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ON APPROXIMATION OF CONJUGATE OF FUNCTIONS BELONGING TO DIFFERENT CLASSES BY PRODUCT MEANS

H.K. Nigam[1], Ajay Sharma[2 §]

[1,2]Department of Mathematics
Faculty of Engineering and Technology
Mody Institute of Technology and Science (Deemed University)
Laxmangarh, 332311, Sikar (Rajasthan), INDIA

Abstract: In this paper, we introduce the concept of (C,1)(E,q) product summability and establish a quite new theorem on degree of approximation of conjugate of the function $f \in W(L_r, \xi(t))$ class by (C,1)(E,q) product summability means of conjugate Fourier series.

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1. Introduction

Let f be $2\pi-$periodic function and Lebesgue integrable. The Fourier series of associated with f at a point is defined by

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=1}^{\infty} A_n(x), \tag{1.1}$$

with n^{th} partial sums $s_n(f;x)$.

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§Correspondence author

The conjugate series of the Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{1.2}$$

with n^{th} partial sums $\overline{s_n}(f;x)$. Throughout this paper, we call (1.2) as conjugate Fourier series of function f.

$L_\infty-$ norm of a function $f : R \to R$ is defined by

$$\|f\|_\infty = \sup\{|f(x)| : x \in R\}$$

L_r- norm is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{\frac{1}{r}}, \; r \geq 1 \tag{1.3}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of degree n under sup norm $\| \;\|_\infty$ is defined by

$$\|t_n - f\|_\infty = \sup\{\,|t_n(x) - f(x)| : x \in R\} \text{ Zygmund[13]} \tag{1.4}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_r$ is given by

$$E_n(f) = min\|t_n - f\|_r \tag{1.5}$$

This method of approximation is called trigonometric Fourier approximation (TFA).

A function $f \in Lip\alpha$ if

$$f(x+t) - f(x) = O(\,|t|^\alpha)\, for\; 0 < \alpha \leq 1 \tag{1.6}$$

$f \in Lip(\alpha, r)$, for $0 \leq x \leq 2\pi$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\,|t|^\alpha)\;, 0 < \alpha \leq 1, \text{ and, } r \geq 1 \tag{1.7}$$

(definition 5.38 of Mc Fadden [7])

Given a positive increasing function $\xi(t)$ and an integer $r \geq 1$, $f(x) \in Lip(\xi(t), r)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{1.8}$$

and that $f(x) \in W(L_r, \xi(t))$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r \sin^{\beta r} x \, dx \right)^{\frac{1}{r}} = O(\xi(t)), \quad \beta \geq 0. \tag{1.9}$$

If $\beta = 0$, our newly defined class $W(L_r, \xi(t))$ reduces to the class $Lip(\xi(t), r)$, if $\xi(t) = t^\alpha$ then $Lip(\xi(t), r)$ class reduces to the class $Lip(\alpha, r)$ and if $r \to \infty$ then $Lip(\alpha, r)$ class reduces to the class $Lip\alpha$.

We observe that

$$Lip\alpha \subseteq Lip(\alpha, r) \subseteq Lip(\xi(t), r) \subseteq W(L_r, \xi(t)) \text{ for } 0 < \alpha \leq 1, \ r \geq 1.$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of its n^{th} partial sums $\{s_n\}$.

The (C,1) transform of (E,q) transform defines (C,1)(E,q) product transform and we denote it by $C_n^1 E_n^q$.

Thus if

$$\begin{aligned} C_n^1 E_n^q &= \frac{1}{n+1} \sum_{k=0}^{n} E_k^q \to s, \ \ as \ \ n \to \infty \\ &= \frac{1}{n+1} \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{v} q^{n-\nu} \right] \to s, \ \ as \ \ n \to \infty, \end{aligned} \tag{1.10}$$

where E_n^q denotes the (E,q) transform of s_n and C_n^1 denotes (C,1) transform of s_n. Then the series $\sum_{n=0}^{\infty} u_n$ is said to be summable by (C,1)(E,q) means or summable (C,1)(E,q) to a definite number s.

