International Journal of Pure and Applied Mathematics
Volume 75 No. 4 2012, 485-492
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

DECOMPOSITIONS OF VARIOUS COMPLETE GRAPHS INTO ISOMORPHIC COPIES OF THE 4-CYCLE WITH A PENDANT EDGE

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Abstract: Necessary and sufficient conditions are given for the existence of isomorphic decompositions of the complete bipartite graph, the complete graph with a hole, and the λ-fold complete graph into copies of a 4-cycle with a pendant edge.

AMS Subject Classification: 05B40
Key Words: graph decompositions, complete graph, complete bipartite graph, complete graphs with a hole

1. Introduction

A *g-decomposition of graph G* is a set of subgraphs of G, $\gamma = \{g_1, g_2, \ldots, g_n\}$, where $g_i \cong g$ for $i \in \{1, 2, \ldots, n\}$, $E(g_i) \cap E(g_j) = \emptyset$ for $i \neq j$, and $\cup_{i=1}^{n} E(g_i) = E(G)$. The g_i are called *blocks* of the decomposition. When G is a complete

Received: December 3, 2011

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graph, the g-decomposition is often called a *graph design*. The study of graph designs and graph decompositions is a vibrant area of research [3, 4, 8]. Several studies have centered on g-decompositions of complete graphs into copies of a given graph g with a small number of vertices [1, 2, 5, 6, 7]. This study takes a slightly different approach and concentrates on g-decompositions of different types of complete graphs for a given g. The g which is the topic of this study is the 4-cycle with a pendant edge. We denote this graph as H. That is, $V(H) = \{a, b, c, d, e\}$ and $E(H) = \{(a, b), (b, c), (c, d), (a, d), (a, e)\}$; we represent this H as $[a, b, c, d; e]$. See Figure 1.1. An H-decomposition of K_v exists if and only if $v \equiv 0$ or 1 (mod 5), $v \geq 10$ [1].

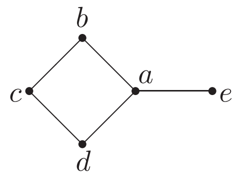

Figure 1.1: We denote this graph as $H = [a, b, c, d; e]$

2. H-Decompositions of $K_{m,n}$

We assume the partite sets of the complete bipartite graph, $K_{m,n}$, are $V_m = \{0_1, 1_1, \ldots, (m-1)_1\}$ and $V_n = \{0_2, 1_2, \ldots, (n-1)_2\}$.

Theorem 2.1. *There is an H-decomposition of $K_{m,n}$ if and only if $mn \equiv 0$ (mod 5), $m \geq 5$, and $n \geq 2$.*

Proof. Since $|E(K_{m,n})| = mn$, and H has 5 edges, $mn \equiv 0$ (mod 5) is necessary. Since H is bipartite with one partite vertex set consisting of 2 vertices, both m and n must be at least 2.

Graph H is bipartite itself and each of its partite sets has a single vertex of odd degree. If $m = 3$ and $n = 5k$ then an H-decomposition of $K_{m,n}$ would require $3k$ copies of H. However, $3k$ copies of H can only produce a bipartite graph with at most $3k$ odd degree vertices in each partite set. But if $m = 5k$ then one of the partite sets contains $5k$ vertices of odd degree. So no H-decomposition of $K_{m,n}$ exists when $m = 3$ and $n = 5k$. Therefore $m \geq 5$.

Case 1. Suppose $m \equiv 0$ (mod 2) and $n \equiv 0$ (mod 5). Then an H-decomposition of $K_{m,n}$ is given by

$\{[(1+2i)_1, (5j)_2, (2i)_1, (1+5j)_2; (2+5j)_2], [(2i)_1, (3+5j)_2, (1+2i)_1, (4+5j)_2;$

$$(2+5j)_2] \mid i = 0, 1, \ldots, m/2-1, j = 0, 1, \ldots, n/5-1\}.$$

Throughout, we reduce vertex labels by a modulus appropriate for the vertex set we use.

