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SUFFICIENT CONDITIONS FOR THE EXISTENCE OF DERIVATE OF THE SOLUTIONS OF PERIODICAL IMPULSE DIFFERENTIAL EQUATIONS

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Abstract: We consider in $\mathbb{R}^N$ the existence and the properties of the shift operator of impulse differential equations. Sufficiently conditions for the existence of periodical solutions are found.

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1. Introduction

The impulse differential equations are flexible mathematical instruments for the simulation of evolutionary processes in the science, in the technics and in many other areas. In the medicine and in the biotechnologies we often consider impulse differential equations with periodical solutions.

In the paper we found sufficiently conditions for the existence of periodical solutions of periodical impulse differential equations.

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2. Problem Statement

Let X is a N-dimensional Eucleadean space with identity I , norm $||.||$ and scalar product $(.,.)$. We consider following impulse differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t,x) \quad \text{for } t \neq t_n \tag{1}$$

$$\Delta x(t = t_n) = x(t_n{}^+) - x(t_n) = I_n(x(t_n)) \quad (n = \pm 1, \pm 2, \pm 3...), \tag{2}$$

where $f(t,x) : \mathbb{R} \times X \to X$ is continuous for $t \neq t_n$ and continuosly extandable on each interval $[t_n, t_{n+1}]$ and $I_n : X \to X$ are continuous maps $(n = \pm 1, \pm 2, \pm 3...)$.

Definition 1. A solution of the impulse differential equation (1), (2) we shall call a function $\varphi(t)$ which is continous for $t \neq t_n$, has discontinuites of the first kind at $t = t_n$, is continous from the left, for the $t \neq t_n$ satisfies the equation (1) and at $t = t_n$ meets the condition of a "jump" (2).

We suppose the validity of the following condition H1 :

H1. Any initial value $x(s) = x_0$ determins an unique solution

$$x(t) = X(t; s, x_0)(t \in \mathbb{R})$$

We consider the shift operator [3], [4]:

$$U(t,s)x_0 = X(t; s, x_0) \ (s \leq t)$$

where $X(t; s, x_0)$ is the solution of (1), (2) with $x(s) = x_0$.

It is not hard to check, that the following equalities hold:

$$U(t,t)x = x \ \ (t \in \mathbb{R}, x \in X),$$

the semigroup conditions

$$U(t,s)U(s,\tau) = U(t,\tau) \ \ (-\infty < \tau \leq s \leq t < \infty)$$

$$U(t+\tau, s) = U(t+\tau, t)U(t,s) \ \ (-\infty < s < t < t+\tau < \infty)$$

and

$$U(t_n{}^+, \tau) = Q_n U(t_n, \tau) \quad (t_n > \tau)$$

$$U(t, t_n{}^+) = Q_n U(t, t_n) \quad (t > t_n)$$

where $n = \pm 1, \pm 2, \pm 3...$ and $Q_n = I_n + I$.

In the paper we will investigate problems about the existence of the Frechet derivate of $U(t,s)$. A conclusion between the existence of periodic solution and the existence of the Frechet derivate is considered.

It is not hard to check, that if (1), (2) is linear, then the shift operator is linear as well.

3. Main Results

Obviously the operator $U(t,s)$ acts in the space $C(\mathbb{R},X)$ of all continuously for $t \neq t_n$ functions, which at $t = t_n$ have jumps of first order and which are continuously from the left.

Lemma 1. (see [1]) *Let the following conditions are fulfilled:*

1. The condition H1. holds.

2. For any $n \in \mathbb{Z}$ the map $\psi_n : X \to X$ with $x \to z,\ z = x + I_n(x)$, is a homeomorphismus.

3. The functions $\frac{\partial f}{\partial x}$ are continuous on $(t_n, t_{n+1}) \times X$.

Then there exists the derivate $\frac{\partial X}{\partial x_0}(t; t_0, x_0)$, which is continuous by $t \neq t_n$.

Corollary 1. *If the conditions of Lemma 1. are fulfilled, then $U(t,s)x$ is continuous on $\mathbb{R} \times \mathbb{R} \times X$.*

To investigate the properties of the operator $U(t,s)(-\infty < s \leq t < \infty)$ we introduce the following condition H2:

H2. Any solution $X(t; s, x_0)$ of (1), (2) is continuous extendable on $[t_n, t_{n+1}]$ for $t \geq s \geq 0$.

