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TUPLE OF OPERATORS AND HYPERCYCLICITY

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Abstract: In this paper we characterize the equivalent conditions for a tuple of commutative bounded linear operators, satisfying the Hypercyclicity Criterion.

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1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+, \ \ i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all k copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the k copies of X, $X \oplus ... \oplus X$.

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For simplicity we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if T_1, T_2 are commutative bounded linear operators on a Banach space X, and $\{m_j\}$, $\{n_j\}$ are two sequences of natural numbers, then we say $\{{T_1}^{m_j}{T_2}^{n_j} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j}{T_2}^{n_j}x : j \geq 0\}$ is dense in X.

Definition 1.2. We say that the pair $\mathcal{T} = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.

Definition 1.3. We say that a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_k\})$ of integers provided for all pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$, the sequence $\{{T_1}^{m_{k_j}}{T_2}^{n_{k_j}} : j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.

The formulation of the Hypercyclicty Criterion in the next section was given by N. S. Feldman ([4]). Here, we want to extend some properties of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–12].

2. Main Results

In this section we characterize the equivalent conditions for a pair of operators, satisfying the Hypercyclicity Criterion.

Theorem 2.1. (The Hypercyclicity Criterion for a Tuple) *Suppose X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on X. If there exist two dense subsets Y and Z in X, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that:*

1. $T_1^{m_j} T_2^{n_j} \to 0$ on Y as $j \to \infty$,

2. There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j} T_2^{n_j} S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.

Lemma 2.2. (see [12]) *Let X be a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ be the pair of operators T_1 and T_2. Then the followings are equivalent:*

(i) $\mathcal{T}$ is hypercyclic.

(ii) for all nonempty open subsets U, V in X, there exists a pair of sequences $(\{m_k\}, \{n_k\})$ of integers such that $T_1^{m_k} T_2{}^{n_k}(U) \cap V \neq \emptyset$ for all $k \geq 0$.

(iii) $\mathcal{T}$ is topologically transitive.

Corollary 2.3. *$\mathcal{T}_d^{(2)}$ is hypercyclic if and only if for given four nonempty open subsets U_1, U_2, $V1$, V_2 of X, there exists a pair of integers (m, n) such that the sets $T_1^m T_2^n(U_1) \cap V_1$ and $T_1^m T_2^n(U_2) \cap V_2$ are nonempty.*

We will use HC($\mathcal{T}$) for the collection of hypercyclic vectors for the tuple $\mathcal{T}$ of operators.

Theorem 2.4. *Let $\mathcal{T}$ be a pair of operators T_1 and T_2 on the an infinite dimensional Banach space X. Also, let T_i^* has no eigenvalues for $i = 1, 2$. Then the followings are equivalent:*

(i) $\mathcal{T}_d^{(2)}$ is hypercyclic.

(ii) for every nonempty open subsets U, V of X, there exists a pair of integers (m, n) such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ and $T_1^{m+1} T_2^{n+1}(U) \cap V \neq \emptyset$.

(iii) there exists a positive integer p such that for any nonempty open subsets U, V of X, there exists a pair of integers (m, n) such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ and $T_1^{m+p} T_2^{n+p}(U) \cap V \neq \emptyset$.

Proof. In Corollary 2.3, put $U_1 = V_1 = U$, $U_2 = V$ and $V_2 = T_1{}^{-1} T_2{}^{-1} V$. Then the sets $T_1^m T_2^n(U) \cap V$ and $T_1^m T_2^n(U) \cap T_1{}^{-1} T_2{}^{-1} V$ are nonempty, from which we conclude that (i) implies (ii). By taking $p = 1$, we see that (ii) implies (iii). Now to prove that (iii) implies (i), consider four nonempty open subsets U_i, V_i, $i = 1, 2$. We want to find integers m and n such that the sets $T_1^m T_2^n(U_1) \cap V_1$ and $T_1^m T_2^n(U_2) \cap V_2$ to be nonempty. By Lemma 2.2, HC($\mathcal{T}$) is dense in X, thus there exists a vector v_1 in V_1 that is a hypercyclic vector of $\mathcal{T}$. Since U_1 is open, there exists a pair of integers (m_1, n_1) such that $T_1^{m_1} T_2^{n_1} v_1 = u_1$ is in U_1. Since T_i^* has no eigenvalues for $i = 1, 2$, $T_1^{m_1} T_2^{n_1}$ has dense range and so it's range intersects U_2. Thus there exists $u_2 \in U$ such that $u_2 = T_1^{m_1} T_2^{n_1} w_2$ for some $w_2 \in X$. Let v_2 be any element of V_2 and choose $\delta > 0$ such that $B(v_2, \delta) \subset V_2$ and $B(u_2, \delta) \subset U_2$. Since $(T_1^p T_2^p - 1)$ has dense range and v_1 is a hypercyclic vector for $\mathcal{T}$, thus clearly $(T_1^p T_2^p - 1)v_1$ is also a hypercyclic vector for $\mathcal{T}$. So there exists a pair of integers (p_1, q_1) such that

$$||T_1^{p_1} T_2^{q_1} (T_1^p T_2^p - 1)v_1 - (v_2 - w_2)|| < \frac{\delta}{2||T_1^{m_1} T_2^{n_1}||},$$

and also there exists a pair of integers (p_2, q_2) such that

$$||T_1^{p_2} T_2^{q_2} v_1 - (v_2 - T_1^{p_1} T_2^{q_1} v_1)|| < \frac{\delta}{2||T_1^{m_1} T_2^{n_1}||}.$$