Case 2. Suppose $m \equiv 1 \pmod 2$, $m \geq 5$, and $n \equiv 0 \pmod 5$. Then and H-decomposition of $K_{m,n}$ is given by

$$\{[0_1, (5j)_2, 1_1, (1+5j)_2; (4+5j)_2], [3_1, (1+5j)_2, 4_1, (2+5j)_2; (5j)_2],$$

$$[2_1, (2+5j)_2, 0_1, (3+5j)_2; (1+5j)_2], [1_1, (3+5j)_2, 3_1, (4+5j)_2; (2+5j)_2],$$

$$[4_1, (5j)_2, 2_1, (4+5j)_2; (3+5j)_2], [(6+2i)_1, (5j)_2, (5+2i)_1, (1+5j)_2; (2+5j)_2],$$

$$[(5+2i)_1, (3+5j)_2, (6+2i)_1, (4+5j)_2; (2+5j)_2] \mid i = 0, 1, \ldots, (m-5)/2-1,$$

$$j = 0, 1, \ldots, n/5-1\}.$$

In both cases, the given set is a decomposition of $K_{m,n}$. □

3. *H*-Decompositions of $K(v, w)$

The *complete graph of order v with a hole of size w*, $K(v, w)$, is the graph with vertex set $V(K(v, w)) = V_{v-w} \cup V_w$, where we assume these sets are $V_{v-w} = \{0_1, 1_1, \ldots, (v-w-1)_1\}$ and $V_w = \{0_2, 1_2, \ldots, (w-1)_2\}$, and edge set $E(K(v, w)) = \{(a, b) \mid a, b \in V(K(v, w)) \text{ and } \{a, b\} \not\subset V_w\}$.

Theorem 3.1. *There is an H-decomposition of $K(v, w)$ if and only if $|E(K(v, w))| \equiv 0 \pmod 5$, $v - w \geq 4$, and $(v, w) \notin \{(5, 1), (6, 1)\}$.*

Proof. Of course, $|E(K(v, w))| \equiv 0 \pmod 5$ is necessary. If $v - w = 1$ then $K(v, w)$ is a star and there clearly is no H-decomposition. We cannot have $v - w = 2$, since there is then no possible H a subgraph of $K(v, w)$ which can contain the edge $(0_1, 1_1)$.

Case 1a. Suppose $(v \pmod 5, w \pmod 5) \in \{(0, 0), (1, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$ and $v - w \geq 10$. Now $K(v, w) = K_{v-w} \cup K_{v-w,w}$ where the vertex set of K_{v-w} is V_{v-w} and the partite sets of $K_{v-w,w}$ are V_{v-w} and V_w. In each case, K_{v-w} can be decomposed [1] and $K_{v-w,w}$ can be decomposed by Theorem 2.1. Therefore $K(v, w)$ can be decomposed.

Case 1b. Suppose $v \equiv w \equiv 0 \pmod 5$ and $v - w = 5$. A decomposition of $K(10, 5)$ is given by the set $\{[0_1, 2_1, 1_1, 4_1; 1_2], [2_2, 3_1, 2_1, 4_1; 1_1], [1_1, 0_2, 2_1, 1_2; 0_1], [3_2, 3_1, 0_2, 4_1; 1_1], [4_2, 3_1, 1_2, 4_1; 2_1], [3_1, 1_1, 4_2, 0_1; 4_1], [0_1, 2_2, 2_1, 3_2; 0_2]\}$. If $v =$

$5+5k$ and $w=5k$, then $K(v,w)=K(10,5)\cup(k-1)\times K_{5,5}$ where the partite sets of the the ith copy of $K_{5,5}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{(5+5i)_2,(6+5i)_2,\ldots,(9+5i)_2\}$. $K(10,5)$ is decomposed above and $K_{5,5}$ can be decomposed by Theorem 2.1.

