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ON SIMPLE SHEAR FOR INCOMPRESSIBLE ISOTROPIC LINEAR ELASTIC MATERIALS

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Abstract: In this paper we discuss some problems involving simple shear in incompressible isotropic linear elastic materials within the framework of the linearized finite theory of elasticity. First we obtain for a simple shear a universal relation in terms of components of the first Piola-Kirchhoff stress tensor. Afterwards for a rectangular block deformed by a simple shear we evaluate the absolute error and the relative error both for the Piola-Kirchhoff tractions and the Cauchy tractions calculated by classical linear elasticity. Finally we discuss two dead load problems corresponding to different Piola-Kirchhoff tractions by using both the linearized finite theory of elasticity and the classical linear elasticity. The first problem can be solved only in linearized finite theory of elasticity and the solution is a simple shear. The second problem admits a simple shear as a solution in both theories, so that we can compare the solutions.

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1. Introduction

Simple shear is one of the main modes of behaviour of incompressible bodies, such as rubber-like materials. Although incompressible materials are usually

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described by the finite elasticity, the constraint of incompressibility can be also used in connection with small-deformation problems. In this case it is worth using a theory which guarantees the accuracy required by a linear model.

In this paper we study some problems concerning simple shear in incompressible isotropic elastic materials according to the linearized finite theory of elasticity, as formulated by Hoger and Johnson in [2], [3]. Such a theory, based on a strict procedure of linearization of the corresponding finite constitutive equations with respect to the displacement gradient, describes the behaviour of constrained linear elastic materials with the accuracy required by a linear theory. In fact, for such materials the classical linear theory of elasticity is commonly adopted, but, as shown in [2], [3], the constitutive equations of linearized finite theory of elasticity contain some terms usually dropped in classical linear elasticity, all of which are first order in the strain. In particular for incompressible isotropic bodies by coincidence the constitutive equations provided by the two theories for the Cauchy stress are the same, while the constitutive equations for the first Piola-Kirchhoff stress are different. In linearized finite theory of elasticity the expression of the Piola-Kirchhoff stress contains a term which is the product of the pressure and the displacement gradient; in classical linear elasticity such a term disappears, since it is understood that small strains correspond to small pressures, while this assumption leads to neglect an essential characteristic of constrained materials.

Motivated by the previous remarks, in this paper we apply the linearized finite theory of elasticity to the study of simple shear. In Section 2 we briefly recall the field equations appropriate for the linearized finite theory of elasticity. In Section 3 we show that in such a theory for a simple shear a universal relation in terms of components of the first Piola-Kirchhoff stress holds; the distinction between Cauchy stress and Piola-Kirchhoff stress imposed by this theory is crucial in order to find such a relation. In Section 4 by means of a consistent procedure of linearization we obtain this universal relation from a corresponding universal relation in finite elasticity. Afterwards in Section 5 we determine the Piola-Kirchhoff tractions and the Cauchy tractions on the boundary of a rectangular block deformed by a simple shear both in linearized finite theory of elasticity and in classical linear elasticity. Following [2], we show that the absolute error in the Piola-Kirchhoff tractions calculated by classical elasticity can be arbitrarily large, but the relative error is first order in the strain; the same applies to Cauchy tractions. Finally, motivated by the unexpected results obtained in [2], [8] for some dead load problems, in Section 6 we discuss two dead load problems corresponding to different Piola-Kirchhoff tractions. We show that the first problem cannot be solved in classical linear elasticity, while in linearized finite theory of elasticity it admits a simple shear as a solution. Afterwards we show that the second problem can be solved in both theories, so that we can compare the solutions which are two simple shears. For both problems we define the range of tractions for which the linearized finite theory of elasticity applies and we show that the shear modulus plays a central role in determining such a range. Section 7 is devoted to concluding remarks.

2. The Linearized Finite Theory of Elasticity for Constrained Materials

In this section we gather the field equations of the so-called linearized finite theory of elasticity (LFTE in the following), derived in 1995 by Hoger and Johnson in [2], [3]. For many reasons extensively exposed in [2], [3], LFTE is the most suitable theory for constrained linear elastic materials, since only this theory is based on constitutive equations having the accuracy required by a linear model. Here we confine our attention to the constitutive equations of LFTE appropriate for solid incompressible isotropic elastic materials.

Let $\mathcal{B}_0$ be a fixed reference configuration of the body; denote by $\mathbf{X}$ a material point in $\mathcal{B}_0$ and by $\mathbf{x} = \mathbf{f}(\mathbf{X})$ the corresponding point in the deformed configuration $\mathcal{B} = \mathbf{f}\,(\mathcal{B}_0)$, where $\mathbf{f}$ is the deformation function. Let $\mathbf{u}$ be the displacement, $\mathbf{F}$ the deformation gradient, $\mathbf{H}$ the displacement gradient, given by

$$\mathbf{u}\,(\mathbf{X}) = \mathbf{f}(\mathbf{X}) - \mathbf{X}, \tag{1}$$

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{f}, \tag{2}$$

$$\mathbf{H} = \mathrm{Grad}\,\mathbf{u} = \mathbf{F} - \mathbf{I}; \tag{3}$$

in (2), (3), Grad denotes the gradient operator taken with respect to $\mathbf{X}$ and $\mathbf{I}$ is the identity tensor.

