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# GRADED PRIME SUBMODULES OVER MULTIPLICATION MODULES

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**Abstract:** Let $G$ be an abelian group with identity $e$, $R$ be a $G-$graded commutative ring and $M$ a graded $R-$module where all modules are unital. Various generalizations of graded prime ideals have been studied. For example, a proper graded ideal $I$ is a graded weakly (resp., almost) prime ideal if $0 \neq ab \in I$ (resp., $ab \in I - I^2$) implies $a \in I$ or $b \in I$. Throughout this work, we define that a proper graded submodule $N$ of $M$ is a graded almost prime if $am \in N - (N : M)N$ implies $a \in (N : M)$ or $m \in N$. We show that graded almost prime submodules enjoy analogs of many of the properties of prime submodules.

**AMS Subject Classification:** 13A02, 16W50
**Key Words:** graded rings, graded multiplication modules, graded almost prime submodules

## 1. Introduction

Several authors have extended the notion of prime ideals to modules [2], [3]. Almost prime ideals were introduced by S. M. Bhatwadekar and P. K. Sharma [6]. Later studied by D. D. Anderson and M. Bataineh [7]. Graded almost prime ideals in a graded commutative ring with non-zero identity have been introduced and studied by A. Jabeer, M. Bataineh and H. Khashan [1, 4].

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In this study, we introduce some properties and characterizations of graded almost prime submodules over a multiplication modules, also we classify all graded modules in which every proper graded submodule can be written as a product of graded almost prime submodules.

## 2. Graded Almost Prime Submodules of Multiplication Modules

A proper graded prime submodule of a unital graded module is graded almost prime if for a$\in h(R)$ and m$\in h(M)$, $am \in N - (N : M)N$ implies that $a \in (N : M)$ or $m \in N$. Throughout this section, we introduce some properties of graded almost prime submodules over a multiplication modules, we investigate what are the conditions of rings and modules to get that every proper graded submodule is graded almost prime submodule, and when we can consider a proper graded submodule as a product of graded almost prime submodules.

Let us introduce some notation and terminology. Let $G$ be an arbitrary abelian group with identity $e$. Let $R$ be a $G-$graded ring and $M$ an $R-$module. We say that $M$ is a **G$-$graded R$-$module** if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of $M$ such that $M = \oplus_{g\in G} M_g$ (as abelian groups), and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also, we write $h(M) = \cup_{g\in G} M_g$. The summands $M_g$ are called homogeneous components and elements of these summands are called homogeneous elements .

**Definition 1.** Let $n$ be a positive integer, $N$ be a graded submodule of $M$ and $g \in G$.

(1) $Ng$ is an $n-$**almost** $g$**-prime submodule** of the $R_e$-module $Mg$, if $Ng \neq Mg$; and whenever $a \in R_e$ and $m \in Mg$ with $am \in Ng - (Ng : Mg)^n Ng$ then either $a \in (Ng : Mg)$ or $m \in Ng$.

(2) $N$ is a **graded** $n-$**almost prime submodule** of $M$, if $N \neq M$; and whenever $a \in h(R)$, and $m \in h(M)$ with $am \in N - (N : M)^n N$, then either $a \in (N : M)$ or $m \in N$.

**Definition 2.** Let $R$ be a $G-$graded commutative ring and $N, K$ are graded submodules of a graded $R-$module $M$ where $G$ an abelian group.

(1) The **residual ideal** of $N$ by $K$ is defined as $(N : K) = \{r \in h(R) : rK \subseteq N\}$.

(2) The **residual ideal** of $N_g$ by $K_g$ is defined as $(N_g : K_g) = \{r \in R_e : rK_g \subseteq N_g\}$.

Theorems 3, 6 and 7 generalize respectively Theorems 2.5, 2.8 and 2.9 in *[8]* and the proof almost the same.

