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WELL-POSEDNESS OF A QUASILINEAR PARABOLIC OPTIMAL CONTROL PROBLEM

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Abstract: In this paper we investigate the existence and uniqueness for the solution of the problem of determining the $v = (v_0, v_1, v_2)$ in the quasilinear parabolic equation $\frac{\partial y}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\lambda_i(y, v_0)\frac{\partial y}{\partial x_i}] + \sum_{i=1}^{n} B_i(y, v_1)\frac{\partial y}{\partial x_i} = f(x, t, v_2)$. For the objective functional $J_\beta(v) = \int_S [y(\zeta, t) - f_0(\zeta, t)]^2 d\zeta dt + \beta \sum_{m=0}^{2} \|v_m - \omega_m\|_{l_2}^2$, it is proven that the problem has at least one solution for $\beta \geq 0$, and has a unique solution for $\beta > 0$.

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1. Introduction

Optimal control problems for partial differential equations are currently of much interest. A larage amount of the theoretical concept which governed by quasilinear parabolic equations [1-5] has been investigated in the field of optimal control problems. These problems have dealt with the processes of hydro- and gasdynamics, heatphysics, filtration, the physics of plasma and others [6-8]. The study and determination of the optimal regimes of heat conduction processes at a long interval of the change of temperture gives rise to optimal control

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problems with respect to a quasilinear equation of parabolic type. In this work, we consider a constrained optimal control problem with respect to a quasilinear parabolic equation with controls in the coefficients of the equation. The existence and uniqueness of the optimal control problem is proved.

2. Formulation of the Problem

Let D is a bounded domain of the N-dimensional Euclidean space E_N; Γ be the boundary of D, assumed to be sufficiently smooth; ν is the exterior unit normal of Γ; $T > 0$ be a fixed time ; $\Omega = D \times (0, T]$; $S = \Gamma \times (0, T]$.

Now we consider a class of optimal control problems governed by the following quasilinear parabolic system.

$$L(v)y(x,t) = f(x,t,v_2), (x,t) \in \Omega,$$

$$y(x,0) = \phi(x), x \in D,$$

$$\textstyle\sum_{i=1}^{n} \lambda_i(y,v_0)\frac{\partial y}{\partial x_i}\cos(\nu,x_i)|_S = g(\zeta,t), (\zeta,t) \in S \tag{1}$$

where $\phi \in L_2(D), g(\zeta,t) \in L_2(S)$ are given functions and the differential operator L takes the following form:

$$L(v)z(x,t) = \frac{\partial z}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}[\lambda_i(z,v_0)\frac{\partial z}{\partial x_i}] + \sum_{i=1}^{n} B_i(z,v_1)\frac{\partial z}{\partial x_i} \tag{2}$$

$y(x,t), v = (v_0, v_1, v_2)$ are the state and the controls rspectively for the system (1).

Furthermore, we consider the functional of the form

$$J_\beta(v) = \int_S [y(\zeta,t) - f_0(\zeta,t)]^2 d\zeta dt + \beta \sum_{m=0}^{2} \|v_m - \omega_m\|_{l_2}^2, \tag{3}$$

which is to minimized under condition (1) and additional restricitions

$$\nu_0 \le \lambda_i(y,v_0) \le \mu_0, \nu_1 \le B_i(y,v_1) \le \mu_1, r_1 \le y(x,t) \le r_2, i = \overline{1,n} \tag{4}$$

on $V = \{v = (v_0, v_1, v_2) : v_m = (v_{0m}, v_{1m}, \cdots, v_{im}, \cdots) \in l_2, \|v_m\|_{l_2} \le R_m, m = \overline{0,2}$, $\omega_m = (\omega_{0m}, \omega_{1m}, \cdots, \omega_{im}, \cdots) \in l_2, m = \overline{0,2}$ are given numbers ,$\beta \ge$

$0, R_m > 0$, $\nu_j, \mu_j, j = 1, 2, r_1, r_2$ are positive numbers and $f_0(\zeta, t) \in L_2(S)$ is a given function.

Throughout this paper, we adopt the following assumptions.

Assumption 2.1. V is closed and bonded subset of l_2.

Assumption 2.2. The function $f(x, t, v_2)$ is given function continuous in v_2 on l_2 for almost all $(x, t) \in \Omega$, bounded and measurable in x, t on Ω $\forall v_2 \in l_2$.