We use the following notations:

$$\psi(t) = f(x+t) + f(x-t)$$

$$\bar{K}_n(t) = \frac{1}{2\pi(n+1)} \sum_{k=0}^{n} \left[\frac{1}{(1+q)^k} \sum_{\nu=0}^{k} \binom{k}{\nu} q^{n-\nu} \frac{\cos\left(\nu + \frac{1}{2}\right) t}{\sin(t/2)} \right]$$

$\tau = \left[\frac{1}{t} \right]$, where τ denotes the greatest integer not greater than $\frac{1}{t}$.

2. Main Theorem

Several researchers like Alexits[1], Sahney and Goel[12], Qureshi and Neha[10], Quershi[8, 9], Chandra[2], Khan [4], Leindler[6] and Rhoades[11] have obtained the degree of approximation of functions belonging to $Lip\alpha$, $Lip\,(\alpha, r)$, $Lip(\xi\,(t), r)$ and $W\,(L^r, \xi\,(t))$ classes using Cesàro, Nörlund and generalized Nörlund single summability methods. But till now nothing seems to have been done so far to obtain the degree of approximation of functions using (C,1)(E,q) product summability method. Therefore, in present paper, we introduce (C,1)(E,q) product summability method and prove a quite new theorem on degree of approximation of conjugate of function $f \in W\,(L_r, \xi\,(t))$ class using (C,1)(E,q) product summability means of conjugate Fourier series in the following form.

Theorem. *If a function $\overline{f}\,(x)$, conjugate to a 2π- periodic function $f(x)$ belonging to class $W\,(L_r, \xi\,(t))\,,\ r \geq 1$, then its degree of approximation by $(C,1)(E,q)$ product means of conjugate Fourier series is given by*

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}\,(x)\right\|_r = O\left[(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{(n+1)}\right)\right] \tag{2.1}$$

provided that $\xi\,(t)$ satisfies the condition (2.2),

$$\left(\frac{\xi\,(t)}{t}\right) \text{ is non-increasing in } t, \tag{2.2}$$

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t\,|\psi\,(t)|}{\xi\,(t)}\right)^r \sin^{\beta r} t\; dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{2.3}$$

and

$$\left\{\int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta}\,|\psi\,(t)|}{\xi\,(t)}\right)^r dt\right\}^{\frac{1}{r}} = O\left\{\,(n+1)^{\delta}\right\} \tag{2.4}$$

where δ is an arbitrary positive number such that $s\,(1-\delta)-1 > 0, \frac{1}{r}+\frac{1}{s} = 1,\quad 1 \leq r \leq \infty$, conditions (2.5) and (2.6) hold uniformly in x, $\overline{C_n^1 E_n^q}$ is $\overline{(\mathrm{C},1)(\mathrm{E},\mathrm{q})}$ means of the series (1.2) and

$$\overline{f}\,(x) = -\frac{1}{2\pi}\int_0^{\pi} \psi\,(t)\cot\frac{1}{2}t\; dt \tag{2.5}$$

provided

$$(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}=O(n+1) \tag{2.6}$$

3. Lemmas

For the proof of our theorem, following lemmas are required.

Lemma 1.

$$\overline{K}_n(t)=O\left(\frac{1}{t}\right)\ for\ 0\le t\le\frac{1}{n+1}$$

Proof. For $0\le t\le\frac{1}{n+1}$, $\sin(t/2)\ge(t/\pi)$ and $|\cos nt|\le 1$

$$\begin{aligned}
\left|\overline{K}_n(t)\right| &= \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right| \\
&\le \frac{1}{2\pi(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\left|\cos\left(\nu+\frac{1}{2}\right)t\right|}{|\sin(t/2)|} \\
&= \frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu} \\
&= \frac{1}{2t(n+1)}\sum_{k=0}^{n}\frac{1}{(1+q)^k}(1+q)^k \quad \text{since}\sum_{\nu=0}^{k}\left(\frac{k}{\nu}\right)q^{k-\nu} \\
&= \frac{1}{2t(n+1)}\sum_{k=0}^{n}1=O\left(\frac{1}{t}\right)
\end{aligned}$$

Lemma 2. *For $0\le a\le b\le\infty$, $0\le t\le\pi$ and any n, we have*

$$\overline{K}_n(t)=O\left(\frac{\tau^2}{(n+1)}\right)+O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right)$$