Let $x^*(t)$ is a solution of (1), (2) and $W(t)$ is the Cauchy operator of the linearized on x^* impulse differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad (t \neq t_n) \tag{3}$$

$$\Delta x \mid_{t=t_n} = J_n(x(t_n)) \quad (n = \pm1, \pm2, \pm3...) \tag{4}$$

where $A(t) = \frac{\partial f}{\partial x}(t, x^*(t)),\quad J_n = \frac{\partial I_n}{\partial x}(x^*(t_n)) \quad (n = \pm1, \pm2, \pm3...)$

It may be noted, that the shift operator $U(t,s)$ of (3),(4) concur with the $W(t)W^{-1}(s)$ $(0 \leq s \leq t < \infty)$, where $W(t)$ is the Cauchy operator of (3), (4).

Other properties of the shift operater are investigated in [2].

Theorem 1. *Let following conditions are fulfilled:*
1. The conditions of Lemma 1. hold.
2. The operators I_n $(n = \pm 1, \pm 2, \pm 3...)$ have continuous derivates.
3. The function $x^*(t) = X(t; 0, x^*)$ is a solution of (1), (2).
4. The condition H2. holds.
Then the derivate of $U(t, s)$ to x in the point x^* concur with $W(t)$.

Proof. From Lemma 1. it follows that the solutions of (1), (2), who lie in an enough little heighbourhood $\Omega(x^*)$ are determined on $[0, \infty)$. That means, that the operator $U(t, s)x = X(t; s, x)$ has a Frechet derivate for $s \leq t$.

Let $t \neq t_n$. We consider the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t; 0, x) = f(t, X(t; 0, x)) \tag{5}$$

and differentiate the two sides. We obtain the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} {X_x}'(t; 0, x) = {f_x}'(t, X(t; 0, x)){X_x}'(t; 0, x)h \tag{6}$$

We set in (6) $x = x^*$ and receive

$$\frac{\mathrm{d}}{\mathrm{d}t} {X_x}'(t; 0, x^*) = {f_x}'(t, X(t; 0, x^*)){X_x}'(t; 0, x^*)h$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} {X_x}'(t; 0, x^*) = A(t){X_x}'(t; 0, x^*)h$$

We consider the function

$$\alpha(t) = {X_x}'(t; 0, x^*)h$$

with initial value $\alpha(0) = h$. The function $\alpha(t)$ is a solution of the linear impulse equation (3), (4) for $t \neq t_n$.

Now we consider the case $t = t_n$. For the difference $\alpha({t_n}^+) - \alpha(t_n)$ we receive the presentation

$$\alpha({t_n}^+) - \alpha(t_n) = ({X_x}'({t_n}^+; 0, x^*) - {X_x}'(t_n; 0, x^*))h \tag{7}$$

We differentiate the two sides of the equality

$$X({t_n}^+; 0, x) - X(t_n; 0, x) = I_n(X(t_n; 0, x))$$

and obtain the equality

$$X_x{}'(t_n{}^+;0,x) - X_x{}'(t_n;0,x) = \frac{\partial I_n}{\partial x}(x(t_n;0,x))X_x{}'(t_n;0,x)$$

For $x = x^*$ we obtain

$$X_x{}'(t_n{}^+;0,x^*) - X_x{}'(t_n;0,x^*) = \frac{\partial I_n}{\partial x}(x(t_n;0,x^*))X_x{}'(t_n;0,x^*)$$

i.e.

$$\alpha(t_n{}^+) - \alpha(t_n) = \frac{\partial I_n}{\partial x}(x(t_n;0,x^*))\alpha(t_n)$$

Hence the function $\alpha(t)$ is a solution of the linear impulse differential equation (3), (4). From the uniqness of this solution follows

$$X_x{}'(t;0,x^*) = W(t)x^*.$$ □

Now we consider the following operator family

$$\Phi(t,s)x = x - U(t,s)x, \quad (-\infty < s \le t < \infty)$$

where $U(t,s)$ is the shift operator of (1), (2).

Let the set $M \subset X$ is bounded and closed.

Lemma 2. (see [3]) *Let following conditions are fulfilled:*

1. *The conditions of Lemma 1. hold.*
2. *The condition H2 holds.*
3. $f(s,x) \neq 0 \quad (s \in \mathbb{R}, x \in M)$

Then there exists a number $\delta > 0$ such that $\Phi(t,s)x \neq 0$ $(x \in M)$ for $s < t \le s+\delta$ and $\Phi(t,s)$ is homotop to $-f(s,x)$.

Proof. The proof is inessential modification of Lemma 13.1 [5]. □

By $\gamma(T,M)$ we shall note the rotation of T on M , where $T : M \to X$ is a continuous map (see [5]). Now we suppose that $M = \partial\Omega$, where $\partial\Omega$ is the boundary of a bounded domain Ω in X.

