International Journal of Pure and Applied Mathematics
Volume 76 No. 3 2012, 317-326
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

HIGH ORDER FINITE DIFFERENCE METHOD FOR NUMERICAL SOLUTION OF GENERAL TWO-POINT BOUNDARY VALUE PROBLEMS INVOLVING SIXTH ORDER DIFFERENTIAL EQUATION

P.K. Pandey
Department of Mathematics
Dyal Singh College (University of Delhi)
Lodhi Road, New Delhi, 110003, INDIA

Abstract: In this paper, the finite difference method is applied to solve a general sixth order boundary value problem. Computational results are presented to demonstrate the order accuracy of the method. Also second and fourth derivatives of solution were obtained as by-product of the method.

AMS Subject Classification: 65L10, 65L12
Key Words: sixth order BVP, finite difference method, numerical solution

1. Introduction

The physical situation is that of an infinite horizontal layer of fluid is heated from below and is subject to the action of rotation, instability sets in. When this instability is as ordinary convection, it is modelled by a sixth order ordinary differential equation (for details see Chandrasekhar [2]). However it is difficult to obtain closed form solution for model problem, especially for nonlinear. In most cases only approximate solution (either numerical solution or analytical solution) can be expected. Some powerful numerical method such as finite

Received: February 6, 2011

element method [1,4], shooting method [3], modified decomposition method [11], the compound matrix method [7], and finite difference method [5] have been developed for obtaining approximate solution to linear boundary value problem. Homotopy analysis method has been proposed to solve problem in [10,12,13].

In the present paper finite difference method is used to solve sixth order boundary value problems. These problems have the general form

$$y^{\mathrm{VI}}(x) = f(x, y(x), \ldots, y^{\mathrm{V}}(x)), \quad -\infty < a \leq x \leq b < \infty \tag{1.1}$$

with the boundary conditions.

$$\begin{aligned} &y(a) = \alpha_0, \quad y^{\mathrm{II}}(a) = \alpha_2, \quad y^{\mathrm{IV}}(a) = \alpha_4, \\ &y(b) = \beta_0, \quad y^{\mathrm{II}}(b) = \beta_2, \quad y^{\mathrm{IV}}(b) = \beta_4, \end{aligned} \tag{1.2}$$

where $y(x)$ and $f(x, y(x)), \ldots, y^{\mathrm{V}}(x))$ are continuous function defined in the interval $x \in [a, b]$, α_i and β_i, $i = 0, 2, 4$ are finite real constants [9]. In detail the existence and uniqueness of solution of such problems were discussed by Agarwal [8] without any numerical method. In this article it is presupposed that unique solution of equation (1.1) exists and specific restriction on f to ensure existence and uniqueness will not be considered [6].

However in practice it is often requirement to consider well suited schemes for different types of non linear problems. The objective of this article is to develop new $O(h^4)$ difference scheme that is well suited for particular types of non linear problems. The computational complexity of the scheme increases with the desired level of accuracy, so for three evaluations of $f(x, y(x), \ldots, y^{\mathrm{V}}(x))$ are required at each mesh point in the our $O(h^4)$ scheme. The Newton-Raphson iterative method, it converges quadratically applied on test problems to get its solution. In making an evaluation of the performance of method, there is balance between the level of accuracy achieved and computational efficiency of the scheme. In Section 2 we describe the finite difference method. In Section 3 we discuss the derivation of the method and also obtain the local truncation error. In Section 4 consider test problems to illustrate the method and its convergence.

A summary of the result and conclusion is given in last Section 5.

2. The Finite Difference Method

Consider a sixth order boundary value problem

$$y^{\mathrm{VI}} = f(x, y, y^{\mathrm{I}}, \ldots, y^{\mathrm{V}}), \quad a \leq x \leq b \tag{2.1}$$

with boundary conditions

$$\begin{aligned} y(a) &= \alpha_0, \quad y^{\mathrm{II}}(a) = \alpha_2, \quad y^{\mathrm{IV}}(a) = \alpha_4 \\ y(b) &= \beta_0, \quad y^{\mathrm{II}}(a) = \beta_2, \quad y^{\mathrm{IV}}(b) = \beta_4 \end{aligned} \tag{2.2}$$