**Theorem 3.** *Let $Ng$ be a graded submodule of $Mg$ and $g \in G$. Then the following are equivalent.*

(1) *$Ng$ is an $n-$almost $g$-prime submodule of $Mg$.*

(2) *For $m \in Mg - Ng$, then $(Ng : m) \equiv (Ng : Mg) \cup ((Ng : Mg)^n Ng : m)$.*

(3) *For $m \in Mg - Ng$, then $(Ng : m) \equiv (Ng : Mg)$ or $(Ng : m) \equiv ((Ng : Mg)^n Ng : m)$.*

(4) *If whenever $PK \subseteq Ng$ an $PK \nsubseteq (Ng : Mg)^n Ng$ with $P$ an ideal of $R_e$ and $K$ a submodule of $Mg$, then $P \subseteq (Ng : Mg)$ or $K \subseteq Ng$.*

*Proof.* $(1 \Rightarrow 2)$ Let $m \in Mg - Ng$ and $a \in (Ng : m)$, then $am \in Ng$. If $am \notin (Ng : Mg)^n Ng$, then $a \in (Ng : Mg)$, since $Ng$ is n-almost $g$-prime and $m \notin Ng$. If $am \in (Ng : Mg)^n Ng$, implies $a \in ((Ng : Mg)^n Ng : m)$. The reverse inclusion if $a \in (Ng : Mg)$, then $aMg \subseteq Ng$, so $am \in Ng$. Hence $a \in (Ng : m)$. If $a \in ((Ng : Mg)^n Ng : m)$, then $am \in (Ng : Mg)^n Ng \subseteq Ng$, $a \in (Ng : m)$. Therefore $(Ng : m) \equiv (Ng : Mg) \cup ((Ng : Mg)^n Ng : m)$.

$(2 \Rightarrow 3)$ Clearly.

$(3 \Rightarrow 4)$ Let $P$ be an ideal of $R_e$ and $K$ be a submodule of $Mg$ such that $PK \subseteq Ng$, $P \nsubseteq (Ng : Mg)$ and $K \nsubseteq Ng$. Want $PK \subseteq (Ng : Mg)^n Ng$. Let $a \in P$ and $m \in K$.

*Case 1.* Assume that $m \notin N_g$. If $a \notin (Ng : Mg)$. Since $am \in Ng$, we have $(Ng : m) \neq (Ng : Mg)$. By assuming $(Ng : m) \equiv ((Ng : Mg)^n Ng : m)$. So $am \in (Ng : Mg)^n Ng$. If $a \in P \cap (Ng : Mg)$. Let $b \in P - (Ng : Mg)$. Then $a + b \in P - (Ng : Mg)$. By previous case, we have $bm \in (Ng : Mg)^n Ng$ and $(a + b)m \in (Ng : Mg)^n Ng$. So $am \in (Ng : Mg)^n Ng$.

*Case 2.* Assume that $m \in N_g$. Let $m\prime \in K - N_g$. Then $m + m\prime \in K - N_g$. By Case 1 $am\prime \in (Ng : Mg)^n Ng$ and $a(m + m\prime) \in (Ng : Mg)^n Ng$. Hence $am \in (Ng : Mg)^n Ng$ for arbitrary $a \in P$ and $m \in K$. Therefore $PK \subseteq (Ng : Mg)^n Ng$.

$(4 \Rightarrow 1)$ Let $am \in Ng - (Ng : Mg)^n Ng$ with $a \in R_e$ and $m \in Mg$. Take $P \equiv \langle a \rangle$ and $K \equiv R_e m$, then $PK \subseteq Ng$ and $PK \nsubseteq (Ng : Mg)^n Ng$. So by assuming $P \subseteq (Ng : Mg)$ or $K \subseteq Ng$. Hence $a \in (Ng : Mg)$ or $m \in Ng$. □

Recall that if $N$ is a graded submodule of a graded $R$-module $M$, then the radical of $N$ (denoted by $\sqrt{N}$) is defined as the intersection of all graded prime submodules of $M$ containing $N$ see [5]. If $M$ is a multiplication graded $R$-module, then $\sqrt{N} = \sqrt{(N : M)}M$, where $\sqrt{(N : M)}$ denotes the radical of the graded ideal $(N : M)$ in $R$ see [5, Theorem 9].

**Definition 4.** An $R$-module $M$ is a **multiplication graded module** provided that for every graded submodule $N$of $M$, there is an ideal $I$of $R$ such that $N = IM$ (or $N = (N : M)M$).

**Theorem 5.** *Let $M$ be a multiplication graded $R$-module. If $N$is a graded submodule of $M$, then $N \subseteq \sqrt{(N : M)N}$. Moreover, if $N$ is a graded prime submodule of $M$, then $N = \sqrt{(N : M)N}$.*

*Proof.* Since $M$ is multiplication, then $\sqrt{(N : M)N} = \sqrt{((N : M)N : M)}M$. Since $(N : M)^2 \subseteq ((N : M)N : M)$, then $(N : M) \subseteq \sqrt{((N : M)N : M)}$ and so $N = (N : M)M \subseteq \sqrt{((N : M)N : M)}M = \sqrt{(N : M)N}$.