Case 1c. Suppose $v\equiv 1 \pmod 5$ and $w\equiv 0 \pmod 5$ and $v-w=6$. A decomposition of $K(11,5)$ is given by the set $[0_1,0_2,1_1,1_2;3_1],[1_1,2_2,0_1,3_2;4_1],[2_1,0_2,3_1,1_2;5_1],[3_1,3_2,2_1,2_2;4_2],[4_1,0_2,5_1,1_2;4_2],[5_1,3_2,4_1,2_2;4_2],[0_1,1_1,3_1,4_1;4_2],[1_1,2_1,4_1,5_1;4_2],[2_1,3_1,5_1,0_1;4_2]\}$. If $v=6+5k$ and $w=5k$, then $K(v,w)=K(11,5)\cup(k-1)\times K_{6,5}$ where the partite sets of the ith copy of $K_{6,5}$ are $\{0_1,1_1,2_1,3_1,4_1,5_1\}$ and $\{(5+5i)_2,(6+5i)_2,\ldots,(9+5i)_2\}$. $K(11,5)$ is decomposed above and $K_{6,5}$ can be decomposed by Theorem 2.1.

Case 1d. Suppose $v\equiv w\equiv 1 \pmod 5$ and $v-w=5$. We know that if $v=6$ and $w=1$, then $K(6,1)=K_6$ and no decomposition of K_6 exists [1]. First, $K(11,6)=K(7,2)\cup K_{5,4}$ where the partite sets of $K_{5,4}$ are $\{0_1,1_1,\ldots,4_1\}$ and $\{2_2,3_2,4_2,5_2\}$. A decomposition of $K(7,2)$ is given by the set $\{[4_1,1_1,2_1,3_1;0_2],[0_1,1_1,3_1,0_2;1_2],[2_1,0_2,1_1,1_2;0_1],[4_1,1_2,3_1,0_1;2_1]\}$, and $K_{5,4}$ can be decomposed by Theorem 2.1. If $v=6+5k$ and $w=1+5k$, then $K(v,w)=K(11,6)\cup(k-1)\times K_{5,5}$ where the partite sets of the ith copy of $K_{5,5}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{(6+5i)_2,(7+5i)_2,\ldots,(10+5i)_2\}$. A decomposition of $K(11,6)$ is given above and $K_{5,5}$ can be decomposed by Theorem 2.1.

Case 1e. Suppose $v\equiv w\equiv 2 \pmod 5$ and $v-w=5$. First, $K(12,7)=K(10,5)\cup K_{5,2}$ where the partite sets of $K_{5,2}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{5_2,6_2\}$. $K(10,5)$ can be decomposed by Case 1b and $K_{5,2}$ can be decomposed by Theorem 2.1. If $v=7+5k$ and $w=2+5k$, then $K(v,w)=K(12,7)\cup(k-1)\times K_{5,5}$ where the partite sets of the the ith copy of $K_{5,5}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{(7+5i)_2,(8+5i)_2,\ldots,(11+5i)_2\}$. $K(12,5)$ can be decomposed as described above and $K_{5,5}$ can be decomposed by Theorem 2.1.

Case 1f. Suppose $v\equiv w\equiv 3 \pmod 5$ and $v-w=5$. A decomposition of $K(8,3)$ is given by the set $\{[0_1,1_2,4_1,2_2;3_1],[2_1,0_2,3_1,1_2;2_2],[0_1,1_1,3_1,2_1;0_2],[4_1,1_1,2_2,3_1;0_1],[1_1,2_1,4_1,0_2;1_2]\}$. If $v=8+5k$ and $w=3+5k$, then $K(v,w)=K(8,3)\cup(k-1)\times K_{5,5}$ where the partite sets of the ith copy of $K_{5,5}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{(3+5i)_2,(4+5i)_2,\ldots,(7+5i)_2\}$. A decomposition of $K(8,3)$ is given above and $K_{5,5}$ can be decomposed by Theorem 2.1.

Case 1g. Suppose $v\equiv w\equiv 4 \pmod 5$ and $v-w=5$. First, $K(9,4)=K(7,2)\cup K_{5,2}$ where the partite sets of $K_{5,2}$ are $\{0_1,1_1,2_1,3_1,4_1\}$ and $\{2_2,3_2\}$. $K(7,2)$ can be decomposed by Case 1d and $K(5,2)$ can be decomposed by Theorem 2.1. If $v=9+5k$ and $w=4+5k$, then $K(v,w)=K(9,4)\cup(k-1)\times K_{5,5}$

where the partite sets of the the ith copy of $K_{5,5}$ are $\{0_1, 1_1, 2_1, 3_1, 4_1\}$ and $\{(4+5i)_2, (5+5i)_2, \ldots, (8+5i)_2\}$. $K(9,4)$ can be decomposed as described above and $K_{5,5}$ can be decomposed by Theorem 2.1.