Since LFTE is a theory appropriate for small deformations, $\mathbf{H}$ is assumed to be small and everywhere only terms that are at most linear in $\mathbf{H}$ are retained.

If we linearize about the zero strain state the finite Green strain tensor

$$\mathbf{E}_G = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \tag{4}$$

and if we use (3), we obtain the infinitesimal strain tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right). \tag{5}$$

In finite elasticity the possible strains $\mathbf{E}_G$ for an elastic material subject to a constraint must satisfy the constraint equation

$$\hat{c}\,(\mathbf{E}_G) = 0. \tag{6}$$

If we linearize (6) we obtain the linear constraint equation

$$\tilde{c}(\mathbf{E}) = 0, \tag{7}$$

where $\tilde{c}(\mathbf{E}) \equiv \dfrac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) \cdot \mathbf{E}$ is the linear constraint function and $\mathbf{O}$ denotes the zero tensor.

We now devote our attention to the constitutive equations of the body, which is assumed to be elastic and homogeneous. In finite elasticity for constrained hyperelastic materials the Cauchy stress $\mathbf{T}$ is the sum of the determinate stress and the reaction stress, that is the constitutive equation for $\mathbf{T}$ is

$$\mathbf{T} = \frac{1}{\det \mathbf{F}}\,\mathbf{F}\,\frac{\partial \hat{W}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T + q\mathbf{F}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{E}_G)\mathbf{F}^T, \tag{8}$$

where $\hat{W}\,(\mathbf{E}_G)$ is the strain energy function and q is a Lagrange multiplier. In (8) both $\hat{W}\,(\mathbf{E}_G)$ and $\hat{c}\,(\mathbf{E}_G)$ are functions of the polynomial invariants of the strain $\mathbf{E}_G$ appropriate for the material symmetry required.

The first Piola-Kirchhoff stress $\mathbf{S}$ is defined in terms of the Cauchy stress $\mathbf{T}$ as follows

$$\mathbf{S} = (\det \mathbf{F})\mathbf{T}\mathbf{F}^{-T}. \tag{9}$$

In LFTE the constitutive equations for $\mathbf{T}$ and $\mathbf{S}$ are derived by linearization of the corresponding finite constitutive equations (8), (9) with respect to the displacement gradient. It is worth noting that the procedure of linearization of LFTE is based on the following requirements: the strain energy function for the constrained material is taken to be that one of the unconstrained material with the same material symmetry; the strain energy function for the unconstrained material is retained until all differentiation is carried out; the linearized constraint equation (7) is substituted after the differentiation is complete; the linearizations of $\dfrac{\partial \hat{W}}{\partial \mathbf{E}_G}$ and $\dfrac{\partial \hat{c}}{\partial \mathbf{E}_G}$ must be parallel.

The final expressions for $\mathbf{T}$ and $\mathbf{S}$ appropriate for LFTE are

$$\begin{aligned}\mathbf{T} = \;& \left.\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_c \mathbf{E} + q\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ & + q\,\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + q\,\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O})\mathbf{H}^T + q\,\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\mathbf{E}\end{aligned} \tag{10}$$

$$\begin{aligned}\mathbf{S} = &\left.\frac{\partial^2 \hat{W}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\right|_c \mathbf{E} + q\,\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + q\,\mathrm{tr}\,\mathbf{E}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + \\ &+ q\,\mathbf{H}\frac{\partial \hat{c}}{\partial \mathbf{E}_G}(\mathbf{O}) + q\,\frac{\partial^2 \hat{c}}{\partial \mathbf{E}_G \partial \mathbf{E}_G}(\mathbf{O})\mathbf{E},\end{aligned} \tag{11}$$

respectively (see [3], formulas (3.22), (3.23)); in (10), (11) the subscript $_c$ indicates evaluation on the linear constraint equation (7).

For the constraint of incompressibility, (6), (7) take the form

$$\det(2\mathbf{E}_G + 1) = 1 \tag{12}$$

$$\mathrm{tr}\,\mathbf{E} = 0, \tag{13}$$

respectively. Moreover, for isotropy $\hat{W}$ is a function of the polynomial invariants $\mathbf{I}\cdot\mathbf{E}_G$, $\mathbf{I}\cdot\mathbf{E}_G^2$, $\mathbf{I}\cdot\mathbf{E}_G^3$. Then, for incompressible isotropic materials in LFTE the Cauchy stress (10) and the Piola-Kirchhoff stress (11) reduce to

$$\mathbf{T} = 2\mu\mathbf{E} - p\mathbf{I} \tag{14}$$

and

$$\mathbf{S} = 2\mu\mathbf{E} - p\left(\mathbf{I} - \mathbf{H}^T\right), \tag{15}$$

respectively (see [3], formulas (4.15), (4.16)); in (14), (15) μ is the shear modulus and $p = -2q$ is the pressure.