Assumption 2.3. The functions $B_i(y, v_1), \lambda_i(y, v_0), i = \overline{1, n}$ are continuous on $(y, v) \in [r_1, r_2] \times l_2$ have continuous derivatives in y at $\forall (y, v) \in [r_1, r_2] \times l_2$ and $\frac{\partial B_i}{\partial y}$, $\frac{\partial \lambda_i}{\partial y}, i = \overline{1, n}$are bounded.

Assumption 2.4. The functions $B_i(y, v_1), \lambda_i(y, v_0), i = \overline{1, n}, f(x, t, v_2)$ satisfy a Lipschitz condition for v_1, v_0, v_2 ,then

$$|B_i(y(x,t), v_1 + \delta v_1) - B_i(y(x,t), v_1)| \le S_0(x,t)\|\delta v_1\|_{l_2}, i = \overline{1, n}$$

$$|\lambda_i(y(x,t), v_0 + \delta v_0) - \lambda_i(y(x,t), v_0)| \le S_1(x,t)\|\delta v_0\|_{l_2}, i = \overline{1, n}$$

$$|f(x, t, v_2 + \delta v_2) - f(x, t, v_2)| \le S_2(x,t)\|\delta v_2\|_{l_2}$$

for almost all $(x, t) \in \Omega, \forall y \in [r_1, r_2], \forall v_m, v_m + \delta v_m \in l_2$ such that $\|v_m\|_{l_2}, \|v_m + \delta v_m\|_{l_2} \le R_m$ where $S_m(x, t) \in L_\infty, m = \overline{0, 2}$.

Assumption 2.5. The first derivatives of the functions $B_i(y, v_0), \lambda_i(y, v_0), i = \overline{1, n}$ and $f(x, t, v_2)$ with respect to v are continuous functions in $[r_1, r_2] \times l_2$ and for any $v_m \in l_2$ such that $\|v_m\|_{l_2} \le R_m, m = \overline{0, 2}$.

Definition 2.1. The problem of finding the function $y = y(x, t) \in V_2^{0,1}(\Omega)$ from condition (1)-(2) at given $v \in V$ is called the reduced problem.

Definition 2.2. A function $y = y(x, t) \in V_2^{1,0}(\Omega)$ is said to be a solution of the problem (1)-(2), if for all $\eta = \eta(x, t) \in W_2^{1,1}(\Omega)$ the equation

$$\int_\Omega [-y\frac{\partial \eta}{\partial t} + \sum_{i=1}^n \lambda_i(y, v_0)\frac{\partial y}{\partial x_i}\frac{\partial \eta}{\partial x_i} + \sum_{i=1}^n B_i(y, v_1)(\frac{\partial y}{\partial x_i})\eta(x,t)$$

$$-f(x, t, v_2)\eta(x,t)]dxdt = \int_D \phi(x)\eta(x, 0)dx + \int_S g(\zeta, t)\eta(\zeta, t)d\zeta dt, \qquad (5)$$

is valid and $\eta(x, T) = 0$.

It is proved in [8] that, under the foregoing assumptions, a reduced problem (1)-(2) has a unique solution and $|\frac{\partial y}{\partial x_i}| \le C_1, i = \overline{1, n}$ almost at all $(x, t) \in \Omega, \forall v \in V$, where C_1 is a certain constant.

3. The Existence Theorem

Optimal control problems of the coefficients of differential equations do not always have solution [9]. Examples in [10] and elswhere of problems of the type (1)-(4) having no solution for $\beta = 0$. A problem of minimization of a functional is said to be unstable, when a minimizing sequare does not converge to an element minimizing the functional [6].