Moreover, suppose that $N$ is prime in $M$. If $r \in \sqrt{((N : M)N : M)}$, then $r^n \in ((N : M)N : M) \subseteq (N : M)$ for some positive integer $n$. Since $(N : M)$ is prime in $R$, then $r \in (N : M)$ and so $\sqrt{((N : M)N : M)} \subseteq (N : M)$. Therefore, $\sqrt{((N : M)N : M)}M \subseteq (N : M)M = N$ and the required equality holds. □

**Theorem 6.** *Let $N$ be a graded submodule of $M$ . Then the following are equivalent.*

(1) *$N$ is a graded $n-$almost prime submodule of $M$ for $n \geq 1$.*

(2) *For $m \in h(M) - h(N)$, then $(N : m) \equiv (N : M) \cup ((N : M)^n N : m)$.*

(3) *For $m \in h(M) - h(N)$, then $(N : m) \equiv (N : M)$ or $(N : m) \equiv ((N : M)^n N : m)$.*

(4) *If whenever $PK \subseteq N$ and $PK \nsubseteq (N : M)^n N$ with $P$ an ideal of $R$ and $K$ a submodule of $M$, then $P \subseteq (N : M)$ or $K \subseteq N$.*

*Proof.* $(1 \Rightarrow 2)$ Let $m \in h(M) - h(N)$ and $a \in (N : m)$, then $am \in N$. If $am \notin (N : M)^n N$, then $a \in (N : M)$ since $N$ is $n-$almost graded prime and $m \notin N$. If $am \in (N : M)^n N$ implies $a \in ((N : M)^n N : m)$. The reverse inclusion, if $a \in (N : M)$, then $aM \subseteq N$, so $am \in N$. Hence $a \in (N : m)$. If $a \in ((N : M)^n N : m)$, then $am \in (N : M)^n N \subseteq N$. So $a \in (N : m)$. Therefore $(N : m) \equiv (N : M) \cup ((N : M)N : m)$.

$(2 \Rightarrow 3)$ Clearly.

$(3 \Rightarrow 4)$ Let $P$ be an ideal of $R$ and $K$ be a submodule of $M$ such that $PK \subseteq N$, $P \nsubseteq (N : M)$ and $K \nsubseteq N$. Want $PK \subseteq (N : M)^n N$. Let $a \in h(P)$ and $m \in h(K)$.

*Case 1.* Assume that $m \notin N$. If $a \notin (N : M)$. Since $am \in N$, we have $(N : m) \neq (N : M)$. By assuming $(N : m) \equiv ((N : M)^n N : m)$. So $am \in (N : M)^n N$. If $a \in P \cap (N : M)$. Let $b \in P - (N : M)$. Then $a + b \in P - (N : M)$. By previous case, we have $bm \in (N : M)^n N$ and $(a + b)m \in (N : M)^n N$. So $am \in (N : M)^n N$.

*Case 2.* Assume that $m \in N$. Let $m\prime \in K - N$. Then $m + m\prime \in K - N$. By Case 1 $am\prime \in (N : M)^n N$ and $a(m+m\prime) \in (N : M)^n N$. Hence $am \in (N : M)N$ for arbitrary $a \in h(P)$ and $m \in h(K)$. So $h(P)h(K) \subseteq (N : M)^n N$. Therefore $PK \subseteq (N : M)^n N$.

($4 \Rightarrow 1$) Let $am \in N - (N : M)^n N$ with $a \in h(R)$ and $m \in h(M)$. Take $P \equiv \langle a \rangle$ and $K \equiv Rm$, then $PK \subseteq N$ and $PK \nsubseteq (N : M)^n N$. So by assuming $P \subseteq (N : M)$ or $K \subseteq N$, then $a \in (N : M)$ or $m \in N$. □

**Theorem 7.** *Let $N$ be a graded prime submodule of $M$. Then the following are equivalent.*

(1) *$N$ is a graded $n-$almost prime submodule of $M$ for $n \geq 1$.*

(2) *For $A \subseteq M$ and $A \nsubseteq N$, then $(N : A) \equiv (N : M) \cup ((N : M)^n N : A)$ where $A$ is a submodule of $M$.*