Introduce a finite set of grid points defined as $x_i = a + ih$, $i = 0(1)N$, where $x_0 = a$, $x_N = b$ and mesh spacing $h = (b-a)/N$. A combination of the values $y(x)$, and derivatives $y^{\mathrm{II}}(x)$, $y^{\mathrm{IV}}(x)$ at three grid points are used to derive difference formulas. We denote $y(x_i)$, $y^{\mathrm{II}}(x_i)$, $y^{\mathrm{IV}}(x_i)$ and $f(x_i)$ at grid point x_i by y_i, y_i^{IV}, y_i^{IV} and f_i respectively. Let y_i be the approximate value of Y_i etc.

Let

$$\bar{Y}^{\mathrm{I}}_{i-1} = \frac{1}{2h}(Y_{i+1} - Y_{i-1}) - \frac{h}{3}(2Y_i^{\mathrm{II}} + Y_{i-1}^{\mathrm{II}}) \tag{2.3}$$

$$\bar{Y}^{\mathrm{I}}_{i} = \frac{1}{2h}(Y_{i+1} - Y_{i-1}) - \frac{h}{12}(Y_{i+1}^{\mathrm{II}} - Y_{i-1}^{\mathrm{II}}) \tag{2.4}$$

$$\bar{Y}^{\mathrm{I}}_{i+1} = \frac{1}{2h}(Y_{i+1} - Y_{i-1}) + \frac{h}{3}(2Y_i^{\mathrm{II}} + Y_{i+1}^{\mathrm{II}}) \tag{2.5}$$

$$\bar{Y}^{\mathrm{III}}_{i-1} = \frac{1}{2h}(Y^{\mathrm{II}}_{i+1} - Y^{\mathrm{II}}_{i-1}) - \frac{h}{3}(2Y_i^{\mathrm{IV}} + Y_{i-1}^{\mathrm{IV}}) \tag{2.6}$$

$$\bar{Y}^{\mathrm{III}}_{i} = \frac{1}{2h}(Y^{\mathrm{II}}_{i+1} - Y^{\mathrm{II}}_{i-1}) - \frac{h}{12}(Y_{i+1}^{\mathrm{IV}} - Y_{i-1}^{\mathrm{IV}}) \tag{2.7}$$

$$\bar{Y}^{\mathrm{III}}_{i+1} = \frac{1}{2h}(Y^{\mathrm{II}}_{i+1} - Y^{\mathrm{II}}_{i-1}) + \frac{h}{3}(2Y_i^{\mathrm{IV}} + Y_{i+1}^{\mathrm{IV}}) \tag{2.8}$$

$$\bar{Y}^{\mathrm{V}}_{i-1} = \frac{-60}{h^3}(Y^{\mathrm{II}}_{i+1} - 2Y^{\mathrm{II}}_{i} + Y^{\mathrm{II}}_{i-1}) + \frac{1}{2h}(7Y_{i+1}^{\mathrm{IV}} + 104Y_i^{\mathrm{IV}} + 9Y_{i-1}^{\mathrm{IV}}) \tag{2.9}$$

$$\bar{Y}^{\mathrm{V}}_{i} = \frac{1}{2h}(Y^{\mathrm{IV}}_{i+1} - Y^{\mathrm{IV}}_{i-1}) \tag{2.10}$$

$$\bar{Y}^{\mathrm{V}}_{i+1} = \frac{60}{h^3}(Y^{\mathrm{II}}_{i+1} - 2Y^{\mathrm{II}}_{i} + Y^{\mathrm{II}}_{i-1}) - \frac{1}{2h}(7Y_{i+1}^{\mathrm{IV}} + 104Y_i^{\mathrm{IV}} + 9Y_{i-1}^{\mathrm{IV}}) \tag{2.11}$$