Case 2. Suppose $v \equiv 0 \pmod 5$ and $w \equiv 1 \pmod 5$. First, $K(5,1) = K_5$ and no decomposition of K_5 exists. A decomposition of $K(10,6)$ is given by $\{[0_1, 1_2, 1_1, 0_2; 3_1], [3_1, 0_2, 2_1, 1_2; 1_1], [2_1, 2_2, 1_1, 3_2, ; 3_1], [0_1, 3_2, 3_1, 2_2; 2_1], [1_1, 5_2, 3_1, 4_2; 2_1], [0_1, 5_2, 2_1, 4_2; 1_1]\}$. If $v = w + 4$ and $w \equiv 1 \pmod 5$, $w \geq 11$, then $K(v,w) = K(10,6) \cup K_{v-w,w-6}$, where $V(K(10,6)) = V_{v-w} \cup \{0_2,\ 1_2, \ldots, 5_2\}$ and the hole is on vertex set $\{0_2, 1_2, \ldots, 5_2\}$, and the partite sets of $K_{v-w,w-6}$ are V_{v-w} and $\{6_2, 7_2, \ldots, (w-1)_2\}$. $K(10,6)$ is decomposed above and $K_{v-w,w-6}$ can be decomposed by Theorem 2.1. For the other values of v and w in this case, $K(v,w) = K_{v-w+1} \cup K_{v-w,w-1}$ where the vertex set of K_{v-w+1} is $V_{v-w} \cup \{0_2\}$ and the partite sets of $K_{v-w,w-1}$ are V_{v-w} and $V_w \setminus \{0_2\}$. K_{v-w+1} can be decomposed [1] and $K_{v-w,w-1}$ can be decomposed by Theorem 2.1.

Case 3. Suppose $v \equiv 2 \pmod 5$ and $w \equiv 4 \pmod 5$. First, if $v - w = 3$, say $w = 4 + 5k$ and $v = 7 + 5k$, then $K(v,w)$ has $15(k+1)$ edges and an H-decomposition of $K(v,w)$ would consist of $3(k+1)$ copies of H. Similar to the proof of the nonexistence of a decomposition of $K_{3,5k}$ in Theorem 2.1, such a decomposition would have at most $3(k+1)$ odd degree vertices in the hole, but each of the $4+5k$ vertices in the hole are of odd degree. So no such decomposition exists. A decomposition of $K(12,4)$ is given by $\{[7_1, 0_1, 6_1, 1_1; 5_1], [3_1, 4_1, 2_1, 5_1; 1_1], [6_1, 2_1, 7_1, 3_1; 4_1], [0_1, 4_1, 1_1, 5_1; 2_1], [7_1, 4_1, 5_1, 6_1; 0_2], [0_1, 1_1, 2_1, 3_1; 3_2][1_2, 0_1, 0_2, 1_1;\ 7_1], [0_2, 2_1, 1_2, 3_1; 6_1], [1_2, 4_1, 0_2, 5_1;\ 6_1], [2_2, 7_1, 3_2, 6_1; 0_1], [3_2, 5_1, 2_2, 4_1; 1_1], [2_2, 3_1, 3_2, 2_1; 1_1]\}$. For the other values of v and w in this case, $K(v,w) = K(12,4) \cup K(v-8,w) \cup K_{8,w-4}$ where $V(K(12,4)) = \{0_1, 1_1, \ldots, 7_1,\ 0_2, 1_2, 2_2, 3_2\}$ and the hole is on the vertex set $\{0_2, 1_2, 2_2, 3_2\}$, $V(K(v-8,w)) = V_{v-w} \cup V_w \setminus \{0_1, 1_1, \ldots, 7_1\}$ and the hole is on the vertex set V_w, and the partite sets of $K_{8,w-4}$ are $\{0_1, 1_1, \ldots, 7_1\}$ and $V_w \setminus \{0_2, 1_2, 2_2, 3_2\}$. $K(12,4)$ is decomposed above, $K(v-8,w)$ can be decomposed by Case 1, and $K_{8,w-4}$ can be decomposed by Theorem 2.1.