(3) *For $A \subseteq M$ and $A \nsubseteq N$, then $(N : A) \equiv (N : M)$ or $(N : A) \equiv ((N : M)^n N : A)$ where $A$ is a submodule of $M$.*

(4) *If whenever $PK \subseteq N$ and $PK \nsubseteq (N : M)^n N$ with $P$ an ideal of $R$ and $K$ a submodule of $M$, then $P \subseteq (N : M)$ or $K \subseteq N$.*

*Proof.* ($1 \Rightarrow 2$) Suppose that $A \subseteq M$ and $A \nsubseteq N$. Let $r \in (N : A)$, then $rA \subseteq N$. If $\langle r \rangle A \nsubseteq (N : M)^n N$, then $\langle r \rangle \subseteq (N : M)$ since $N$ is graded $n-$almost prime and $A \nsubseteq N$. And so $r \in (N : M)$. If $\langle r \rangle A \subseteq (N : M)^n N$ implies $r \in ((N : M)^n N : A)$. The reverse inclusion, if $r \in (N : M)$, then $rM \subseteq N$, so $rm \in N$ for all $m \in M$. Hence $r \in (N : A)$. If $r \in ((N : M)^n N : A)$, then $rA \subseteq (N : M)^n N \subseteq N$, $r \in (N : A)$. Therefore $(N : A) \equiv (N : M) \cup ((N : M)^n N : A)$.

($2 \Rightarrow 3$) Clearly.

($3 \Rightarrow 4$) Let $P$ be an ideal of $R$ and $K$ be a submodule of $M$ such that $PK \subseteq N$, $P \nsubseteq (N : M)$ and $K \nsubseteq N$. Want $PK \subseteq (N : M)^n N$. Let $a \in h(P)$ and $m \in h(K)$.

*Case 1.* assume that $m \notin N$. If $a \notin (N : M)$. Since $am \in N$, we have $(N : m) \neq (N : M)$. By assuming $(N : m) \equiv ((N : M)^n N : m)$. So $am \in (N : M)^n N$. If $a \in P \cap (N : M)$. Let $b \in P - (N : M)$. Then $a + b \in P - (N : M)$. By previous case, we have $bm \in (N : M)^n N$ and $(a + b)m \in (N : M)^n N$. So $am \in (N : M)^n N$.

*Case 2.* assume that $m \in N$. Let $m\prime \in K - N$. Then $m + m\prime \in K - N$. By Case 1 $am\prime \in (N : M)^n N$ and $a(m + m\prime) \in (N : M)^n N$. Hence $am \in (N : M)^n N$ for arbitrary $a \in h(P)$ and $m \in h(K)$. And so $h(P)h(K) \subseteq (N : M)^n N$. Therefore $PK \subseteq (N : M)^n N$.

($4 \Rightarrow 1$) Let $am \in N - (N:M)^nN$ with $a \in h(R)$ and $m \in h(M)$. Take $P \equiv \langle a \rangle$ and $K \equiv Rm$, then $PK \subseteq N$ and $PK \nsubseteq (N:M)^nN$. So by assuming $P \subseteq (N:M)$ or $K \subseteq N$. Hence $a \in (N:M)$ or $m \in N$. □

**Theorem 8.** *Let $M$ be a graded multiplication $R$-module and $n \geq 1$. If $N$ is a graded $n$-almost prime submodule of $M$, then $\sqrt{((N:M)^nN:M)}N = (N:M)N$.*