set

$$\begin{aligned} \bar{f}_{i-1} &= f(x_{i-1}, Y_{i-1}, \bar{Y}^{\mathrm{I}}_{i-1}, Y^{\mathrm{II}}_{i-1}, \bar{Y}^{\mathrm{III}}_{i-1}, Y^{\mathrm{IV}}_{i-1}, \bar{Y}^{\mathrm{V}}_{i-1}) \\ \bar{f}_{i+1} &= f(x_{i+1}, Y_{i+1}, \bar{Y}^{\mathrm{I}}_{i+1}, Y^{\mathrm{II}}_{i+1}, \bar{Y}^{\mathrm{III}}_{i+1}, Y^{\mathrm{IV}}_{i+1}, \bar{Y}^{\mathrm{V}}_{i+1}) \\ \bar{f}_{i} &= f(x_{i}, Y_{i}, \bar{Y}^{\mathrm{I}}_{i}, Y^{\mathrm{II}}_{i}, \bar{Y}^{\mathrm{III}}_{i}, Y^{\mathrm{IV}}_{i}, \bar{Y}^{\mathrm{V}}_{i}) \end{aligned}$$

Finally let

$$\bar{\bar{Y}}^{\mathrm{V}}_{i-1} = \bar{Y}^{\mathrm{V}}_{i-1} + \frac{1}{3}h(\bar{f}_{i+1} - \bar{f}_i) \tag{2.12}$$

$$\bar{\bar{Y}}^{\mathrm{V}}_{i} = \bar{Y}^{\mathrm{V}}_{i} - \frac{739}{16620}h(\bar{f}_{i+1} - \bar{f}_{i-1}) \tag{2.13}$$

$$\bar{\bar{Y}}^{\text{V}}_{i+1} = \bar{Y}^{\text{V}}_{i+1} + \frac{1}{3}h(\bar{f}_i - \bar{f}_{i-1}) \tag{2.14}$$

set

$$\begin{aligned}
\bar{\bar{f}}_{i-1} &= f(x_{i-1}, Y_{i-1}, \bar{Y}^{\text{I}}_{i-1}, Y^{\text{II}}_{i-1}, \bar{Y}^{\text{III}}_{i-1}, Y^{\text{IV}}_{i-1}, \bar{\bar{Y}}^{\text{V}}_{i-1}) \\
\bar{\bar{f}}_{i} &= f(x_{i}, Y_{i}, \bar{Y}^{\text{I}}_{i}, \bar{Y}^{\text{II}}_{i}, \bar{Y}^{\text{III}}_{i}, \bar{Y}^{\text{IV}}_{i}, \bar{\bar{Y}}^{\text{V}}_{i}) \\
\bar{\bar{f}}_{i+1} &= f(x_{i+1}, Y_{i+1}, \bar{Y}^{\text{I}}_{i+1}, Y^{\text{III}}_{i+1}, \bar{Y}^{\text{III}}_{i+1}, Y^{\text{IV}}_{i+1}, \bar{\bar{Y}}^{\text{V}}_{i+1})
\end{aligned}$$

Then at each mesh point x_i, the differential equations (2.1)-(2.2) are discretized as

$$\begin{aligned}
& Y_{i-1} - 2Y_i + Y_{i+1} - \frac{h^2}{2}(Y^{\text{II}}_{i-1} + Y^{\text{II}}_{i+1}) + \frac{5h^4}{24}(Y^{\text{IV}}_{i-1} + Y^{\text{IV}}_{i+1}) \\
= \; & \frac{h^6}{40320}(646\bar{f}_{i-1} + 5540\bar{\bar{f}}_i + 646\bar{f}_{i+1}) \qquad (2.15) \\
& Y_{i-1} - 2Y_i + Y_{i+1} - \frac{h^2}{1229760}(116280Y^{\text{II}}_{i-1} + 997200Y^{\text{II}}_{i} + 116280Y^{\text{II}}_{i+1}) \\
& + \frac{6900h^4}{1229760}(Y^{\text{IV}}_{i-1} + Y^{\text{IV}}_{i+1}) = \frac{313h^6}{1229760}(\bar{\bar{f}}_{i-1} + \bar{\bar{f}}_{i+1}) \qquad (2.16) \\
& Y_{i-1} - 2Y_i + Y_{i+1} - \frac{5040h^2}{10080}(Y^{\text{II}}_{i-1} + Y^{\text{II}}_{i+1}) \qquad (2.17) \\
& + \frac{6h^4}{10080}(73Y^{\text{IV}}_{i-1} + 554Y^{\text{IV}}_{i} + 73Y^{\text{IV}}_{i+1}) \\
= \; & \frac{23h^6}{10080}(\bar{\bar{f}}_{i-1} + \bar{\bar{f}}_{i+1}) \qquad (2.18)
\end{aligned}$$

The difference schemes (2.15), (2.16) and (2.17) form a system of linear/ non linear equations in unknown $Y_i, Y^{\text{II}}_i, Y^{\text{IV}}_i, 1 \le i \le N$, depend on if forcing function f is linear/non linear.