Case 4. Suppose $v \equiv 4 \pmod 5$ and $w \equiv 2 \pmod 5$. A decomposition $K(9,2)$ is given by $\{[0_2, 0_1, 1_2, 1_1; 2_1], [0_2, 3_1, 1_2, 4_1; 5_1], [5_1, 6_1, 0_1, 2_1; 1_1], [2_1, 1_1, 4_1,\ 3_1;\ 6_1], [1_1, 3_1, 5_1, 0_1; 6_1], [6_1, 4_1, 0_1, 3_1; 0_2], [1_2, 2_1, 4_1, 5_1; 6_1]\}$. For the other values of v and w in this case, $K(v,w) = K(9,2) \cup K(v-7,w) \cup K_{7,w-2}$ where $V(K(9,2)) = \{0_1, 1_1, \ldots, 6_1, 0_2, 1_2\}$ and the hole is on the vertex set $\{0_2, 1_2\}$, $V(K(v-7,w)) = V_{v-w} \cup V_w \setminus \{0_1, 1_1, \ldots 6_1\}$ and the hole is on the vertex set V_w, and the partite sets of $K_{7,w-2}$ are $\{0_1, 1_1, \ldots, 6_1\}$ and $V_w \setminus \{0_2, 1_2\}$. $K(9,2)$ is decomposed above, $K(v-7,w)$ can be decomposed by Case 1, and $K_{7,w-2}$

can be decomposed by Theorem 2.1. □

4. Decompositions of λK_v

The λ-fold complete graph, λK_v, is the multigraph with edge multiset $E(\lambda K_v))$ $= \{\lambda \times (a,b) \mid a \neq b$ and $\{a,b\} \subset V(\lambda K_v)\}$.

Theorem 4.1. *There is an H-decomposition of λK_v if and only if*

(a) $v \equiv 0$ *or* 1 $(mod\ 5)$ *and* $v \geq 10$ *when* $\lambda = 1$, *or*

(b) $\lambda \equiv 0$ $(mod\ 5)$ *and* $v \geq 5$.

Proof. Since $|E(\lambda K_v)| = \lambda v(v-1)/2$ and $|E(H)| = 5$, then a necessary condition for an H-decomposition of λK_v is that $\lambda v(v-1)/2 \equiv 0$ (mod 5), and the necessary conditions follow. For $v = 5$ and $\lambda = 2$, the set $\{[0,2,3,4;1],[3,1,2,4;0],[2,0,1,4;3],[1,3,0,4;2]\}$ forms a decomposition where $V(2K_5) = \{0,1,2,3,4\}$. For $v = 5$ and $\lambda = 3$, the set $\{[0,3,1,4;2],[1,2,3,4;0],$ $[4,3,0,2;1],[2,4,0,1;3],[2,1,\ 3,0;4],[3,4,0,1;2]\}$ forms a decomposition. For $v = 5$ and $\lambda \geq 4$, a decomposition follows by taking repeated copies of the decompositions from the $\lambda = 2$ and $\lambda = 3$ cases. For $v = 6$ and $\lambda = 2$, the set $\{[i,1{+}i,2{+}i,4{+}i;3{+}i] \mid i = 0,1,\ldots,5\}$ forms a decomposition where $V(2K_6) =$ $\{0,1,2,3,4,5\}$. For $v = 6$ and $\lambda = 3$, the set $\{[5,2,4,3;1],[2,0,4,1;3],[0,2,5,4;$ $3],[2,1,5,3;4],[5,2,3,4;1],[2,0,3,1;4],[1,4,\ 5,0;3],[0,1,4,3;5],[0,1,3,5;4]\}$ forms a decomposition. For $v = 6$ and $\lambda \geq 4$, a decomposition follows similarly to the case of $v = 5$. For $v \equiv 0$ or 1 (mod 5), $v \geq 10$, an H-decomposition of K_v exists, and hence an H-decomposition of λK_v exists. For the remaining values of v, we have $\lambda \equiv 0$ (mod 5), so in these cases it is sufficient to present the constructions for $\lambda = 5$ only.