*Proof.* Let $r \in \sqrt{((N:M)^nN:M)}$. If $r \in (N:M)$, then $rN \subseteq (N:M)N$. Suppose that $r \notin (N:M)$, then $rM \nsubseteq N$. Then we get $(N:rM) = (N:M)$ or $(N:rM) = ((N:M)^nN:rM)$ by Theorem 2.7. If $(N:rM) = (N:M)$, let $k$ be the smallest positive integer such that $r^k \in ((N:M)^nN:M)$, then $r^kM \subseteq (N:M)^nN \subseteq N$. If $k=1$, then $rM \subseteq N$ which is a contradiction, then assume that $k \geq 2$. $r^kM \subseteq N$ such that $r^jM \nsubseteq (N:M)^nN$ $\forall j \leq k-1$. $r^kM \subseteq N$ implies $r^{k-1}rM \subseteq N$ then $r^{k-1} \in (N:rM) = (N:M)$, $r^{k-1}M \subseteq N$ and $r^{k-1}M \nsubseteq (N:M)^nN$. If $k=2$, then $r \in (N:M)$ which is a contradiction. If $k \geq 3$, then $r(r^{k-2}M) \subseteq N$ and $r(r^{k-2}M) \nsubseteq (N:M)^nN$. Since $N$ is a graded $n$-almost prime submodule, then $r \in (N:M)$ or $r^{k-2}M \subseteq N$. Continue in this process we conclude that $r \in (N:M)$ which is a contradiction. If $(N:rM) = ((N:M)^nN:rM)$, then $rMN \subseteq rM(N:rM) = rM((N:M)^nN:rM) \subseteq (N:M)N$. Then $rMN \subseteq (N:M)N$. But $N = NM$, so $rN \subseteq (N:M)N$. The reverse inclusion is always true, since $(N:M)^{n+1} \subseteq ((N:M)^nN:M)$, then $(N:M) \subseteq \sqrt{((N:M)^nN:M)}N$ and so $(N:M)N \subseteq \sqrt{((N:M)^nN:M)}N$. Therefore, $\sqrt{((N:M)^nN:M)}N \subseteq (N:M)N$.

□

In the next theorems, we give a characterizations of graded $n$-almost prime submodules in one kind of cancellation graded modules. We need the following definitions and lemma to prove the next theorems.

**Definition 9.** An $R$-module $M$ is a **multiplication graded module** provided that for every graded submodule $N$of $M$, there is an ideal $I$of $R$ such that $N = IM$ (or $N = (N:M)M$).

**Definition 10.** An $R$-module $M$ is a **cancellation graded module** of $R$ if for all graded ideals $I$ and $J$ of $R$, $IM = JM$, implies that $I = J$.

**Definition 11.** An $R$-module $M$ is a **faithful graded module** if it is annihilator $\mathrm{ann}(M)$ is 0.

**Definition 12.** A graded $R$-module $M$ is said to be **finitely generated** if there exists a finite set $A = \{m_1, m_2, ...m_n\}$ in $M$ such that any element $m$ in $M$ can be written as $m = a_1m_1 + a_2m_2 + ... + a_nm_n$, where $a_i$'s $\in R$ .

**Lemma 13.** *Let $N$ be a graded submodule of a cancellation graded $R$-module $M$. Then $(IN : M) = I(N : M)$.*

*Proof.* It is enough to prove that $(IN : M)M = I(N : M)M$, since $M$ is a cancellation module. Let $rm \in (IN : M)M$ such that $rM \subseteq IN$ and $m \in M$. Then $rm = an$ where $a \in I$ and $n \in N = (N : M)M$. Therefore $rm \in I(N : M)M$.

The reverse inclusion, let $arm \in I(N : M)M$ such that $rM \subseteq N$ and $a \in I$. So $ar \in (IN : M)$. Hence $arm \in (IN : M)M$. □

**Theorem 14.** *Let $M$ be a finitely generated faithful multiplication graded $R$-module and $N$ be a proper graded submodule of $M$. Then $N$ is a graded $n$-almost prime submodule of $M$ if and only if $(N : M)$ is a graded $(n+1)$-almost prime ideal of $R$.*

*Proof.* Let $ab \in (N : M) - (N : M)^{n+1}$ where $a,b \in h(R)$. Then $abM \nsubseteq (N : M)^n N$ because $ab \notin (N : M)^{(n+1)} = (N : M)^n(N : M) = ((N : M)^n N : M)$ since $M$ is cancellation and by Lemma 13. Then $abM \subseteq N$ and $abM \nsubseteq (N : M)^n N$ . Since $N$ is a graded $n$-almost prime submodule we get $a \in (N : M)$ or $bM \subseteq N$. Hence $a \in (N : M)$ or $b \in (N : M)$. Therefore $(N : M)$ is a graded $(n + 1)$-almost prime ideal.