3. Derivation of the Finite Difference Scheme

In this section we discuss the derivation of finite difference method and the local truncation error associated with it. We need $O(h^4)$-approximation for Y^{I}_i. Let

$$h\bar{Y}^{\text{I}}_i = a_0 Y_{i+1} + a_1 Y_{i-1} + h^2(b_0 Y^{\text{II}}_{i+1} + b_1 Y^{\text{II}}_{i-1}) \tag{3.1}$$

write in Taylor series, each term of (3.1) about mesh point x_i. Comparing the coefficients of $h^p, p = 0(1)4$ both sides and solving the system of equations. We have

$$(a_0, a_1, b_0, b_1) = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{12}, \frac{1}{12}\right) \tag{3.2}$$

So, (3.1) can be rewritten as

$$\bar{Y}_i^{\mathrm{I}} = (Y_{i+1} - Y_{i-1})|(2h) - h(Y_{i+1}^{\mathrm{II}} - Y_{i-1}^{\mathrm{II}})/12 = Y_i^{\mathrm{II}} + O(h^4) \tag{3.3}$$

Thus from (3.3) we conclude that $\bar{Y}_i^{\mathrm{I}}$ provide an $O(h^4)$ approximation for Y_i^{I}. Similarly we can find an $O(h^4)$ approximation for $Y_{i+1}^{\mathrm{I}} \dots Y_{i-1}^{\mathrm{III}}$ etc. Now we need $O(h^2)$ approximation for Y_i^{V}. Let

$$h^5 \bar{Y}_i^{\mathrm{V}} = h^4(b_2 Y_{i+1}^{\mathrm{IV}} + b_3 Y_{i-1}^{\mathrm{IV}}) \tag{3.4}$$

following the steps similar to above, we obtained

$$(b_2, b_3) = \left(\frac{1}{2}, -\frac{1}{2}\right) \tag{3.5}$$

Thus (3.5) can be rewritten as

$$\bar{Y}_i^{\mathrm{V}} = \frac{1}{2h}(Y_{i+1}^{\mathrm{IV}} - Y_{i-1}^{\mathrm{IV}}) = Y_i^{\mathrm{V}} + \frac{h^2}{6} Y_i^{\mathrm{VII}} + O(h^4) \tag{3.6}$$

from (3.6), we find that $\bar{Y}_i^{\mathrm{V}}$ provide an $O(h^2)$ approximation for Y_i^{V}.

Let define

$$\bar{\bar{Y}}_i^{\mathrm{V}} = \bar{Y}_i^{\mathrm{V}} + b_4 h(\bar{f}_{i+1} - \bar{f}_{i-1}) \tag{3.7}$$

where b_4 is free parameter to be determined.

With the help of (3.6) and from (3.7) we have

$$\bar{\bar{Y}}_i^{\mathrm{V}} = Y_i^{\mathrm{V}} + h^2\left(\frac{1}{6} + 2b_4\right) Y_i^{\mathrm{VII}} + O(h^4) \tag{3.8}$$

$323(\bar{f}_{i+1} + \bar{f}_{i-1}) + 2770\bar{\bar{f}}_i$, the term in (2.15), with help of (2.5), (2.11) and (3.8) can be written as

$$\begin{aligned} & 323(\bar{f}_{i+1} + \bar{f}_{i-1}) + 2770\bar{\bar{f}}_i \\ = \; & 323(f_{i+1} + f_{i-1}) - \frac{h^2}{3}\left[\left(y^{\mathrm{VII}}\frac{\partial f}{\partial y^{\mathrm{V}}}\right)_{i+1} + \left(y^{\mathrm{VII}}\frac{\partial f}{\partial y^{\mathrm{V}}}\right)_{i-1}\right] \\ & +2770\left[f_i + h^2\left(\frac{1}{6} + 2b_4\right)\left(y^{\mathrm{VII}}\frac{\partial f}{\partial y^{\mathrm{V}}}\right)_i\right] + O(h^4) \end{aligned} \tag{3.9}$$