To begin with, we need

Theorem 3.1. *Under the above assumptions for every solution of the reduced problem (1)-(2) the following estimate is valid:*

$$\|\delta y\|_{V_2^{1,0}(\Omega)} \leq C_2[\|\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}\|_{L_2(\Omega)} + \|\Delta f - \sum_{i=1}^{n}\Delta B_i\frac{\partial y}{\partial x_i}\|_{L_2(\Omega)}], \quad (6)$$

where $\delta y(x,t) = y(x,t;v+\delta v) - y(x,t;v), \delta y(x,t) \in W_2^{1,1}(\Omega)$, $\Delta\lambda_i = \lambda_i(u,v_0+\delta v_0) - \lambda_i(u,v_0)$,$\Delta B_i = B_i(u,v_1+\delta v_1) - B_i(u,v_1)$, $\Delta f = f(x,t,v_2+\delta v_2) - f(x,t,v_2)$ *and* $C_2 \geq 0$ *is a constant not dependent on* $\delta v = (\delta v_0, \delta v_1, \delta v_2), \delta v_m \in l_2, m = \overline{0,2}$.

Proof. Set $\delta y(x,t) = y(x,t,v+\delta v) - y(x,t;v), y = y(x,t;v), \overline{y} = y(x,t;v+\delta v)$. From (5) it follows that

$$\int_\Omega[-\delta y\frac{\partial\eta}{\partial t} + \sum_{i=1}^n \overline{\lambda_i}\frac{\partial\delta y}{\partial x_i}\frac{\partial\eta}{\partial x_i} + \sum_{i=1}^n \frac{\partial\lambda_i(y+\theta_1\delta y, v_0+\delta v_0)}{\partial y}\frac{\partial y}{\partial x_i}\frac{\partial\eta}{\partial x_i}\delta y$$

$$+\sum_{i=1}^n \Delta\lambda_i\frac{\partial y}{\partial x_i}\frac{\partial\eta}{\partial x_i} + \sum_{i=1}^n \overline{B_i}\frac{\partial\delta y}{\partial x_i}\eta + \sum_{i=1}^n \Delta B_i(\frac{\partial y}{\partial x_i})\eta$$

$$+\sum_{i=1}^n \frac{\partial B_i(y+\theta_2\delta y, v_1+\delta v_1)}{\partial y}\frac{\partial y}{\partial x_i}\delta y\eta - \Delta f\eta]dxdt = 0 \quad (7)$$

for all $\eta = \eta(x,t) \in W_2^{1,1}(\Omega)$ and $\eta(x,T) = 0$.

Here $\theta_1, \theta_2 \in (0,1), i = \overline{1,n}$ is some number, $\overline{\lambda_i} \equiv \lambda_i(y+\delta y, v_0+\delta v_0)$,$\Delta\lambda_i \equiv \lambda_i(y, v_0+\delta v_0) - \lambda_i(y,v_0)$, $\overline{B_i} \equiv B_i(y+\delta y, v_1+\delta v_1)$,$\Delta B_i \equiv B_i(y, v_1 + \delta v_1) - \lambda_i(y,v_1), i = \overline{1,n}, i = \overline{1,n}$, $\Delta f \equiv f(x,t,v_2+\delta v_2) - f(x,t,v_2)$.

Let $\eta_h(x,t) = \frac{1}{h}\int_{t-h}^t \overline{\eta}(x,\tau)d\tau, 0 < h < \tau$ where $\overline{\eta} = \delta y(x,t)$ at $(x,t) \in \Omega_{t_1}$, zero at $t > t_1(t_1 \leq T-h)$ and $\Omega_{t_1} = D \times (0,t_1]$. In identity (5) put $\eta(x,t)$ instead of $\eta_h(x,t)$, and following the method in [11,p. 166-168] we obtain

$$\frac{1}{2}\int_D(\delta y)^2dx + \int_{\Omega_{t_1}}[\sum_{i=1}^n \overline{\lambda_i}(\frac{\partial\delta y}{\partial x_i})^2 + \sum_{i=1}^n \frac{\partial\lambda_i(y+\theta_1\delta y, v_0+\delta v_0)}{\partial y}\frac{\partial y}{\partial x_i}\frac{\partial\delta y}{\partial x_i}\delta y]dxdt$$

$$+\int_{\Omega_{t_1}}\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}\frac{\partial \delta y}{\partial x_i}dxdt+\sum_{i=1}^{n}\frac{\partial B_i(y+\theta_2\delta y,v_1+\delta v_1)}{\partial y}\frac{\partial y}{\partial x_i}(\delta y)^2dxdt$$

$$+\int_{\Omega_{t_1}}\sum_{i=1}^{n}\overline{B_i}\frac{\partial \delta y}{\partial x_i}\delta y+\int_{\Omega_{t_1}}\sum_{i=1}^{n}\Delta B_i(\frac{\partial y}{\partial x_i})\delta ydxdt-\int_{\Omega_{t_1}}\Delta f\delta ydxdt=0 \quad (8)$$

