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ON (COFINITELY) GENERALIZED AMPLY WEAK SUPPLEMENTED MODULES

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Abstract: Let R be a ring and M be a left $R-$module. In this paper, we will study some properties of (cofinitely) generalized amply weak supplemented modules (CGAWS) as a generalization of (cofinitely) amply weak supplemented and give a new characterization of semilocal rings using CGAWS-modules. Nevertheless, we will show that (1) M is Artinian if and only if M is a GAWS-module and satisfies DCC on generalized weak supplement submodules and on small submodules. (2) A ring R is semilocal if and only if every left R-module is CGAWS-module.

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1. Introduction and Preliminiaries

Throughout the paper, R will be an associative ring with identity and all modules are unital left $R-$modules unless otherwise specified. Let M be an $R-$module. The symbol $N \leq M$ means that N is a submodule of M. Recall that a submodule $N \leq M$ is called *small* and denoted by $N \ll M$ if $N+K \neq M$

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for every proper submodule K of M. $Rad(M)$ will indicate Jacobson radical of M. K is a supplement of N in M if and only if $N + K = M$ and $N \cap K \ll K$ (see [7]) where K and N are submodules of M. M is called *supplemented*, if every submodule N of M has a *supplement* in M, i.e. a submodule K minimal with respect to $N + K = M$.If $N + K = M$ and $N \cap K \ll M$, then K is called a *weak supplement* of N in M, (see [5], [9]), and clearly in this situation N is a weak supplement of K, too. M is a *weakly supplemented* module if every submodule of M has a weak supplement in M.

A submodule N of a M is said to be *cofinite* if $\frac{M}{N}$ is finitely generated. M is called a *cofinitely (weak) supplemented* module if every cofinite submodule of M has a (weak) supplement in M (see [1], [2]). Clearly supplemented modules are cofinitely supplemented and weakly supplemented modules are cofinitely weak supplemented.

A submodule N of a module M has *ample (weak) supplements* in M if for all $K \leq M$ with $M = N + K$, there is a (weak) supplement $K^{'}$of N with $K^{'} \leq K$. If every submodule of M has ample *(weak) supplements* in M, then M is called *amply (weak) supplemented*. Similarly, if every cofinite submodule of M has ample *(weak) supplements* in M, M is called *cofinitely amply (weak) supplemented*.

Let M be an $R-$module and N, K be any submodules of M with $M = N + K$. If $N \cap K \leq Rad(K)$ $(N \cap K \leq Rad(M))$ then K is called a *generalized (weak) supplement* of N in M . Following [6], M is called *generalized supplemented* module or briefly a GS-module if every submodule N of M has a generalized supplemented K in M. In [6], an $R-$module M is called *generalized weakly supplemented* or briefly a GWS-module (WGS-module in [6]) if every submodule K of M has a generalized weak supplement N in M. M is called a *generalized amply supplemented* module or briefly a GAS-module in case $M = N + K$ implies that N has a generalized supplement $N^{'} \leq K$. For characterizations of generalized (amply) supplemented and generalized weakly supplemented modules we refer to [6] and [8]. M is called *cofinitely generalized supplemented* if every cofinite submodule of M has a generalized supplement [4].

In this paper, we introduce generalized amply weak supplemented modules and cofinitely generalized (amply) weak supplemented modules. We obtain some properties of these modules and have the following implications of these modules:

$$\begin{gathered} Generalized\ Amply\ Supplemented \\ \Downarrow \\ Generalized\ Amply\ Weak\ Supplemented \\ \Downarrow \\ Generalized\ Weak\ Supplemented \\ \Downarrow \\ Cofinitely\ Generalized\ Weak\ Supplemented \end{gathered}$$

2. Cofinitely Generalized Weak Supplemented Modules

Definition 1. A module M is called a cofinitely generalized weak supplemented or briefly a CGWS-module if every cofinite submodule of M has a generalized weak supplement.

To prove that an arbitrary sum of CGWS-modules is a CGWS-module, we use the following standard lemma.