from (3.9), we conclude that $323(\bar{f}_{i+1} + \bar{f}_{i-1}) + 2770\bar{\bar{f}}_i$ will provide an $O(h^4)$ approximation for 323 $(f_{i+1} + f_{i-1}) + 2770 f_i$ if

$$b_4 = -739/16620. \tag{3.10}$$

As above we can find $\bar{Y}^{\mathrm{V}}_{i+1}$ and $\bar{Y}^{\mathrm{V}}_{i-1}$ which provide an $O(h^2)$ approximation for Y^{V}_{i+1} and Y^{V}_{i-1} respectively. Let again define.

$$\bar{\bar{Y}}^{\mathrm{V}}_{i+1} = \bar{Y}^{\mathrm{V}}_{i+1} + b_5 \cdot h(\bar{f}_i - \bar{f}_{i-1}) \tag{3.11}$$

and

$$\bar{\bar{Y}}^{\mathrm{V}}_{i-1} = \bar{Y}^{\mathrm{V}}_{i-1} + b_6 \cdot h(\bar{f}_{i+1} - \bar{f}_i) \tag{3.12}$$

where b_5, b_6 are free parameters to be determined. With the helps of approximation (2.9), (2.10) and (2.11), from (3.11) and (3.12) respectively, we have

$$\bar{\bar{Y}}^{\mathrm{V}}_{i+1} = Y^{\mathrm{V}}_{i+1} - \frac{h^2}{3} Y^{\mathrm{VII}}_{i+1} + b_5 h^2 Y^{\mathrm{VII}}_{i+1} + O(h^4) \tag{3.13}$$

and

$$\bar{\bar{Y}}^{\mathrm{V}}_{i-1} = Y^{\mathrm{V}}_{i-1} - \frac{h^2}{3} Y^{\mathrm{VII}}_{i-1} + b_6 h^2 Y^{\mathrm{VII}}_{i-1} + O(h^4) \tag{3.14}$$

$\bar{\bar{f}}_{i+1} + \bar{\bar{f}}_{i-1}$, the term in (2.16) and (2.17), with help of (3.11), (3.12), (3.13) and (3.14), can be written as

$$\begin{aligned} \bar{\bar{f}}_{i+1} + \bar{\bar{f}}_{i-1} &= f_{i+1} + f_{i-1} + \left(-\frac{1}{3} + b_5\right) h^2 \left(y^{\mathrm{VII}} \frac{\partial f}{\partial y^{\mathrm{V}}}\right)_{i+1} \\ &\quad + \left(-\frac{1}{3} + b_6\right) h^2 \left(y^{\mathrm{VII}} \frac{\partial f}{\partial y^{\mathrm{V}}}\right)_{i-1} \end{aligned} \tag{3.15}$$

from (3.15), we conclude that $\bar{\bar{f}}_{i+1} + \bar{\bar{f}}_{i-1}$ will provide an $O(h^4)$ approximation for $f_{i+1} + f_{i-1}$ if

$$-\frac{2}{3} + b_5 + b_6 = 0 \tag{3.16}$$

$$b_5 - b_6 = 0 \tag{3.17}$$

solving system of equations (3.16) and (3.17), we have

$$(b_5, b_6) = \left(\frac{1}{3}, \frac{1}{3}\right). \tag{3.18}$$

Finally let

$$\begin{aligned} & A_0(Y_{i+1}-2Y_i+Y_{i-1})+B_0h^2(Y^{\mathrm{II}}_{i+1}+Y^{\mathrm{II}}_{i-1})+c_0h^4(Y^{\mathrm{IV}}_{i+1}+Y^{\mathrm{IV}}_{i-1}) \\ = \; & h^6(D_0(f_{i+1}+f_{i-1})+D_1f_i) \end{aligned} \tag{3.19}$$