Corollary 2. *We assume, that the conditions of Lemma 2. are fulfilled. Then for t near to s $(t \ge s)$ we have*

$$\Phi(t,s)x \neq 0 \quad (x \in \Omega)$$

$$\gamma(\Phi(t,s),\Omega) = \gamma(-f(s,x),\Omega) \tag{8}$$

Let x_0 be an isolated singular point of T. By $ind(x_0)$ we shall denote the index of the point x_0 (see [5]).

Corollary 3. *Let the conditions of Lemma 2. are fulfilled and let* $\Phi(t,s)x \neq 0$ $(x \in \partial\Omega, s < t \leq s_1))$. *Then following equality holds*

$$\gamma(\Phi(s_1,s),\Omega) = \gamma(-f(s,x),\Omega)$$

The proofs of Corrolary 2. and Corrolary 3. can be found in [5].

We shall call the continuously differentiate function $V(x)$ $(x \in X)$ to be a leading potential for the solution $x(t) = X(t;s,x_0)$ of (1), (2) if

$$(grad\ V(x), f(t,x)) > 0 \quad (||x|| \geq \rho_0),$$

where ρ_0 is some positiv number (see [5]).

Lemma 3. *Let the following conditions are fulfilled:*

1. The condition H2 holds.

2. There exists a leading potential $V(x)$. *(see [5])*

3. For any solution $x(t)$ *of (1), (2) we have*

$$V(x({t_n}^+)) \leq V(Q_n x(t_n)) \quad (n = \pm 1, \pm 2, \pm 3...)$$

4. The conditions of Lemma 1. are fulfilled.

Then for fixed t *and* s $(t > s)$ *there exists a number* $\rho_0(t,s) \geq \rho_0$ *such that*

$$\Phi(t,s)x \neq 0\ (||x|| = \rho \geq \rho_0(t,s)$$

moreover following equality holds

$$\gamma(\Phi(t,s),S_\rho) = (-1)^N indV(x) \quad (\rho \geq \rho_0(t,s))$$

Proof. From condition 2. of Lemma 3. it follows that the two vectors $gradV(x)$ and $f(t,x)$ are not opposite orientated for all x for which $||x|| \geq \rho_0$ and $t \in \mathbb{R}$. That's why on any sphere S_ρ with radius $\rho \geq \rho_0$ the operator $f(t,x)$ is homotop to $grad\ V(x)$. Hence

$$\gamma(f(t,s),S_\rho) = ind\ V(x) \quad (t \in \mathbb{R})$$

i.e.

$$\gamma(-f(t,s),S_\rho) = (-1)^N indV(x).$$

From the last equality and Corolaries 2. and 3. it follows, that to proove Lemma 3. it is sufficient to show that the operators $\Phi(t,s)$ $(s < \tau \leq t)$ have no zero values on any sphere S_ρ. We set

$$\rho_1 = \max ||U(\tau,\sigma)x|| \quad (s \leq \sigma \leq \tau \leq t, \ \ ||x|| \leq \rho_0) \tag{9}$$

We shall show that $\Phi(\tau,s)x \neq 0$ for $s < \tau \leq t$, $||x|| \geq \rho_1$. Let suppose the oposite. Then there exists x_0 with $||x_0|| > \rho_1$ and a number $\tau_0 \in (s,t]$ such that $U(\tau_0,s)x_0 = x_0$. From (9) it follows

$$||U(\tau,s)x_0|| \geq \rho_0 \quad (s \leq \tau \leq \tau_0) \tag{10}$$

We consider the scalar function

$$v(\tau) = V(U(\tau,s)x_0) \quad (s \leq \tau \leq \tau_0).$$

Because for $\tau \neq t_i$ we have

$$v'(\tau) = (gradV(U(\tau,s)x_0), \frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau,s)x_0) = (gradV(U(\tau,s)x_0), f(\tau, U(\tau,s)x_0))$$

It follows from (10) that $v'(\tau) > 0$, i.e. the function $v(\tau)$ is increasing on any subinterval $[t_n, t_{n+1}]$.

Let $\tau = t_n$. Then we have

$$v({t_n}^+) = V(U({t_n}^+,s)x_0), \quad v(t_n) = V(U(t_n,s)x_0),$$

i.e.

$$v({t_n}^+) = V(Q_n(X(t_n;s,x_0))), \quad v(t_n) = V(X(t_n;s,x_0).$$

From condition 3. of Lemma 3. it follows, that $v({t_n}^+) \geq v(t_n)$, i.e. the function $v(\tau)$ is increasing on the whole interval $(s,t]$. Hence $V(U(\tau_0,s),x_0) > V(x_0)$, i.e. $U(\tau_0,s)x_0 \neq x_0$. The received contradiction ends the proof. □

Now we shall consider periodical impulse differential equations.