Lemma 2. *Let M be a module, N and U be submodules of M with cofinitely generalized weak suplemented N and cofinite U. If $N+U$ has a generalized weak supplement in M, then U also has a generalized weak supplement in M.*

Proof. Let X be a generalized weak supplement of $N+U$ in M. Then we have

$$\frac{N}{[N\cap(X+U)]}\cong\frac{N+(X+U)}{X+U}=\frac{M}{X+U}\cong\frac{\left(\frac{M}{U}\right)}{\left(\frac{(X+U)}{U}\right)}.$$

Since U is a cofinite submodule, $\frac{M}{U}$ is a finitely generated module. The last module in the right hand side of the preceding equation is a finitely generated module hence $N\cap(X+U)$ has a generalized weak supplement Y in N, i.e.

$$\begin{aligned} Y+[N\cap(X+U)] &= N \\ Y\cap[N\cap(X+U)] &= Y\cap(X+U)\leq Rad(N)\leq Rad(M). \end{aligned}$$

Since

$$M=U+X+N=U+X+Y+[N\cap(X+U)]=X+U+Y,$$

Y is a generalized weak supplement of $X+U$ in M. Therefore

$$U \cap (X+Y) \leq [X \cap (Y+U)] + [Y \cap (X+U)] \leq Rad(M).$$

This means that $X+Y$ is a generalized weak supplement of U in M. □

Proposition 3. *Any arbitrary sum of CGWS-modules is a CGWS-module.*

Proof. Let $M = \sum_{i \in I} M_i$ where each module M_i is a cofinitely generalized weak supplemented and N be a cofinite submodule of M. Then $\frac{M}{N}$ is generated by some finite set $\{x_1+N, x_2+N, ..., x_n+N\}$ and therefore $M = Rx_1 + Rx_2 + ... + Rx_n + N$. Since each x_i is contained in the sum $\sum_{j \in J} M_j$ for some finite subset $J = \left\{1_1, ..., 1_{s(1)}, .., n_{s(n)}\right\}$ of I, $M = M_{1_1} + \sum_{j \in J - \{1_1\}} M_j + N$ has a trivial generalized weak supplement 0 in M and since M_{1_1} is a CGWS-module, $N + \sum_{j \in J} M_j$ has a generalized weak supplement by Lemma 2. Continuing in this way we will obtain (after we have used Lemma 2 $\sum_{i=1}^{n} s(i)$ times) at last N has a generalized weak supplement in M. □

3. Generalized Amply Weak Supplemented Modules

In this section, we define the concept of generalized amply weak supplemented modules, which is adapted from amply weak supplemented modules, and we give the properties of these modules.

Definition 4. Let M be a module and $U \leq M$. If every $V \leq M$ with $U+V=M$ there exists a generalized weak supplement V' of U with $V' \leq V$, then we call U has generalized ample weak supplements in M.

Definition 5. Let M be a module. If every submodule of M has a generalized ample weak supplements in M, then M is called a generalized amply weak supplemented module or briefly GAWS-module.

Proposition 6. *Any factor module of a GAWS-module is a GAWS-module.*

Proof. Let M be a GAWS-module, N be any submodule of M and $\frac{K}{N}$ be any submodule of $\frac{M}{N}$. For $\frac{V}{N} \leq \frac{M}{N}$, let $\frac{K}{N} + \frac{V}{N} = \frac{M}{N}$. Then $K+V=M$ and there exists a generalized weak supplement V' of K with $V' \leq V$ since M is a GAWS-module. Therefore $\frac{M}{N} = \frac{K}{N} + \frac{\left(V'+N\right)}{N}$.

Now, let $f : M \to \frac{M}{N}$ be a canonical epimorphism. Since $K \cap V' \leq Rad(M)$, we obtain

$$\begin{aligned} \frac{K}{N} \cap \frac{(V'+N)}{N} &= \frac{K \cap (V'+N)}{N} = \frac{((K \cap V') + N)}{N} \\ &= f\left(K \cap V'\right) \leq f(Rad(M)) \leq Rad(\frac{M}{N}). \end{aligned}$$

Also $\frac{(V'+N)}{N} \leq \frac{V}{N}$ implies that $\frac{K}{N}$ has a generalized weak supplement in $\frac{M}{N}$. Therefore $\frac{M}{N}$ is a GAWS-module. $\square$

Corollary 7. *Any homomorphic image of a GAWS-module is a GAWS-module.*

Lemma 8. *Every supplement submodule of a GAWS-module is generalized amply weak supplemented.*