Hence,from the above assumptions and applying Cauchy Bunyakoviskii inequality, we obtain

$$\frac{1}{2}\int_D(\delta y(x,t_1)^2dx+\nu_0\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\frac{\partial \delta y}{\partial x_i}]^2dxdt$$

$$\leq (C_3+C_4)(\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\frac{\partial \delta y}{\partial x_i}]^2dxdt)^{\frac{1}{2}}(\int_{\Omega_{t_1}}(\delta y(x,t))^2dxdt)^{\frac{1}{2}}$$

$$+\{\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}]^2dxdt\}^{\frac{1}{2}}(\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\frac{\partial \delta y}{\partial x_i}]^2dxdt)^{\frac{1}{2}}+C_5\int_{\Omega_{t_1}}(\delta y(x,t))^2dxdt$$

$$+\int_0^{t_1}\{\int_D[\Delta f-\sum_{i=1}^{n}\Delta B_i(\frac{\partial y}{\partial x_i})]dx(\int_D\delta ydx)\}dt, \quad (9)$$

where C_3, C_4, C_5 are positive constants not depending on δv.

Applying Cauchy's inequality with ε and combine similar terms, then multiply both sides by two, we obtain

$$\|\delta y(x,t_1)\|^2_{L_2(D)}+\frac{\nu_0}{2}\|\sum_{i=1}^{n}\frac{\partial \delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})}\leq C_6\|\delta y(x,t)\|^2_{L_2(\Omega_{t_1})}$$

$$+\frac{2}{\nu_0}\{\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}]^2dxdt\}^{\frac{1}{2}}+\frac{2}{\nu_0}\int_{\Omega_{t_1}}[\Delta f-\sum_{i=1}^{n}\Delta B_i(\frac{\partial y}{\partial x_i})]dxdt, \quad (10)$$

where C_6 is positive constant not depending on δv.

Now we replace $\|\delta y(x,t)\|^2_{L_2(\Omega_{t_1})}=t_1(y(t_1))^2$. This gives us the inequality (10) yields the two inequalities

$$y(t_1)\leq C_6\int_0^{t_1}y(t)dt$$
$$+\frac{2}{\nu_0}[\{\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}]^2dxdt\}^{\frac{1}{2}}+\int_{\Omega_{t_1}}[\Delta f-\sum_{i=1}^{n}\Delta B_i(\frac{\partial y}{\partial x_i})]dxdt] \quad (11)$$

$$\|\sum_{i=1}^{n}\frac{\partial \delta y}{\partial x_i}\|^2_{L_2(\Omega_{t_1})}\leq\frac{2C_6}{\nu_0}\|\delta y\|^2_{L_2(\Omega_{t_1})}+\frac{4}{\nu_0^2}[\{\int_{\Omega_{t_1}}[\sum_{i=1}^{n}\Delta\lambda_i\frac{\partial y}{\partial x_i}]^2dxdt\}^{\frac{1}{2}}$$

$$+\int_{\Omega_{t_1}}[\Delta f-\sum_{i=1}^{n}\Delta B_i(\frac{\partial y}{\partial x_i})]dxdt] \quad (12)$$

From the known estimate [12,pp. 117-118] it follows that

$$y(t_1) \leq C_7[\{\int_{\Omega_{t_1}} [\sum_{i=1}^{n} \Delta\lambda_i \frac{\partial y}{\partial x_i}]^2 dxdt\}^{\frac{1}{2}} + \int_{\Omega_{t_1}} [\Delta f - \sum_{i=1}^{n} \Delta B_i(\frac{\partial y}{\partial x_i})]dxdt] \quad (13)$$

where C_7 is positive constant not depending on δv. Consequently,

$$\max_{0\leq t\leq t_1} \|\delta y\|_{L_2(D)} \leq C_7[\{\int_{\Omega_{t_1}} [\sum_{i=1}^{n} \Delta\lambda_i \frac{\partial y}{\partial x_i}]^2 dxdt\}^{\frac{1}{2}} + \int_{\Omega_{t_1}} [\Delta f - \sum_{i=1}^{n} \Delta B_i(\frac{\partial y}{\partial x_i})]dxdt]^{\frac{1}{2}} \quad (14)$$