Conversely, let $rm \in N - (N : M)^n N$ where $r \in h(R)$ and $m \in h(M)$. We not that $r(\langle m\rangle : M) \subseteq (\langle rm\rangle : M) \subseteq (N : M)$ and $r(\langle m\rangle : M) \nsubseteq (N : M)^{n+1}$. To see that $r(\langle m\rangle : M) \subseteq (\langle rm\rangle : M)$, let $rs \in r(\langle m\rangle : M)$ where $s \in (\langle m\rangle : M)$, then $sM \subseteq \langle m\rangle$. So $rsM \subseteq r\langle m\rangle \equiv \langle rm\rangle$. Therefore $rs \in (\langle rm\rangle : M)$. Also to see that $(\langle rm\rangle : M) \subseteq (N : M)$, since $rm \in N$, then $\langle rm\rangle \subseteq N$. So $(\langle rm\rangle : M) \subseteq (N : M)$. If $r(\langle m\rangle : M) \subseteq (N : M)^{n+1} \subseteq ((N : M)^n N : M)$ then $r(\langle m\rangle : M)M \subseteq (N : M)^n N$. But $r(\langle m\rangle : M)M = r\langle m\rangle$, since $M$ is a cancellation module which is a contradiction. Since $(N : M)$ is graded $(n + 1)$-almost prime, we obtain $r \in (N : M)$ or $(\langle m\rangle : M) \subseteq (N : M)$, then $(\langle m\rangle : M)M \subseteq (N : M)M = N$. Therefore, $N$ is a graded $n$-almost prime submodule. □

**Theorem 15.** *Let $M$ be a finitely generated faithful multiplication graded $R$-module and $N$ be a proper graded submodule of $M$. Then $N$ is a graded $n$-almost prime submodule of $M$ if and only if whenever $A$ and $B$ are graded submodules of $M$ such that $AB \subseteq N$ and $AB \nsubseteq (N : M)^n N$, then either $A \subseteq N$ or $B \subseteq N$.*

*Proof.* Let $A$ and $B$ be two submodules of $M$. Since $M$ is a cancellation module then $A = (A : M)M$ and $B = (B : M)M$ and so $AB = (A : M)(B :$

$M)M$. Suppose that $AB \subseteq N$ and $AB \nsubseteq (N:M)^nN$ but $A \nsubseteq N$ and $B \nsubseteq N$. Then $(A:M) \nsubseteq (N:M)$ and $(B:M) \nsubseteq (N:M)$. Since $(N:M)$ is a graded $(n+1)$-almost prime ideal then either $(A:M)(B:M) \nsubseteq (N:M)$ or $(A:M)(B:M) \subseteq (N:M)^{n+1}$. If $(A:M)(B:M) \nsubseteq (N:M)$, we get $AB = (A:M)(B:M)M \nsubseteq (N:M)M = N$ which is a contradiction. If $(A:M)(B:M) \subseteq (N:M)^{n+1} = ((N:M)^nN:M)$ then $AB = (A:M)(B:M)M \subseteq ((N:M)^nN:M\ )M = (N:M)^nN$ which is a contradiction.

Conversely, by Theorem 14 it is enough to prove that $(N:M)$ is a graded $(n+1)$-almost prime ideal of $R$. Let $rs \in (N:M) - (N:M)^{n+1}$ where $r, s \in h(R)$ want $r \in (N:M)$ or $s \in (N:M)$. Let $A = \langle r\rangle M$ and $B = \langle s\rangle M$. Then $AB = \langle r\rangle\langle s\rangle M \subseteq N$ and $AB \nsubseteq (N:M)^nN$. If $AB \subseteq (N:M)^nN$, then $\langle r\rangle\langle s\rangle \subseteq ((N:M)^nN:M\ ) = (N:M)^{n+1}$ which is a contradction. By assuming either $\langle r\rangle M \subseteq N$ or $\langle s\rangle M \subseteq N$ implies $r \in (N:M)$ or $s \in (N:M)$. Therefore, $(N:M)$ is a graded $(n+1)$-almost prime ideal and so $N$ is a graded $n$-almost prime submodule of $M$. □

**Corollary 16.** *Let $M$ be a finitely generated faithful multiplication graded $R$-module and $N$ be a proper graded submodule of $M$. Then $N$ is a graded $n$-almost prime submodule of $M$ if and only if $mm\prime \in N - (N:M)N$ implies $m \in N$ or $m\prime \in N$ for any $m, m\prime \in h(M)$.*

H. Khashan, A. Jaber and M. Bataineh [4] had proved that every proper graded ideal is a graded almost prime ideal. Now we want to classify the properties of rings and modules in which every proper graded submodule is a graded almost prime submodule.

**Definition 17.** A graded ring $R$ is called **graded regular** if for every $a \in h(R)$, there is $x \in h(R)$ satisfying $a = axa$.