Proof. Let M be a GAWS-module and V be any supplement submodule of M. Suppose that V is a supplement of U in M. Let $K \leq V$ and $K + T = V$ for $T \leq V$. Then $U + K + T = M$. Since M is generalized amply weak supplemented, $U + K$ has a generalized weak supplement T' in M with $T' \leq T$. In this case

$$U + K + T' = M \text{ and } (U + K) \cap T' \leq Rad(M).$$

Since $K + T' \leq K + T = V$ and V is a supplement of U in M, one can see that $K + T' = V$. Therefore $K \cap T' \leq Rad(M) \cap V = Rad(V)$. (See [7], 41.1) for $Rad(M) \cap V = Rad(V)$ and V is generalized amply weak supplemented. $\square$

Corollary 9. *Every direct summand of a GAWS-module is generalized amply weak supplemented.*

Theorem 10. *Let M be a module and $M = U_1 + U_2$. If U_1 and U_2 have generalized ample weak supplements in M, then $U_1 \cap U_2$ has also generalized ample weak supplements in M.*

Proof. Let $V \leq M$ and $M = (U_1 \cap U_2) + V$. Then we have

$$\begin{aligned} U_1 &= (U_1 \cap U_2) + (V \cap U_1) \\ U_2 &= (U_1 \cap U_2) + (V \cap U_2) \end{aligned}$$

and so

$$M = U_1 + V \cap U_2 \ \text{ and } \ M = U_2 + V \cap U_1.$$

Since U_1 and U_2 have generalized ample weak supplements in M, there exist $V_2^{'} \leq V \cap U_2$ and $V_1^{'} \leq V \cap U_1$ such that

$$M = U_1 + V_2^{'} \ \text{ and } \ U_1 \cap V_2^{'} \leq Rad(M),$$

and

$$M = U_2 + V_1^{'} \ \text{ and } \ U_2 \cap V_1^{'} \leq Rad(M).$$

Therefore $V_1^{'} + V_2^{'} \leq V$ and $U_1 = (U_1 \cap U_2) + V_1^{'}$ and $U_2 = (U_1 \cap U_2) + V_2^{'}$. As a result, we get

$$M = (U_1 \cap U_2) + (V_1^{'} + V_2^{'})$$

and

$$\begin{aligned} (U_1 \cap U_2) \cap (V_1^{'} + V_2^{'}) &\leq (U_1 \cap U_2 + V_1^{'}) \cap V_2^{'} + (U_1 \cap U_2 + V_2^{'}) \cap V_1^{'} \\ &= (U_1 \cap V_2^{'}) + (U_2 \cap V_1^{'}) \leq Rad(M) \end{aligned}$$

which completes the proof. □

A module M is said to be $\pi-$projective if for any two submodules U and V of M with $M = U + V$ there exists $f \in End(M)$ with $f(M) \leq U$ and $(1 - f)(M) \leq V$.

Proposition 11. *Let M be a module. If M is a $\pi-$projective GWS-module, then M is a GAWS-module.*

Proof. Let U and V be two submodules of M, such that $M = U + V$. Since M is $\pi-$projective, there exists an endomorphism f of M such that $f(M) \leq U$ and $(1 - f)(M) \leq V$. Let X be a generalized weak supplement of U in M. Then we have

$$M = f(M) + (1 - f)(M) = f(M) + (1 - f)(U + X) \leq U + (1 - f)(X) \leq M$$

and so $M = U + (1 - f)(X)$. It is easy to see that $(1 - f)(X) \leq V$. Let $u \in U \cap (1 - f)(X)$. Then $u \in U$ and $u = (1 - f)(x) = x - f(x)$, for some $x \in X$. Being $x = u + f(x) \in U$ implies that $u \in (1 - f)(U \cap X)$. However $U \cap X \leq Rad(M)$ gives that

$$U \cap (1 - f)(X) = (1 - f)(U \cap X) \leq Rad((1 - f)(M)).$$

Thus $(1 - f)(X)$ is a generalized weak supplement of U in M and M is a GAWS-module. □

Corollary 12. *Every projective and GWS-module is generalized amply weak supplemented.*

Corollary 13. *Let $M_1, M_2, ...M_n$ be projective modules. Then $\oplus_{i=1}^{n} M_i$ generalized amply weak supplemented if and only if M_i is a generalized amply weak supplemented for every $1 \leq i \leq n$.*