In classical linear elasticity for constrained materials the typical method followed in order to construct linear constitutive equations is very different. As shown in [3], Section 5, it is *assumed* that the constitutive equation for the Cauchy stress, denoted by $\mathbf{T}_{cl}$, is

$$\mathbf{T}_{cl} = \frac{\partial \tilde{W}_c}{\partial \mathbf{E}}(\mathbf{E}) + q\frac{\partial \tilde{c}}{\partial \mathbf{E}}(\mathbf{E}), \tag{16}$$

where $\tilde{c}(\mathbf{E})$ is the linear constraint function and $\tilde{W}_c$ is the quadratic strain energy function of the corresponding unconstrained material that has been evaluated with $\tilde{c}(\mathbf{E}) = 0$ (see [3], formula (5.1)). In general, there are several terms missing from (16) as compared to (10), both for the determinate stress and the reaction stress; such terms are first order in the strain, so that the constitutive equation (16) usually adopted in classical linear elasticity is not correct at first order in the displacement gradient.

For incompressible isotropic materials equation (16) reduces to (14), so that casually the constitutive equations provided by classical linear elasticity and LFTE coincide (see [3], Section 5, for more details).

Now we turn to first Piola-Kirchhoff stress tensor $\mathbf{S}$. In classical linear elasticity for constrained materials it is *stated* that the Piola-Kirchhoff stress and the Cauchy stress coincide (see [2], Section 7), as occurs for unconstrained materials. Then for isotropic linear elastic bodies the classical constitutive equation for the Piola-Kirchhoff stress is

$$\mathbf{S}_{cl} = 2\mu\mathbf{E} - p\mathbf{I}. \tag{17}$$

If we compare (15) to (17), we see that the two constitutive equations differ by a term that is linear in $\mathbf{H}$. Then, only if the pressure p is small, so that in (15) the term $p\mathbf{H}^T$ can be neglected, the two constitutive equations coincide. In classical linear elasticity it is *a priori* assumed that small strains correspond to small pressures. Since for incompressible materials the strains can be small even under large pressures, LFTE must be adopted.

Finally, we list the field equations of linearized elastostatics for incompressible isotropic materials according to LFTE

$$\begin{aligned}
&\mathbf{H} = \operatorname{Grad}\mathbf{u}\\
&\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right)\\
&\mathbf{W} = \frac{1}{2}\left(\mathbf{H} - \mathbf{H}^T\right)\\
&\operatorname{tr}\mathbf{E} = 0\\
&\mathbf{S} = 2\mu\mathbf{E} - p\,\mathbf{I} + p\left(\mathbf{E} - \mathbf{W}\right), \quad p = -\operatorname{tr}\mathbf{S}\\
&\operatorname{Div}\mathbf{S} + \mathbf{b} = \mathbf{0}.
\end{aligned} \tag{18}$$

These equations hold on the undeformed body. In $(18)_3$, $\mathbf{W}$ is the infinitesimal rotation tensor; in $(18)_6$ Div denotes the divergence operator taken with respect to $\mathbf{X}$, while $\mathbf{b}$ is the body force density measured per unit volume of $\mathcal{B}_0$.

For completeness' sake we also recall the corresponding field equations appropriate for classical linear elasticity

$$\begin{aligned}
&\mathbf{H} = \operatorname{Grad}\mathbf{u}\\
&\mathbf{E} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^T\right)\\
&\operatorname{tr}\mathbf{E} = 0\\
&\mathbf{S}_{cl} = 2\mu\mathbf{E} - p\,\mathbf{I}, \quad p = -\operatorname{tr}\mathbf{S}_{cl}\\
&\operatorname{Div}\mathbf{S}_{cl} + \mathbf{b} = \mathbf{0}.
\end{aligned} \tag{19}$$

3. Simple Shear According to LFTE. A New Universal Relation. Comparison to Classical Linear Elasticity

In this section we study the problem of simple shear within the context of LFTE; we obtain a new universal relation in terms of components of the Piola-Kirchhoff stress $\mathbf{S}$. Finally we compare our results to those of classical linear elasticity.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis. A simple shear of a rectangular block is a homogeneous deformation $\mathbf{x} = \mathbf{f}(\mathbf{X})$ defined with respect to such a basis as

$$\begin{aligned} x_1 &= X_1 + \gamma X_2 \\ x_2 &= X_2 \\ x_3 &= X_3, \end{aligned} \tag{20}$$

where $\gamma > 0$ is an arbitrary dimensionless constant called the amount of shear. Simple shear was first considered in 1948 by Rivlin [5] within the framework of the nonlinear elasticity; various problems related to simple shear have been analyzed since, both for compressible and incompressible isotropic solid elastic materials (see, among many others, [1], [4], [9]). An important result concerning simple shear is the Poynting effect, which is a typical feature of the nonlinear elasticity. In fact, constitutive equations for compressible or incompressible isotropic nonlinear elastic bodies provide for the components of the Cauchy stress $\mathbf{T}$ the well-known *universal relation* due to Rivlin

$$T_{11} - T_{22} = \gamma T_{12} : \tag{21}$$

not only are the normal stresses not equal to zero, but in general they cannot even equal one another (Poynting effect). Moreover (21) shows that the normal Cauchy stresses T_{11} and T_{22} determine the shear stress T_{12}, while T_{11} and T_{22} cannot be determined by T_{12}.