Expand in Taylor series about point x_i each term of (3.19). Comparing the coefficients of h^p, $p=1,2,\dots,9$ solving the system of equation so obtained in $A_0,\dots,$ we have

$$(A_0,B_0,C_0,D_0,D_1)=\left(1,-\frac{1}{2},\frac{1}{12},\frac{646}{40320},\frac{5540}{40320}\right) \tag{3.20}$$

So (3.19) can be written as

$$\begin{aligned} & Y_{i-1}-2Y_i+Y_{i+1}-\frac{h^2}{2}(Y^{\mathrm{II}}_{i-1}+Y^{\mathrm{II}}_{i+1})+\frac{5h^4}{24}(Y^{\mathrm{IV}}_{i+1}+Y^{\mathrm{IV}}_{i-1}) \\ = \; & \frac{h^6}{40320}(646(f_{i+1}+f_{i-1})+5540f_i)+\phi_i(h) \end{aligned} \tag{3.21}$$

where $\phi_i(h)=O(h^{10}), i=1,2,\dots N$.

With the help of (3.9) we have

$$\begin{aligned} & Y_{i-1}-2Y_i+Y_{i+1}-\frac{h^2}{2}(Y^{\mathrm{II}}_{i-1}+Y^{\mathrm{II}}_{i+1})+\frac{5h^4}{24}(Y^{\mathrm{IV}}_{i+1}+Y^{\mathrm{IV}}_{i-1}) \\ = \; & \frac{h^6}{40320}(646(\bar{f}_{i+1}+\bar{f}_{i-1})+5540\bar{\bar{f}}_i)+\theta_i(h) \end{aligned} \tag{3.22}$$

where $\theta_i(h)=O(h^{10}), i=1,2,\dots,N$.

Thus (3.22) discretize the differential equation (2.1). Similarly we can discretize (2.2) at each mesh point x_i and obtain

$$Y_i-2Y_i+Y_{i+1}-\frac{h^2}{1229760}(116280Y^{\mathrm{II}}_{i+1}+997200Y^{\mathrm{II}}_i+116280Y^{\mathrm{II}}_{i-1}) \tag{3.23}$$

$$\begin{aligned} & +\frac{6900h^4}{1229760}(Y^{\mathrm{IV}}_{i-1}+Y^{\mathrm{IV}}_{i+1}) \\ = & \frac{313h^6}{1229760}(\bar{\bar{f}}_{i-1}+\bar{\bar{f}}_{i+1})+r_i(h) \end{aligned} \tag{3.24}$$

$$\begin{aligned} & Y_{i-1}-2Y_i+Y_{i+1}-\frac{5040h^2}{10080}(Y^{\mathrm{II}}_{i-1}+Y^{\mathrm{II}}_{i+1}) \\ & +\frac{6h^4}{10080}(73Y^{\mathrm{IV}}_{i-1}+554Y^{\mathrm{IV}}_i+73Y^{\mathrm{IV}}_{i+1}) \end{aligned}$$

$$=\frac{23h^6}{10080}(\bar{\bar{f}}_{i+1}+\bar{\bar{f}}_{i-1})+\beta_i(h) \tag{3.25}$$

where $r_i(h)=O(h^{10})$ and $\beta_i(h)=O(h^{10})$, $i=1,2,\ldots,N$.

Thus we define discretization (2.15) for differential equation (1.1) and with the help of (3.21) estimate local truncation error associated with (2.15) which is of $O(h^4)$. The emphasis here is an the actual formula required for computation, complete expansions for the local truncation errors and bound on errors are leaved out of this section.

4. Numerical Results

In this section, the method discussed in Section 2 were tested on three model problems. The maximum absolute errors (MAE) in the analytical solutions and derivatives of solution for model problems were calculated by

$$MAE=\max|\text{Exact value}-\text{Approximate value}|$$

and are given in Tables 1-3.

Model Problem 1. Consider the non linear boundary value problem

$$y^{\mathrm{VI}}=y^{\mathrm{I}}\cdot y^{\mathrm{V}}-(y^{\mathrm{III}})^2+f(x),0\le x\le 1$$

with exact solution $y(x)=\sin(\pi x)$, we use method discussed in Section 2 and obtain the results, given in Table 1.