Definition 2. The impulse differential equation is called periodical if there exists a positiv number ω and a number $p \in \mathbb{N}$ such that following conditions are fulfilled

H3. $f(t+\omega,x) = f(t,x) \quad (t \in \mathbb{R}, x \in X)$

H4. $t_{n+p} = t_n + \omega \quad (n = \pm1, \pm2, \pm3...)$, where p is the number of the point, which lies in the interval $[0,\omega]$

H5. $I_{n+p} = I_n \quad (n = \pm1, \pm2, \pm3...)$

We introduce following condition

H6. The function $f(t,x)$ satisfy on x and for any $t \in \mathbb{R}$ the Lipschitz condition.

Theorem 2. *Let following conditions are fulfilled:*

1. The conditions H3.-H6.

2. The conditions of Lemma 3.

3. $indV(x) \neq 0$

Then the periodic impulse differentioal equation has an ω-periodical solution.

Proof. We consider the operator $\Phi(\omega, 0)$. From Lemma 3. and condition 3. of Theorem 2. it follows, that $\Phi(\omega, 0)$ has a zeropoint $x_0 \in X$,

i.e. $\Phi(\omega, 0)x_0 = 0$, i.e. $U(\omega, 0)x_0 = x_0$.

Hence x_0 is the first value of the ω-periodical solution of (1), (2). □

Theorem 3. *Let following conditions are fulfilled:*

1. The conditions of Theorem 2.

2. The impulse differential equation (1), (2) has an ω-periodical solution $x^*(t)$ and $ind(x^*, \Phi) \neq (-1)^N indV(x)$.

Then the equation (1), (2) has an ω-periodical solution, which is different as x^*.

Proof. The proof of Theorem 3. is a corollary from the Theorem of Kronecker in [1] and [5]. □

Remark 1. It may be noted, that the solution of a periodical equation can be periodical with another period. This situation is possible even in $\mathbb{R}$ [7].

4. Examples

Now we shall give an example, where we can calculate the number $ind(x^*, \Phi)$.

We consider the linearized on x^* impulse differential equation (1), (2). We suppose, that all eigenvalues of the martix $W(\omega)$ are inequal 1. Then there exists the martix $(I - W(\omega))^{-1}$ and hence $det|I - W(\omega)| \neq 0$.

From Lemma 3. follows the equality

$$\frac{\partial}{\partial x}\Phi(t,s) = I - W(t,s).$$

From the theorems for the calculation of the index on the linearized operator (see [3]) we receive that x^* is an isolated zero point of $\Phi(\omega, 0)$ and

$$ind(x^*, \Phi) = (-1)^\beta,$$

where β is the sum of rate frequency of all real negativ eigenvalues of the matrix $\frac{\partial}{\partial x}\Phi(\omega, 0)$.

To determine the eigenvalues of $W(\omega)$ we must have an explizit formula for $W(t, s)$. We shall consider the simply case when $f'_x(t, x^*(t)) = A = const$ and $AI_i = I_iA \quad (i = 1, ..., p)$. In this case we obtain for $W(t)$ the explizit formula

$$W(t) = e^{At} \prod_{j=1}^{n(t)} Q_j,$$

where $n(t) = \max\{n : t_n < t\}$ and $Q_j = I + I_j$. For $t = \omega$ we obtain

$$W(\omega) = e^{A\omega} \prod_{j=1}^{p} Q_j.$$

Let λ_i are the eigenvalues of A. It is not hard to check that the eigenvalues μ_i of $W(\omega)$ concure with the eigenvalues of the operator

$$W(\omega) = e^{2i\omega} \prod_{j=1}^{p} Q_j.$$

If σ_i are the eigenvalues of

$$\prod_{j=1}^{p} Q_j,$$

then μ_i concure with the eigenvalues of the operators $\sigma_i e^{A\omega} \quad (i = 1, ..., N)$.

Analogously we can consider the case where

$$A(t)A(s) = A(s)A(t) \quad (-\infty < s, t < \infty)$$

and

$$AI_i = I_iA \quad (i = 1, 2, 3, ..., p).$$

In this case it is not hard to check that for $W(t, s)$ is valid the following formula

$$W(t, s) = e^{\int_s^t A(\tau)\mathrm{d}\tau} \prod_{s \le t_j \le t} Q_j.$$

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