In classical linear elasticity the normal Cauchy stresses are of the order of terms neglected, the Poynting effect disappears and shear stress suffices to produce simple shear.

The previous remarks hold both for compressible and incompressible isotropic materials.

Let us now consider the homogeneous deformation of simple shear (20) according to LFTE. From (1) we have

$$\begin{aligned} u_1 &= \gamma X_2 \\ u_2 &= 0 \\ u_3 &= 0\,; \end{aligned} \tag{22}$$

then by (3), (22) we obtain

$$[\mathbf{H}] = \begin{bmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

In virtue of (5), (23) the infinitesimal strain tensor is

$$[\mathbf{E}] = \begin{bmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

For an incompressible isotropic material subject to simple shear (20) constitutive equation (15) for the Piola-Kirchhoff stress provides

$$[\mathbf{S}] = \begin{bmatrix} -p & \mu\gamma & 0 \\ p\gamma + \mu\gamma & -p & 0 \\ 0 & 0 & -p \end{bmatrix}. \tag{25}$$

Since the deformation (20) is homogeneous and the body is assumed to be homogeneous, the equilibrium equation $(18)_6$ in the absence of body force is satisfied if and only if p is constant.

We now consider the components of $\mathbf{S}$ in (25): matrix (25) shows that the non-zero components of $\mathbf{S}$ satisfy the relations

$$S_{12} - S_{21} = \gamma\, S_{11} = \gamma\, S_{22} = \gamma\, S_{33}. \tag{26}$$

Moreover the following relation holds

$$S_{12} - S_{21} = \frac{1}{3}(\mathrm{tr}\,\mathbf{S})\,\gamma. \tag{27}$$

Relations (26), (27) are first obtained in this paper; when a simple shear is applied, they are satisfied by every isotropic incompressible linear elastic body in LFTE, so that they are *universal relations*. From (26) we see that the components S_{12} and S_{21} of $\mathbf{S}$ determine the normal components S_{11}, S_{22}, S_{33}, while the opposite is untrue.

It is worth noting that (26), (27) exhibit terms as γS_{ii} (i not summed), or $(\mathrm{tr}\,\mathbf{S})\,\gamma$; since $\gamma S_{ii} = -pH_{12}$ and $(\mathrm{tr}\,\mathbf{S})\,\gamma = -3\,p\,H_{12}$, such terms must be retained in order to have accuracy to first order in the strain, since the pressure can be large even if the strain is small.

A second remark concerns the comparison of LFTE to classical linear elasticity. The classical linear elasticity for constrained materials is valid if the pressure is small, so that the constitutive equation for the Piola-Kirchhoff stress is given by (17). In this case the symmetry of $\mathbf{S}_{cl}$ forces to vanish the left-hand side of (26); on the other hand, for small pressures the terms in the right-hand sides of (26) must be neglected, so that in (26) all sides reduce to zero.

In this sense, the well-known results for simple shear in classical linear elasticity can be obtained by the more general results for simple shear in LFTE.

A final remark concerns the choice of the stress tensor. The distinction between $\mathbf{S}$ and $\mathbf{T}$, as clearly claimed in LFTE, is crucial in order to find universal relation (26), since no relation involving the components of $\mathbf{T}$ can be obtained.

4. Universal Relation for Simple Shear: Comparison to Finite Elasticity

In this section we show that universal relation (26) obtained for $\mathbf{S}$ in LFTE follows from a universal relation for $\mathbf{S}$ in finite theory, if a suitable procedure of linearization is applied.

First we recall the constitutive equations for an incompressible isotropic elastic material within the framework of the finite elasticity, both for the Cauchy stress $\mathbf{T}$ and the Piola-Kirchhoff stress $\mathbf{S}$. For the Cauchy stress $\mathbf{T}$ the constitutive equation is

$$\mathbf{T} = -p\mathbf{I} + \varphi_1 \mathbf{F}\mathbf{F}^T + \varphi_{-1}\mathbf{F}^{-T}\mathbf{F}^{-1}, \tag{28}$$

where the coefficients φ_1 and φ_{-1} are functions of the first two invariants of the tensor $\mathbf{F}\mathbf{F}^T$ (see [9], formula (49.5)).