Proof. For $0\le\frac{1}{n+1}\le t\le\pi$, $\sin(t/2)\ge(t/\pi)$

$$\left|\overline{K}_n(t)\right| = \frac{1}{2\pi(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^k}\sum_{\nu=0}^{k}\binom{k}{\nu}q^{k-\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin(t/2)}\right]\right|$$

$$\leq \frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right|\left|e^{\frac{it}{2}}\right|$$

$$\leq \frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{n}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right|$$

$$\leq \frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right|$$
$$+\ \frac{1}{2t\,(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^{k}}\ \mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right| \quad (3.1)$$

Now considering first term of (3.1)

$$\frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right|$$
$$\leq \frac{1}{2t\,(n+1)}\left|\sum_{k=0}^{\tau-1}\frac{1}{(1+q)^{k}}\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\right|\left|e^{i\nu t}\right|$$
$$\leq \frac{1}{2t\,(n+1)}\sum_{k=0}^{\tau-1}\left[\frac{1}{(1+q)^{k}}\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\right]$$
$$= \frac{1}{2t\,(n+1)}\sum_{k=0}^{\tau-1}1$$
$$= \frac{\tau}{2t\,(n+1)} = O\left(\frac{\tau^{2}}{(n+1)}\right) \quad (3.2)$$

Now considering second term of (3.1) and using Abel's Lemma

$$\frac{1}{2t\,(n+1)}\left|\sum_{k=\tau}^{n}\left[\frac{1}{(1+q)^{k}}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right\}\right]\right|$$
$$\leq \frac{1}{2t\,(n+1)}\sum_{k=\tau}^{n}\frac{1}{(1+q)^{k}}\ \max_{0\leq\ m\leq\ k}\ \left|\sum_{\nu=0}^{m}\binom{k}{\nu}\,q^{k-\nu}\,e^{i\nu t}\right|$$

$$\leq \frac{1}{2t\,(n+1)}\,(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^{k}}$$

$$= O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right) \tag{3.3}$$

Combining (3.1), (3.2) and (3.3), we get

$$\overline{K}_n\,(t) = O\left(\frac{\tau^2}{(n+1)}\right) + O\left(\frac{\tau}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}(1+q)^{-k}\right) \tag{3.4}$$

4. Proof of the Theorem

Let $\overline{s}_n\,(f;x)$ denotes the n^{th} partial sum of the series (1.2), then, following Lal[5], we have

$$\overline{s}_n\,(x) - \overline{f}\,(x) = \frac{1}{2\pi}\int_0^{\pi}\psi\,(t)\,\frac{\cos\left(n+\frac{1}{2}\right)\,t}{\sin\frac{t}{2}}dt$$

Therefore using (1.2), the (E, q) transform E_n^q of $\overline{s}_n\,(f;x)$ is given by

$$\overline{E_n^q} - \overline{f}\,(x) = \frac{1}{2\pi(1+q)^n}\int_0^{\pi}\frac{\psi\,(t)}{\sin\,(t/2)}\,\left\{\sum_{k=0}^{n}\binom{n}{k}\,q^{n-k}\cos\left(k+\frac{1}{2}\right)t\right\}dt$$

Now denoting $\overline{(\mathrm{C},1)(\mathrm{E},\mathrm{q})}$ transform of $\overline{s}_n$ by $\left(\overline{C_n^1E_n^q}\right)$, we write

$$\overline{C_n^1E_n^q} - \overline{f\,(x)} = \frac{1}{2\pi\,(n+1)}\left[\sum_{k=0}^{n}\frac{1}{(1+q)^k}\int_0^{\pi}\left(\frac{\phi\,(t)}{\sin\left(\frac{t}{2}\right)}\right)\right.$$
$$\left.\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}\,q^{n-k}\sin\left(\nu+\frac{1}{2}\right)t\right\}dt\right]$$
$$= \int_0^{\pi}\psi\,(t)\,\overline{K}_n\,(t)\;dt$$
$$= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi\,(t)\;\overline{K}_n\,(t)\;dt$$

$$= I_{1.1} + I_{1.2} \text{ (say)} \tag{4.1}$$

we have

$$|\ \phi(x+t) - \phi(x)\ | \ \leqq \ |\ f(u+x+t) - f(u+x)\ | + |\ f(u-x-t) - f(u-x)\ | \ .$$