Model Problem 2. Consider the non linear boundary value problem

$$y^{\mathrm{VI}}=y^{\mathrm{III}}\cdot y^{\mathrm{V}}+f(x),0\le x\le 1$$

with exact solution $y(x)=e^{-x}\sin(x)$, result given in Table 2.

Model Problem 3. Consider the non linear boundary value problem

$$y^{\mathrm{VI}}=(y)^2y^{\mathrm{V}}+f(x),\quad 0\le x\le 1$$

with exact solution $y(x)=e^{-x}$, result given in Table 3.

MAE	h				
	8	16	32	64	128
y	.101414(-2)	.569422(-4)	.346105(-5)	.205712(-6)	.250685(-9)
y^{II}	.100500(-1)	.564359(-3)	.343047(-4)	.204113(-5)	.273880(-8)
y^{IV}	.100796(0)	.573325(-2)	.349710(-3)	.208685(-4)	.119195(-6)

Table 1

MAE	h				
	4	8	16	32	64
y	.110709(-5)	.551079(-7)	.323775(-8)	.155442(-9)	.320077(-11)
y^{II}	.109557(-4)	.593518(-6)	.350615(-7)	.173936(-8)	.444749(-10)
y^{IV}	.150117(-3)	.857085(-5)	.530639(-6)	.288172(-7)	.249525(-8)

Table 2

MAE	h			
	4	8	16	32
y	.409158(-6)	.254517(-7)	.158674(-8)	.249504(-10)
y^{II}	.401237(-5)	.249774(-6)	.155758(-7)	.259844(-9)
y^{IV}	.373227(-4)	.245577(-5)	.153605(-6)	.325405(-8)

Table 3

5. Conclusions

A linear combination of known values with application of free parameter, create a method for the numerical solution of sixth order boundary value problems. It is interesting to note that the method solve nonlinear boundary value problem with these boundary conditions, of $O(h^4)$ accuracy. Also it is interesting to note that we got numerical value of second and fourth derivative of solution. We applied Newton Raphson method to solve the discretized system of non linear equations. It is evident from the results, method has higher computational accuracy and efficiency.

References

[1] R.C.D. Prima, Application of the Galerkin method to problems in hydrodynamic stability, *Quart. Appl. Math.*, **30** (1955), 55-62.

[2] S. Chandrasekhar, *Hydrodynamics and Hydromagnetic Stability*, Clarendon Press, Oxford (1961).

[3] S.M. Roberts, J.S. Shipman, *Two Point Boundary Value Problems: Shooting Methods*, American Elsevier, New York (1972).

[4] A.G. Deacon, S. Osher, Finite element method for a boundary value problem of mixed type, *SIAM J. Numer. Anal.*, **16** (1979), 756-778.

[5] M.M. Chawla, C.P. Katti, Finite differnece method for two point boundary value problems involving higher order differential equations, *BIT*, **19** (1979), 27-33.

[6] J.V. Baxley, Nonlinear two point boundary value problems, In: *Ordinary and Partial Differential Equations* (Ed-s: W.N. Everitt, B.D. Sleeman), New York, Springer-Verlag (1981), 46-54.

[7] B.S. Ng, W.H. Reid, The compound matrix method for ordinary differntial equations, *J. Comput. Phys.*, **58** (1985), 209-228.

[8] R.P. Agarwal, *Boundary Value Problems for Higher Order Ordinary Differential Equations*, World Scientific, Singapore (1986).

[9] S.S. Siddiqi, E.H. Twizel Spline solution of linear sixth order boundary value problems, *Int. J. Comput. Math.*, **60** (1996), 295-304.

[10] L. Shijun, Homotopy analysis method: A new analytic method for non linear problems, *Appl. Math. and Mech.*, **19**, No. 10 (1998), 957-962.

[11] A.M. Wazwaz, The numerical solution of sixth order boundary value problems by the modified decomposition method, *Applied Mathematics and Computation*, **18** (2001), 311-325.

[12] M.A. Noor, S.T. Mohyuddlin, Homotopy perturbation method for solving sixth-order boundary value problems, *Comput. Math Appl.*, **55** (2008), 2953-2972.

[13] S.J. Liao, *Beyond Perturbation: Introduction to the Homotopy Analysis Method*, Chapman and Hall, CRC Press, Boca Raton (2003).