If we substitute in (9) constitutive equation (28) and the constraint condition $\det \mathbf{F} = 1$ we obtain the following constitutive equation for the Piola-Kirchhoff stress $\mathbf{S}$

$$\mathbf{S} = -p\mathbf{F}^{-T} + \varphi_1 \mathbf{F} + \varphi_{-1}\mathbf{F}^{-T}\mathbf{F}^{-1}\mathbf{F}^{-T}. \tag{29}$$

Let us now consider simple shear (20); in virtue of (2) the matrix corresponding to the deformation gradient $\mathbf{F}$ is

$$[\mathbf{F}] = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{30}$$

Then equation (28) provides for $\mathbf{T}$ the matrix

$$[\mathbf{T}] = \begin{bmatrix} -p+\varphi_1(1+\gamma^2)+\varphi_{-1} & (\varphi_1-\varphi_{-1})\gamma & 0 \\ (\varphi_1-\varphi_{-1})\gamma & -p+\varphi_1+\varphi_{-1}(1+\gamma^2) & 0 \\ 0 & 0 & -p+\varphi_1+\varphi_{-1} \end{bmatrix}, \tag{31}$$

while (29) gives for $\mathbf{S}$ the matrix

$$[\mathbf{S}]=\begin{bmatrix} -p+\varphi_1+\varphi_{-1}(1+\gamma^2) & (\varphi_1-\varphi_{-1})\gamma & 0 \\ p\gamma-2\varphi_{-1}\gamma-\varphi_{-1}\gamma^3 & -p+\varphi_1+\varphi_{-1}(1+\gamma^2) & 0 \\ 0 & 0 & -p+\varphi_1+\varphi_{-1} \end{bmatrix}. \tag{32}$$

Note that in (31), (32) according to finite elasticity the terms of second order and third order in γ must be retained.

Of course the components of $\mathbf{T}$ in (31) satisfy the Rivlin's universal relation (21); in addition to this well-known relation matrix (32) shows that another relation involving some components of $\mathbf{S}$ holds

$$S_{12} - S_{21} = \gamma S_{11} = \gamma S_{22}. \tag{33}$$

Now we linearize relation (33). To this aim we first linearize the components of $\mathbf{S}$ in (32). The requirement that the undeformed configuration be a natural state provides in linear theory the condition

$$\varphi_1(3,3) - \varphi_{-1}(3,3) = \mu \tag{34}$$

(see [9], formula (50.14)); moreover the requirement that the residual stress be zero in the undeformed configuration (see [3], formulas (3.11), (3.12)) imposes the further restriction

$$\varphi_1(3,3) + \varphi_{-1}(3,3) = 0. \tag{35}$$

Then linearization of (33), obtained by using (34), (35) and retaining only terms that are of first order in γ, provides the first three sides of (26).

The anomaly represented by the last term in (26) is due to the different powers of γ appearing in (32).

The results obtained for simple shear exhibit that, unlike the classical linear elasticity for constrained materials, in one sense LFTE retains a clear memory of finite elasticity. This particular feature of LFTE is emphasized also in [2], where Hoger and Johnson show that in LFTE a dead load traction problem may admit multiple solutions, as occurs in finite elasticity.

5. The Piola-Kirchhoff Tractions and the Cauchy Tractions in LFTE. Absolute Error and Relative Error with Respect to Classical Linear Elasticity

In this section we determine the Piola-Kirchhoff tractions and the Cauchy tractions on the boundary of a rectangular block deformed by a simple shear both in LFTE and in classical linear elasticity. Moreover we show that the absolute error in the Piola-Kirchhoff tractions calculated by classical elasticity can be arbitrarily large, but the relative error is first order in the strain; such a discrepancy will be significant when very high accuracy is required, for instance in numerical simulations; the same applies to Cauchy tractions.

Let $\mathbf{n}_0$ be the outward unit normal to the surface of the undeformed body. The Piola-Kirchhoff traction is defined as $\mathbf{S}\mathbf{n}_0$. For a simple shear in LFTE the Piola-Kirchhoff stress $\mathbf{S}$ is given by (25), so that the Piola-Kirchhoff tractions on the boundary of the rectangular block are

$$\mathbf{S}\mathbf{e}_1 = \begin{pmatrix} -p \\ p\gamma + \mu\gamma \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_2 = \begin{pmatrix} \mu\gamma \\ -p \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix}. \tag{36}$$

As noted in Section 2, in classical linear elasticity the Piola-Kirchhoff stress tensor is given by $(19)_4$. Since the values of the pressure in $(18)_5$ and $(19)_4$ are determined by appropriate boundary conditions, we give boundary conditions such that $\operatorname{tr}\mathbf{S} = \operatorname{tr}\mathbf{S}_{cl}$; then the classical Piola-Kirchhoff tractions on the boundary are

$$\mathbf{S}_{cl}\mathbf{e}_1 = \begin{pmatrix} -p \\ \mu\gamma \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_2 = \begin{pmatrix} \mu\gamma \\ -p \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix}. \tag{37}$$

Therefore we note that in LFTE simple shear (20) produces traction vectors $\mathbf{S}\mathbf{e}_1$ and $\mathbf{S}\mathbf{e}_2$ such that $|\mathbf{S}\mathbf{e}_1| \neq |\mathbf{S}\mathbf{e}_2|$, while in classical linear elasticity $|\mathbf{S}_{cl}\mathbf{e}_1| = |\mathbf{S}_{cl}\mathbf{e}_2|$.