Hence, by Minkowiski's inequality,

$$\begin{aligned} &[\int_0^{2\pi} |\ \{\phi(x+t) - \phi(x)\} \sin^\beta x\ |^r\ dx]^{\frac{1}{r}} \\ &\qquad \leqq [\int_0^{2\pi} |\ \{f(u+x+t) - f(u+x)\} \sin^\beta x\ |^r\ dx]^{\frac{1}{r}} \\ &\qquad + [\int_0^{2\pi} |\ \{f(u-x-t) - f(u-x)\} \sin^\beta x\ |^r\ dx]^{\frac{1}{r}} = O\{\xi(t)\}. \end{aligned}$$

Then $f \in W(L^r, \xi(t)) \Rightarrow \phi \in W(L^r, \xi(t))$.

Now considering,

$$|I_{1.1}| \leq \int_0^{\frac{1}{n+1}} |\psi(t)|\ \left|\overline{K}_n(t)\right| dt$$

Using Hölder's inequality and the fact that $\phi \in W(L_r, \xi(t))$,

$$\begin{aligned} |I_{1.1}| &\leq \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{t\,|\,\psi(t)\,|\sin^\beta t}{\xi(t)} \right\}^r dt \right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\ \left|\,\overline{K}_n(t)\,\right|}{t\ \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \\ &= O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)\ \left|\,\overline{K}_n(t)\,\right|}{t\ \sin^\beta t} \right\}^s dt \right]^{\frac{1}{s}} \quad \text{by (2.3)} \end{aligned}$$

Since $\sin t \geq \left(\frac{2t}{\pi}\right)$ and using Lemma 1,

$$I_{1.1} = O\left(\frac{1}{n+1}\right) \left[\int_0^{\frac{1}{n+1}} \left\{ \frac{\xi(t)}{t^{2+\beta}} \right\}^s dt \right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals,

$$I_{1.1} = O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{\int_{\in}^{\frac{1}{n+1}}\left(\frac{dt}{t^{(2+\beta)s}}\right)\right\}^{\frac{1}{s}} \text{ for some } 0<\in<\frac{1}{n+1}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(2+\beta)s+1}}{(2+\beta)s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}$$

$$= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{2+\beta-\frac{1}{s}}\right\}$$

$$= O\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1-\frac{1}{s}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1,\ 1\le r\le\infty. \tag{4.2}$$

Now we take,

$$|I_{1.2}| \le \int_{\frac{1}{n+1}}^{\pi} |\psi(t)|\left|\overline{K}_n(t)\right| dt$$

Now using Lemma 2,

$$|I_{1.2}| = O\left[\int_{\frac{1}{n+1}}^{\pi}\frac{|\psi(t)|}{t^2(n+1)}dt\right] + O\left[\int_{\frac{1}{n+1}}^{\pi}\frac{|\psi(t)|}{(n+1)t}(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^k}dt\right]^{\frac{1}{s}}$$

$$= O(I_{1.2.1}) + O(I_{1.2.2}) \text{ say} \tag{4.3}$$

Using Hölder's inequality, $|\sin t| \le 1,\ \sin t \ge (2t/\pi)$, conditions (2.2) and (2.4) and using second mean value theorem for integral,

$$|I_{1.2.1}| \le \left(\frac{1}{n+1}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|\sin^{\beta}t}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+2}\sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= \left(\frac{\pi}{2(n+1)}\right)\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+\beta+2}}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta-1}\right\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-2-\beta}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-2-\beta)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi}\leq\eta\leq n+1$$

$$= O\left\{(n+1)^{\delta-1}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-2-\beta)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi}\leq 1\leq n+1$$

$$= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(2+\beta-\delta)-1}}{s\,(2+\beta-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{(n+1)^{\delta-1}\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(2+\beta-\delta)-\frac{1}{s}}\right]$$

$$= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1-\frac{1}{s}}\right\}$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since } \frac{1}{r}+\frac{1}{s}=1 \tag{4.4}$$