Case 1. Suppose $v \equiv 0$ (mod 4), $v \geq 8$, say $v = 4k$ and $\lambda = 5$. For $v = 8$, consider the set $B_1 = \{2 \times [0,1,3,2;4],[\infty,0,3,6;1],[0,3,\infty,5;1]\}$. For $v \geq 12$, consider the set:

$$B_1 = \{[\infty,0,2k-5,4k-9;1],[0,2k-3,\infty,2k-2;2k-1]\}$$

$$\cup\{2 \times [0,1,3,2;2k-1], 2 \times [0,3,7,4;2k-1]\}$$

$$\cup\{[0,5+2i,11+4i,6+2i;1+i] \mid i = 0,1,\ldots,k-4\}$$

$$\cup\{[0,5+2i,11+4i,6+2i;k-2+i] \mid i = 0,1,\ldots,k-4\}.$$

Define the permutation π on $\{0, 1, 2, \ldots, v-2, \infty\}$ as $\pi = (\infty)(0, 1, 2, \cdots, v-2)$. Then the set $\gamma = \{\pi^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_1 \text{ and } i = 0, 1, \ldots, v-2\}$ is an H-decomposition of λK_v where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-2, \infty\}$.

Case 2. Suppose $v \equiv 1 \pmod 4$, say $v = 4k+1$ and $\lambda = 5$. Consider the set:

$$B_2 = \{[0, 1+2i, 3+4i, 2+2i; 1+i] \mid i = 0, 1, \ldots, k-1\}$$
$$\cup\{[0, 1+2i, 3+4i, 2+2i; k+1+i] \mid i = 0, 1, \ldots, k-1\}.$$

Define the permutation ρ on $\{0, 1, 2, \ldots, v-1\}$ as $\rho = (0, 1, 2, \cdots, v-1)$. Then the set $\gamma = \{\rho^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_2 \text{ and } i = 0, 1, \ldots, v-1\}$ is an H-decomposition of λK_v where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-1\}$.

Case 3. Suppose $v \equiv 2 \pmod 4$, $v \geq 10$, say $v = 4k+2$ and $\lambda = 5$. Consider the set:

$$B_3 = \{[\infty, 0, 2k, 4k; 1], [0, 2k, \infty, 2k-1; 2k+2]\}$$
$$\cup\{[0, 1+2i, 3+4i, 2+2i; 1+i] \mid i = 0, 1, \ldots, k-1\}$$
$$\cup\{[0, 1+2i, 3+4i, 2+2i; k+1+i] \mid i = 0, 1, \ldots, k-2\}.$$

Then the set $\gamma = \{\pi^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_3 \text{ and } i = 0, 1, \ldots, v-2\}$ is an H-decomposition of λK_v where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-2, \infty\}$, where π is defined in Case 1.

Case 4. Suppose $v \equiv 3 \pmod 4$, say $v = 4k+3$ and $\lambda = 5$. For $v = 7$, consider the set $B_4 = \{2 \times [0, 1, 3, 2; 4], [0, 3, 6, 2; 1]\}$. For $v \geq 11$, consider the set:

$$B_4 = \{2 \times [0, 1, 3, 2; 2k+1], 2 \times [0, 3, 7, 4; 2k+1], [0, 2k-3, 4k-3, 2k-2; 2k+1]\}$$
$$\cup\{[0, 5+2i, 11+4i, 6+2i; 1+i] \mid i = 0, 1, \ldots, k-3\}$$
$$\cup\{[0, 5+2i, 11+4i, 6+2i; k-1+i] \mid i = 0, 1, \ldots, k-3\}.$$

Then the set $\gamma = \{\rho^i([a, b, c, d; e]) \mid [a, b, c, d; e] \in B_4 \text{ and } i = 0, 1, \ldots, v-1\}$ is an H-decomposition of λK_v where $V(\lambda K_v) = \{0, 1, 2, \ldots, v-1\}$, where ρ is defined in Case 2. $\square$

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