and

$$\|\sum_{i=1}^{n} \frac{\partial \delta y}{\partial x_i}\|_{L_2(\Omega_{t_1})} \leq C_8[\{\int_{\Omega_{t_1}} [\sum_{i=1}^{n} \Delta\lambda_i \frac{\partial y}{\partial x_i}]^2 dxdt\}^{\frac{1}{2}} + \int_{\Omega_{t_1}} [\Delta f - \sum_{i=1}^{n} \Delta B_i(\frac{\partial y}{\partial x_i})]dxdt]^{\frac{1}{2}}, \quad (15)$$

where C_8 is positive constant not depending on δv.

If we combine the last two estimates, this proves the estimate (6). This completes the proof of the theorm.

Corollary 3.1. *Under the above assumptions, the right part of estimate (6) converges to zero at* $\sum_{m=0}^{2} \|\delta v\|_{l_2} \to 0$, *therefore*

$$\|\delta y\|_{V_2^{1,0}(\Omega)} \to 0 \ \ at \ \ \sum_{m=0}^{2} \|\delta v\|_{l_2} \to 0. \quad (16)$$

Hence from the theorem on trace, see [13], we get

$$\|\delta y\|_{L_2(\Omega)} \to 0, \|\delta y\|_{L_2(S)} \to 0 \ \text{ at } \sum_{m=0}^{2} \|\delta v_m\|_{l_2} \to 0. \quad (17)$$

Now we consider the functional $J_0(v) = \int_S [y(\zeta,t) - f_0(\zeta,t)]^2 d\zeta dt$.

Theorem 3.2. *The functional* $J_0(v)$ *is continuous on* V.

Proof. Let $\delta v = (\delta v_0, \delta v_1, \delta v_2), \delta v_m \in l_2, m = \overline{0,2}$ be an increment of control on an element $v \in V$ such that $v + \delta v \in V$. For the increment of $J_0(v)$ we have

$$\Delta J_0(v) = J_0(v+\delta v) - J_0(v)$$
$$= 2\int_S [y(\zeta,t)f_0(\zeta,t)]\delta y(\zeta,t)d\zeta dt + \int_S \delta y(\zeta,t)d\zeta dt \quad (18)$$

Applying the Cauchy-Bunyakovskii inequality, we obtain

$$|\Delta J_0(v)| \leq 2\|y(\zeta,t) - f_0(\zeta,t)\|_{L_2(S)}\|\delta y(\zeta,t)\|_{L_2(S)} + \|\delta y(\zeta,t)\|^2_{L_2(S)} \quad (19)$$

An application of the Corollary 3.1 completes the proof.

Theorem 3.3. *For any $\beta \geq 0$ the problem (1)-(4) has a least one solution.*

Proof. The set of V is closed and bounded in l_2. Since $J_0(v)$ is continuous on V by Theorem 3.2, so is

$$J_\beta(v) = J_0(v) + \beta \sum_{m=0}^{2} \|v_m - w_m\|^2_{l_2}. \quad (20)$$

Then from the Weierstrass theorem [14] it follows that the problem (1)-(4) has a least one solution. This completes the proof of the theorm.

4. The Uniqueness Theorem

According to the above discussions, we ca easily obtain a theorem concerning solution uniqueness for the considering optimal control problem (1)-(4).

Theorem 4.1. *There exists a dense set K of l_2 such that for any $\omega_m \in K, m = \overline{0,2}$ the problem (1)-(4) for $\beta > 0$ has a unique solution.*

Proof. The functional $J_0(v)$ is bounded below, and the foreging establishes that it is continues on V. Furthermore, l_2 is uniformaly convex [12]. It thus follows from a theorm in [16] that the space l_2 contains an everywhere-dense subset K such that the problem (1)-(4) has a unque solution when $\omega_m \in K, m = \overline{0,2}$ and $\beta > 0$. This completes the proof of the theorm.

5. Conclusion

We have investigated a constrained optimal control problems governed by quasilinear parabolic equations with controls in the coefficients of the equation. The existence and uniqueness of the optimal control problem is proved.

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