Similarly using (2.6), conditions (2.2) and (2.4), $|\sin t|\leq 1,\ \sin t\geq(2t/\pi)$ and second mean value theorem for integrals,

$$|I_{1.2.2}|\leq\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,|\,\psi(t)\,|\sin^{\beta}t}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}$$

$$\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\frac{1}{(n+1)}(1+q)^{\tau}\sum_{k=\tau}^{n}\frac{1}{(1+q)^{k}}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\,|\,\psi(t)\,|}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^{s}dt\right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta+\beta}}\right\}^{s} dt\right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta}\right\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^{s} \frac{dy}{y^{2}}\right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{\in_1}^{n+1} \left\{\frac{dy}{y^{s(\delta-1-\beta)+2}}\right\}\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi} < \in_1 < n+1$$

$$= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi} \le 1 \le n+1$$

$$= O\left\{ (n+1)^{\delta}\xi\left(\frac{1}{n+1}\right)\right\} \left[\left\{\frac{y^{s(1+\beta-\delta)-1}}{s(1+\beta-\delta)-1}\right\}_{1}^{n+1}\right]^{\frac{1}{s}}$$

$$= O\left\{ (n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\} \left[(n+1)^{\beta+1-\delta-\frac{1}{s}}\right]$$

$$= O\left\{ (n+1)^{\beta+1-\frac{1}{s}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

$$= O\left\{ (n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \text{ since} \frac{1}{r}+\frac{1}{s}=1 \tag{4.5}$$

Combining (4.3), (4.4) and (4.5),

$$\left|\overline{C_n^1 E_n^q} - \overline{f}(x)\right| = O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

Now, using L_r-norm, we get

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}(x)\right\|_r = \left\{\int_0^{2\pi} \left|\overline{C_n^1 E_n^q} - \overline{f}(x)\right|^r dx\right\}^{\frac{1}{r}}$$

$$= O\left[\left\{\int_0^{2\pi} \left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right]$$

$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right]$$
$$= O\left\{(n+1)^{\beta+\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}$$

This completes the proof of theorem.

5. Applications

Following corollaries can be derived from our main Theorem:

Corollary 1. *If* $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, *then the weighted class* $W(L_r, \xi(t))$, $r \geq 1$, *reduces to the class* $Lip(\alpha, r)$ *and the degree of approximation of a function* $\overline{f}(x)$, *conjugate to a* 2π *- periodic function* $f \in Lip(\alpha, r)$, $\frac{1}{r} \leq \alpha < 1$, *is given by*

$$\mid \overline{C_n^1 E_n^q} - \overline{f}(x) \mid = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right).$$

Proof. The result follows by setting $\beta = 0$ in (2.1).

5.1. Corollary 2

If $\xi(t) = t^{\alpha}$ *for* $0 < \alpha < 1$ and $r = \infty$ in corollary 5.1, then $f \in Lip\alpha$ and we have

$$\left\|\overline{C_n^1 E_n^q} - \overline{f}(x)\right\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha}}\right)$$

Corollary 3. *If* $\xi(t) = t^{\alpha}$, $0 < \alpha \leq 1$, *then the weighted class* $W(L_r, \xi(t))$, $r \geq 1$, *reduces to the class* $Lip(\alpha, r)$ *and the degree of approximation of a function* $\overline{f}(x)$, *conjugate to a* 2π *- periodic function* $f \in Lip(\alpha, r)$, $\frac{1}{r} \leq \alpha < 1$, *is given by*

$$\mid \left(\overline{CE}\right)_n^1 - \overline{f}(x) \mid = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right).$$

Proof. The result follows by setting $\beta = 0$ and $q_n = 1\,\forall\, n$ in (2.3).

Corollary 4. *If $\xi(t) = t^{\alpha}$ for $0 < \alpha < 1$ and $r = \infty$ in corollary 5.3, then $f \in Lip\alpha$ and we have*

$$\left\| \left(\overline{CE}\right)_n^1 - \overline{f}(x) \right\|_\infty = O\left(\frac{1}{(n+1)^\alpha} \right).$$

Remark. Independent proofs of above Corollaries 5.1 and 5.3 can be obtained along the same lines of our theorem.

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