{\mathbf{i}}{4} \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} &= -\frac{g^2}{2} B^{a\alpha\beta}, \\ D_\beta (\epsilon^{\mu\nu\alpha\beta} B_{a\mu\nu}) &= 0, \end{aligned} \tag{26}$$

where the derivation D_β is defined as above. We observe that the action (24) yields on shell YM theory (just as the action (13)).

Thus, the Hamiltonian analysis of the action (25) leads to

$$\begin{aligned} & S_E[A^a_\mu, \Pi^\mu_a, B^a_{\mu\nu}, \Pi^{\mu\nu}_a, \lambda^a_0, \lambda^a, u^a_i, u^a_{0i}, u^a_{ij}, v^a_{0i}] \\ &= \int d^4x (\dot{A}^a_\mu \Pi^\mu_a + \dot{B}^a_{\mu\nu} \Pi^{\mu\nu}_a - \frac{1}{2} \Pi^i_a \Pi^a_i + 2g^2 B^a_{0i} B^{0i}_a + A^a_0 D_i \Pi^i_a \\ &+ \mathbf{i} \eta^{ijk} B_{a0i} F^a_{jk} - \lambda^a_0 \gamma^0_a - \lambda^a \gamma_a - u^a_i \phi^i_a - u^a_{0i} \phi^{0i}_a - u^a_{ij} \phi^{ij}_a - v^a_{0i} \psi^{0i}_a), \end{aligned} \tag{27}$$

where $(\Pi^\alpha_a, \Pi^{\alpha\beta}_a)$ are canonically conjugate to $(A^a_\alpha, B^a_{\alpha\beta})$, and we are able to identify the symplectic structure

$$\begin{aligned} \{A^a_\alpha(x), \Pi^\mu_b(y)\} &= \delta^\mu_\alpha \delta^a_b \delta^3(x-y), \\ \{B^a_{\alpha\beta}(x), \Pi^{\mu\nu}_b(y)\} &= \frac{1}{2} \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha \right) \delta^a_b \delta^3(x-y). \end{aligned} \tag{28}$$

On the other hand, λ_0^a, λ^a are Lagrange multipliers enforcing the following $2(N^2-1)$ first class constraints

$$\begin{aligned} \gamma_a^0 &= \Pi_a^0 \approx 0, \\ \gamma_a &= D_i\Pi_a^i + 2f_{abc}B_{0i}^b\Pi^{0ic} + f_{abc}B_{ij}^b\Pi^{ijc} \approx 0, \end{aligned} \tag{29}$$

and u_i^a, u_{0i}^a, u_{ij}^a, v_{0i}^a are Lagrange multipliers enforcing the following $12(N^2-1)$ second class constraints

$$\begin{aligned} \phi_a^i &= \Pi_a^i - \mathbf{i}\eta^{ijk}B_{jka} \approx 0, \\ \phi_a^{0i} &= \Pi_a^{0i} \approx 0, \\ \phi_a^{ij} &= \Pi_a^{ij} \approx 0, \\ \psi_a^{0i} &= 2g^2B_a^{0i} + \frac{\mathbf{i}}{2}\eta^{ijk}F_{ajk} \approx 0. \end{aligned} \tag{30}$$

It is important to remark that in (29) the second first class constraint corresponds to the Gauss constraint for this theory; on the other side, in virtue of (30) the symplectic structure (17) and (28) are different since in (14) the definition of the momenta is $\Pi_a^i = -B_a^{0i}$, while for (25) is $\Pi_a^i = \mathbf{i}\eta^{ijk}B_{jka}$, and this fact will be important in the quantum treatment. So, in spite of both actions (14) and (24) yield *on shell* YM theory and the fact that their corresponding symplectic structures are different from each other, we could expect a similar situation just as Second-Chern and Euler invariants studied in Sect. III, where the quantum theories are different. However, the quantum study of (14) is still in progress and we can not say yet if at quantum level the action (14) corresponds to YM theory; nevertheless, for an abelian group the action (14) is equivalent to Maxwell theory at classical and quantum level [16].

On the other hand, the extended Hamiltonian for the theory (25) is given by

$$H_E = \frac{1}{2}\Pi_a^i\Pi_i^a - 2g^2B_{0i}^aB_a^{0i} - A_0^aD_i\Pi_a^i - \mathbf{i}\eta^{ijk}B_{a0i}F_{jk}^a - \lambda_0^a\gamma_a^0 - \lambda^a\gamma_a. \tag{31}$$

With all this information at hand, we are able to carry out the counting of physical degrees of freedom as follows; There are $20(N^2-1)$ phase space variables, $2(N^2-1)$ independent first class constraints and $12(N^2-1)$ second class constraints, thus the theory given in (24) has $2(N^2-1)$ degrees of freedom.

Now, it is straightforward to prove that the gauge transformations on the phase space for the theory under study are those for YM theory. For this aim we define the gauge generator in the form

$$G = \int_{\Sigma} \left[D_0 \epsilon_0^a \gamma_a^0 + \epsilon^a \gamma_a \right] d^3x, \tag{32}$$

hence, the gauge transformations are given by

$$\begin{aligned} \{A_\alpha^a, G\} &= -D_\alpha \epsilon^a, \\ \{\Pi_a^\alpha, G\} &= -f_{ca}{}^b \Pi_b^\alpha \epsilon^c, \end{aligned} \tag{33}$$

or

$$\begin{aligned} A_\alpha^a &\rightarrow A_\alpha^a - D_\alpha \epsilon^a, \\ \Pi_a^\alpha &\rightarrow \Pi_a^\alpha - \epsilon^c f_{ca}{}^b \Pi_b^\alpha. \end{aligned} \tag{34}$$

Therefore, we have presented the Hamiltonian study for two actions that give rise to YM equations of motion, nevertheless, their respective symplectic structures are different. The actions have a close relation with topological theories, and we believe that these facts will be relevant in the quantum scenario. The analysis presented in this paper has been performed at classical level and the quantum approach will be reported in forthcoming works [16], expecting to confirm a similar situation as was found for the Second-Chern and the Euler invariants [18].

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