From (36), (37) we see that for $\mathbf{n}_0 = \mathbf{e}_1$ the absolute error in the Piola-Kirchhoff tractions is

$$|\mathbf{S}\mathbf{e}_1 - \mathbf{S}_{cl}\mathbf{e}_1| = |p|\,\gamma, \tag{38}$$

while for $\mathbf{n}_0 = \mathbf{e}_2$ and $\mathbf{n}_0 = \mathbf{e}_3$ the tractions in the two theories coincide.

Thus, because p can be arbitrarily large, also the absolute error can be arbitrarily large.

The corresponding relative error is defined as $\dfrac{|\mathbf{S}\mathbf{e}_1 - \mathbf{S}_{cl}\mathbf{e}_1|}{|\mathbf{S}\mathbf{e}_1|}$, where before linearization

$$|\mathbf{S}\mathbf{e}_1| = |p| \left[1 + \left(\frac{p+\mu}{p} \right)^2 \gamma^2 \right]^{\frac{1}{2}}. \tag{39}$$

Then the relative error in the Piola-Kirchhoff tractions consistent with a linear theory is

$$\frac{|\mathbf{S}\mathbf{e}_1 - \mathbf{S}_{cl}\mathbf{e}_1|}{|\mathbf{S}\mathbf{e}_1|} = \gamma, \tag{40}$$

that is the relative error is first order in the strain.

We now determine the Cauchy tractions according to LFTE. In general, in finite elasticity the outward unit normal $\mathbf{n}$ to the surface of the deformed body is related to the normal $\mathbf{n}_0$ of the undeformed body through

$$\mathbf{n} = \left(\mathbf{F}^{-T}\mathbf{n}_0 \cdot \mathbf{F}^{-T}\mathbf{n}_0\right)^{-\frac{1}{2}} \mathbf{F}^{-T}\mathbf{n}_0\,; \tag{41}$$

linearization of (41) yields

$$\mathbf{n} = \left[(1 + \mathbf{n}_0 \cdot \mathbf{E}\mathbf{n}_0)\,\mathbf{I} - \mathbf{H}^T\right] \mathbf{n_0} \tag{42}$$

(see [2], formula (2.8)).

Then by linearizing (9) and using (42), $(18)_5$ we obtain the Cauchy tractions $\mathbf{Tn}$ on the deformed body in terms of the Piola-Kirchhoff tractions $\mathbf{S}\mathbf{n}_0$ on the undeformed body

$$\mathbf{Tn} = \mathbf{S}\mathbf{n}_0 - p\,(\mathbf{n}_0 \cdot \mathbf{E}\mathbf{n}_0)\,\mathbf{n}_0 \tag{43}$$

(see [2], formula (4.3)); (43) shows that in general in LFTE the difference between $\mathbf{Tn}$ and $\mathbf{S}\mathbf{n}_0$ is first order in the strain.

We consider now simple shear (20). Since $\mathbf{E}$ is given by (24), we see from (42) that the normals corresponding to $\mathbf{n}_0 = \mathbf{e}_1$, $\mathbf{n}_0 = \mathbf{e}_2$, $\mathbf{n}_0 = \mathbf{e}_3$, are $\mathbf{n} = \begin{pmatrix} 1 \\ -\gamma \\ 0 \end{pmatrix}$, $\mathbf{n} = \mathbf{e}_2$, $\mathbf{n} = \mathbf{e}_3$. The vector $\begin{pmatrix} 1 \\ -\gamma \\ 0 \end{pmatrix}$ is a unit vector, since we deal with a linear theory. Then in LFTE by (43), (36) we obtain for the Cauchy tractions the expressions

$$\mathbf{T}\begin{pmatrix} 1 \\ -\gamma \\ 0 \end{pmatrix} = \begin{pmatrix} -p \\ p\gamma + \mu\gamma \\ 0 \end{pmatrix}, \quad \mathbf{T}\mathbf{e}_2 = \begin{pmatrix} \mu\gamma \\ -p \\ 0 \end{pmatrix}, \quad \mathbf{T}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix}; \tag{44}$$

comparison of (44) to (36) shows that for a simple shear in LFTE the Cauchy tractions and the Piola-Kirchhoff tractions coincide.