Now consider the vector $z_2 = T_1^{p_2}T_2^{q_2}u_1 + T_1^{p_1+p}T_2^{q_1+p}u_1$. Then we have

$$\begin{aligned}\|z_2 - u_2\| &= \|z_2 - T_1^{m_1}T_2^{n_1}w_2\| \\ &= \|T_1^{m_1}T_2^{n_1}(T_1^{p_2}T_2^{q_2}v_1 + T_1^{p_1+p}T_2^{q_1+p}v_1) - w_2\| \\ &\leq \|T_1^{m_1}T_2^{n_1}\| \, \|T_1^{p_2}T_2^{q_2}v_1 - (v_2 - T_1^{p_1}T_2^{q_1}v_1)\| \\ &\quad + \|v_2 - T_1^{p_1}T_2^{q_1}v_1 + T_1^{p_1+p}T_2^{q_1+p}v_1 - w_2\| \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

Since $B(u_2, \delta) \subset U_2$, we get $z_2 \in U_2$ and by the same way we can see that $y_2 = T_1^{p_2}T_2^{q_2}v_1 + T_1^{p_1}T_2^{q_1}v_1$ is in V_2. We now apply the condition (iii) of the Theorem to the pairs of open sets (U_k, V_k) with $U_k = B(u_1, 2^{-k})$ and $V_k = B(v_1, 2^{-k})$. Thus there exists a pair of sequences $(\{m_k\}, \{n_k\})$ of integers such that $T_1^{m_k}T_2^{n_k}(U_k) \cap V_k \neq \emptyset$ and $T_1^{m_k+p}T_2^{n_k+p}(U_k) \cap V_k \neq \emptyset$. Thus, there exist the elements u_k and u'_k in U_k such that $T_1^{m_k}T_2^{n_k}u_k \in V_k$ and $T_1^{m_k+p}T_2^{n_k+p}u'_k \in V_k$. Therefore, the sequences $\{u_k\}$, $\{u'_k\}$ converges to u_1, and the sequences $\{T_1^{m_k}T_2^{n_k}u_k\}$, $\{T_1^{m_k+p}T_2^{n_k+p}u'_k\}$ converges to v_1. Hence $T_1^{m_k}T_2^{n_k}(T_1^{p_2}T_2^{q_2}u_k + T_1^{p_1+p}T_2^{q_1+p}u'_k)$ converges to $T_1^{p_2}T_2^{q_2}v_1 + T_1^{p_1}T_2^{q_1}v_1$ which is equal to y_2. But $y_2 \in V_2$ that is open, thus there exists an integer k_0 such that $T_1^{m_k}T_2^{n_k}(T_1^{p_2}T_2^{q_2}u_k + T_1^{p_1+p}T_2^{q_1+p}u'_k) \in V_2$ for all $k \geq k_0$. Note that since u_k, $u'_k \longrightarrow u_1$, thus $T_1^{p_2}T_2^{q_2}u_k + T_1^{p_1+p}T_2^{q_1+p}u'_k$ converges to $T_1^{p_2}T_2^{q_2}u_1 + T_1^{p_1+p}T_2^{q_1+p}u_1$ which is equal to z_2. Since U_2 is open and $z_2 \in U_2$, thus there exists $k_1 > k_0$ such that $T_1^{p_2}T_2^{q_2}u_k + T_1^{p_1+p}T_2^{q_1+p}u'_k \in U_2$ for all $k > k_1$. This implies that $T_1^{m_k}T_2^{n_k}(U_2) \cap V_2 \neq \emptyset$. Moreover, since $u_k \longrightarrow u_1 \in U_1$ and $T_1^{m_k}T_2^{n_k}u_k \longrightarrow v_1 \in v_1$, thus there exists $k_2 > 0$ such that $T_1^{m_k}T_2^{n_k}u_k \in T_1^{m_k}T_2^{n_k}(U_1) \cap V_1$ for all $k > k_2$. Put $k_3 = \max\{k_1, k_2\}$. Thus for $k > k_3$ we have $T_1^{m_k}T_2^{n_k}(U_i) \cap V_i \neq \emptyset$ for $i = 1, 2$. This completes the proof. □

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