**Theorem 18.** *Let $R$ be graded regular or graded local with $P^2 = 0$ where $P$ is the graded maximal ideal of $R$. And $M$ be a finitely generated faithful multiplication graded $R$-module. Then every proper graded submodule of $M$ is a graded almost prime submodule.*

*Proof.* Let $N$ be a graded submodule of $M$. Since $M$ is multiplication we can write $N = (N:M)M$. So $(N:M)$ is a proper graded ideal of $R$ beacuse $N$ is proper. By [4,Theorem 2.15] we obtain $(N:M)$ is a graded almost prime ideal of $R$. Hence $N$ is a graded almost prime submodule of $M$, by Theorem 14. □

Now we need the following lemma to point out the main results of this section.

**Lemma 19.** *Let $M$ be a finitely generated faithful multiplication graded $R$-module. Then a proper graded almost prime submodule $N$ of $M$ is a graded prime submodule in $M$ if and only if $(N : M)$ is a graded prime ideal in $R$ .*

*Proof.* Let $N$ be a graded prime submodule of $M$. Want $(N : M)$ is a graded prime ideal of $R$. Let $ab \in (N : M)$, want $a$ or $b \in (N : M)$.

Then $abM \subseteq N$. Since $N$ is prime, $a \in (N : M)$ or $bM \subseteq N$ (implies $b \in (N : M)$).

Conversely, let $am \in N$. Assume that $m \notin N$, then by Theorem 2.6, $(N : m) \equiv (N : M) \cup ((N : M)N : m)$. If $(N : m) \equiv (N : M)$ done.

If $(N : m) \equiv ((N : M)N : m)$.

*Case 1.* $a \in ((N : M)N : M) = (N : M)^2 \subseteq (N : M)$, since $M$ is a cancellation module and so $a \in (N : M)$.

*Case 2.* If $a \notin ((N : M)N : M) = (N : M)^2$. Hence $a \in (N : M) - (N : M)^2$. Since $(N : M)$ is a graded almost prime ideal, we obtain $a \in (N : M)$. Therefore $N$ is a graded prime submodule. □

**Theorem 20.** *Let $R$ be a Noetherian graded domain and $M$ be a finitely generated faithful multiplication graded $R$-module. Then a graded submodule $N$ is a graded prime submodule if and only if its a graded $n$-almost prime submodule for all $n \geq 1$.*

*Proof.* Let $N$ be a graded $n-$almost prime submodule of $M$ for all $n \geq 1$. Then $N = (N : M)M$ , since $M$ is multiplication. By Theorem 14, $N$ is graded $n-$almost if and only if $(N : M)$ is graded $(n + 1)-$almost prime. So $(N : M)$ is graded $(n + 1)-$almost prime for all $n \geq 1$ if and only if $(N : M)$ is graded prime, by [4,Theorem 3.6]. Hence $N$ is a graded prime submodule, by using Lemma 19. □

**Theorem 21.** *Let $R$ be a valuation graded domain, and $M$ be a finitely generated faithful multiplication graded $R$-module. Then a graded submodule $N$ is a graded prime submodule if and only if it is a graded almost prime submodule.*

*Proof.* Let $N$ be a proper graded almost prime submodule of $M$. Then $(N : M)$ is a graded almost prime ideal in $R$. By [4,Theorem 3.8], $(N : M)$ is a graded prime ideal. So $N$ is a graded prime submodule. □

**Theorem 22.** *Let* $(R, P)$ *be a local graded ring and either* $P$ *is principle or (i) for each* $x \in h(P) - h(P^2)$, $\langle x^2 \rangle = P^2$. *(ii)* $P^3 = 0$. *And* $M$ *be a finitely generated faithful multiplication graded* $R$*-module. Then every proper graded submodule of* $M$ *is a product of graded almost prime submodules.*

*Proof.* Let $N$ be a graded submodule of $M$. Then $N = (N : M)M$ , since $M$ is multiplication. By assuming and [4,Theorem 3.9] we get $(N : M) = I_1I_2....I_k$ where $I_i$ is graded almost prime ideals of $R$ , $k$ is a positive integer and $1 \leq i \leq k$. So $N = I_1I_2....I_kM = (I_1M)(I_2M)...(I_kM\ )$, where $I_iM$ is graded almost prime submodules of $M$, for all $1 \leq i \leq k$. □

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