Finally we determine the Cauchy tractions according to classical linear elasticity. The usual assumption is

$$\mathbf{T}_{cl}\mathbf{n} = \mathbf{S}_{cl}\mathbf{n}_0 \tag{45}$$

(see [2], formula (7.4)): in classical linear elasticity for any deformation the Cauchy tractions and the Piola-Kirchhoff tractions coincide. Then for simple shear (20) by (45), (37) we have

$$\mathbf{T}_{cl}\begin{pmatrix} 1 \\ -\gamma \\ 0 \end{pmatrix} = \begin{pmatrix} -p \\ \mu\gamma \\ 0 \end{pmatrix}, \quad \mathbf{T}_{cl}\mathbf{e}_2 = \begin{pmatrix} \mu\gamma \\ -p \\ 0 \end{pmatrix}, \quad \mathbf{T}_{cl}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ -p \end{pmatrix}. \tag{46}$$

It follows that for the absolute error and the relative error in Cauchy tractions calculated by classical elasticity the remarks concerning Piola-Kirchhoff tractions hold.

6. Two Piola-Kirchhoff Traction Problems in LFTE and Classical Linear Elasticity

In this section we consider two dead load problems corresponding to different Piola-Kirchhoff tractions which are prescribed on the boundary of the rectangular block.

We show that the first problem can be solved in LFTE and the solution is a simple shear, while no solution exists in classical linear elasticity.

Afterwards we show that the second problem can be solved both in LFTE and in classical linear elasticity; moreover we note that the simple shear obtained as a solution in LFTE reduces to the simple shear which solves the problem in classical linear elasticity when the pressure is small.

Finally for both problems we determine the range of tractions for which LFTE applies.

(i) First we consider within the framework of LFTE the dead load problem corresponding to the following Piola-Kirchhoff tractions

$$\mathbf{S}\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_2 = \begin{pmatrix} \delta \\ \alpha \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \tag{47}$$

where $\beta \neq \delta$. We will discuss the case $\beta = \delta$ in the second problem.

A homogeneous Piola-Kirchhoff stress which is in agreement with tractions (47) is

$$[\mathbf{S}] = \begin{bmatrix} \alpha & \delta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}. \tag{48}$$

If we compare (48) to $(18)_5$, we see that $p = -\alpha$; moreover if $\alpha \neq 2\mu$ we obtain the components of the infinitesimal strain tensor $(18)_2$ in terms of tractions (47)

$$[\mathbf{E}] = \begin{bmatrix} 0 & \dfrac{1}{2}\dfrac{\beta+\delta}{2\mu-\alpha} & 0 \\ \dfrac{1}{2}\dfrac{\beta+\delta}{2\mu-\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{49}$$

We now recall that the infinitesimal rotation tensor $\mathbf{W}$ is defined by $(18)_3$; then from $(18)_5$, (48) we see that the components of $\mathbf{W}$ in terms of the tractions are

$$[\mathbf{W}] = \begin{bmatrix} 0 & \dfrac{1}{2}\dfrac{\delta-\beta}{\alpha} & 0 \\ \dfrac{1}{2}\dfrac{\beta-\delta}{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{50}$$

Finally the displacement $\mathbf{u}$ from the origin is

$$\mathbf{u} = (\mathbf{E} + \mathbf{W})\,\mathbf{X}, \tag{51}$$

where $\mathbf{E}$ and $\mathbf{W}$ are given by (49) and (50), respectively.

Then in LFTE simple shear (49) is the solution of the dead load problem (47). When the pressure is such that $\alpha = 2\,\mu$, we have $\beta = -\delta$; it follows that $\mathbf{E}$ is arbitrary up to $\mathrm{tr}\,\mathbf{E} = 0$, while $W_{12} = -W_{21} = \dfrac{\delta}{2\mu}$.

We now prescribe the same tractions in classical linear elasticity, that is we set

$$\mathbf{S}_{cl}\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_2 = \begin{pmatrix} \delta \\ \alpha \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}, \tag{52}$$

where $\beta \neq \delta$. In this case no solution exists for the dead load problem (52), because the Piola-Kirchhoff stress $\mathbf{S}_{cl}$ given by $(19)_4$ is symmetric, while in (52) we have $\beta \neq \delta$.

(ii) We now prescribe in LFTE the Piola-Kirchhoff tractions

$$\mathbf{S}\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_2 = \begin{pmatrix} \beta \\ \alpha \\ 0 \end{pmatrix}, \quad \mathbf{S}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}. \tag{53}$$

Tractions (53) provide a symmetric stress $\mathbf{S}$; moreover by $(18)_5$ we have $p = -\alpha$ and

$$[\mathbf{E}] = \begin{bmatrix} 0 & \dfrac{\beta}{2\mu - \alpha} & 0 \\ \dfrac{\beta}{2\mu - \alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{54}$$

if $\alpha \neq 2\mu$.

Of course (54) can be also obtained from (49) by setting $\beta = \delta$, but we discuss the case $\beta \neq \delta$ and the case $\beta = \delta$ separately, in order to compare our results with those ones of the classical linear elasticity.

Note that in this case the symmetry of $\mathbf{S}$ has as a consequence the symmetry of the tensor $\mathbf{H}$, so that $\mathbf{W} = \mathbf{O}$; the displacement from the origin is then

$$\mathbf{u} = \mathbf{E}\mathbf{X}, \tag{55}$$

where $\mathbf{E}$ is given by (54).

If $\alpha = 2\mu$, we obtain $\beta = 0$, $\mathbf{E}$ arbitrary up to $\operatorname{tr}\mathbf{E} = 0$, and $\mathbf{W} = \mathbf{O}$.

Finally we solve in classical linear elasticity the dead load problem corresponding to the same tractions, that is we set

$$\mathbf{S}_{cl}\mathbf{e}_1 = \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_2 = \begin{pmatrix} \beta \\ \alpha \\ 0 \end{pmatrix}, \quad \mathbf{S}_{cl}\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix}. \tag{56}$$

Equation $(19)_4$ provides $p = -\alpha$; moreover, denoting by $\mathbf{E}_{cl}$ the infinitesimal strain obtained by classical linear elasticity, from $(19)_4$ we have

$$[\mathbf{E}_{cl}] = \begin{bmatrix} 0 & \dfrac{\beta}{2\mu} & 0 \\ \dfrac{\beta}{2\mu} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{57}$$

Since the infinitesimal rotation $\mathbf{W}$ is arbitrary, the displacement from the origin, denoted by $\mathbf{u}_{cl}$, is given in terms of the infinitesimal strain (57) by the formula

$$\mathbf{u}_{cl} = \mathbf{E}_{cl}\mathbf{X}, \tag{58}$$

to within an arbitrary infinitesimal rigid body displacement.

Therefore the second Piola-Kirchhoff traction problem can be solved both in LFTE and in classical linear elasticity. The corresponding solutions are the simple shear (54) and the simple shear (57), respectively. Our aim is now to compare solution (54) to solution (57). By (54) we see that $\beta = 2\mu E_{12} + pE_{12}$; in classical linear elasticity the pressure p is supposed to be small, the product pE_{12} is neglected, so that strain (54) reduces to strain (57).

The same occurs also in [8], where another dead load problem is solved both in LFTE and in classical linear elasticity and the corresponding solutions are compared.

In one sense the qualitative behaviour of the solutions for LFTE and classical linear elasticity parallels the behaviour of the corresponding constitutive equations.

Finally we turn our attention to the solutions obtained for LFTE both for the first problem and the second problem in order to define the range of tractions for which LFTE applies.

If we use $(18)_2$, $(18)_3$ and the condition $\mathbf{E} \cdot \mathbf{W} = 0$, we obtain

$$|\mathbf{E}|^2 + |\mathbf{W}|^2 = |\mathbf{H}|^2 . \tag{59}$$

Since we deal with a linear theory, we impose a suitable restriction on $|\mathbf{H}|$; suppose for instance that $|\mathbf{H}| \leq \tilde{H}$ for a particular material. Then $|\mathbf{E}|$ and $|\mathbf{W}|$ must satisfy the restriction

$$|\mathbf{E}|^2 + |\mathbf{W}|^2 \leq \tilde{H}^2. \tag{60}$$

We now consider the first problem, for which $\mathbf{E}$ and $\mathbf{W}$ are given by (49), (50) respectively; restriction (60) takes the form

$$\frac{(\beta+\delta)^2}{2\,(2\mu-\alpha)^2\,\tilde{H}^2} + \frac{(\beta-\delta)^2}{2\alpha^2\tilde{H}^2} \leq 1. \tag{61}$$

The range of tractions (47) for which LFTE applies is then represented by the region whose boundary is the ellipse

$$\frac{(\beta+\delta)^2}{2\,(2\mu-\alpha)^2\,\tilde{H}^2} + \frac{(\beta-\delta)^2}{2\alpha^2\tilde{H}^2} = 1. \tag{62}$$

Finally we consider the second problem, for which $\mathbf{E}$ is given by (54), while $\mathbf{W} = \mathbf{O}$. In this case restriction (60) shows that the region in which LFTE applies is defined by

$$|\beta| \leq \frac{1}{\sqrt{2}}\,|2\mu-\alpha|\,\tilde{H}. \tag{63}$$

Conditions (61), (63) emphasize that the shear modulus μ plays a central role in determining the range of tractions for which LFTE applies; for instance for rubber materials the shear modulus μ is $4.225 \cdot 10^5$ N/m^2.

7. Conclusions

The stress-strain relations usually adopted to describe the behaviour of constrained materials within the framework of the linear elasticity are those of the so-called classical linear elasticity. For many reasons such a theory is inadequate in order to have the accuracy required by a linear theory, so that for constrained linear elastic materials the constitutive equations of the linearized finite theory of elasticity must be adopted (see [2], [3]). In this paper, by using the linearized finite theory of elasticity, we obtain unexpected results concerning simple shear in incompressible linear elastic materials.

Other results provided by such a theory which are unexpected for a linear theory can be found in [2], [8] as regards static problems, and in [6], [7] as concerns wave propagation; for instance, in [2] Hoger and Johnson show that a dead load problem for incompressible linear elastic materials may have multiple solutions. Therefore this paper represents a natural continuation of [2], [3], [6], [7], [8].

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