Proof. The necessity is obvious by Corollary 9. Since for every $1 \leq i \leq n$, M_i is a generalized amply weak supplemented, M_i is a generalized weak supplemented. Therefore $\oplus_{i=1}^{n} M_i$ is also generalized weak supplemented by ([6], Proposition 3.7). Since for every $1 \leq i \leq n$, M_i is projective, $\oplus_{i=1}^{n} M_i$ is also projective. Then by Corollary 12, $\oplus_{i=1}^{n} M_i$ generalized amply weak supplemented. □

Theorem 14. *Let M be a module. Then M is Artinian if and only if M is a GAWS-module and satisfies DCC on generalized weak supplement submodules and on small submodules.*

Proof. The necessity is clear. Conversely, suppose that M is a GAWS-module which satisfies DCC on generalized weak supplement submodules and on small submodules. Then $Rad(M)$ is Artinian by ([3], Theorem 5). Next it suffices to show that $\frac{M}{Rad(M)}$ is Artinian. Let N be any submodule of M containing $Rad(M)$. Then there exists a generalized weak supplement K of N in M, i.e. $M = N + K$ and $N \cap K \leq Rad(M)$. Therefore, we have

$$\frac{M}{Rad(M)} = \frac{N}{Rad(M)} \oplus \frac{(K + Rad(M)}{Rad(M)}$$

and that every submodule of $\frac{M}{Rad(M)}$ is a direct summand. It means $\frac{M}{Rad(M)}$ is semisimple.

Now suppose that $Rad(M) \leq N_1 \leq N_2 \leq .....$ is an ascending chain of submodules of M. Because M is a GAWS-module, there exists a descending chain of submodules $K_1 \geq K_2 \geq ....$ such that K_i is a generalized weak supplement of N_i in M for each $i \geq 1$. By hypothesis, there exists a positive integer t such that $K_t = K_{t+1} = K_{t+2} = ...$ Because of

$$\frac{M}{Rad(M)} = \frac{N_i}{Rad(M)} \oplus \frac{(K_i + Rad(M))}{Rad(M)}$$

for all $i \geq t$, it follows that $N_t = N_{t+1} = N_{t+2} = ...$ Thus $\frac{M}{Rad(M)}$ is Noetherian and finitely generated. This means $\frac{M}{Rad(M)}$ is Artinian by ([7], 31.3). □

4. Cofinitely Generalized Amply Weak Supplemented Modules

In this section we define and study cofinitely generalized amply weak supplemented modules.

Definition 15. An $R-$module M is called cofinitely generalized amply weak supplemented, or briefly CGAWS-module if every cofinite submodule of M has a generalized ample weak supplement in M.

Proposition 16. *Let M be a CGAWS-module. Then*
(1) Every supplement submodule of M is a CGAWS-module.
(2) Every factor module of M is a CGAWS-module.

Proof. (1) Let V be a supplement of U in M and K is a cofinite submodule of V. Then we have

$$\frac{V}{(U \cap V) + K} \cong \frac{U+V}{U+K} = \frac{M}{U+K} \cong \frac{\left(\frac{M}{K}\right)}{\left(\frac{(U+K)}{K}\right)}.$$

Since the last module in the right hand-side of the preceding equation is a finitely generated module, we get that $(U \cap V) + K$ is also a cofinite submodule of V. Let $K + T = V$ for any $T \leq V$. Then $U + K + T = M$. Since $U + K$ is a cofinite submodule of M and M is a CGAWS-module, $U + K$ has a generalized weak supplement $T^{'}$ in M with $T^{'} \leq T$, i.e.

$$\begin{aligned} U + K + T^{'} &= M, \\ (U + K) \cap T^{'} &\leq Rad(M). \end{aligned}$$

Also we have $K + T^{'} = V$ because of $K + T^{'} \leq V$ and V is a supplement of U in M. Thus $K \cap T^{'} \leq Rad(M) \cap V = Rad(V)$. Hence V is a CGAWS-module.

(2) Let N be a submodule of M and $\frac{K}{N}$ is a cofinite submodule of $\frac{M}{N}$. Note that $\frac{\left(\frac{M}{N}\right)}{\left(\frac{K}{N}\right)} \cong \frac{M}{K}$. Hence K is a cofinite submodule of M. For $\frac{V}{N} \leq \frac{M}{N}$, let $\frac{K}{N} + \frac{V}{N} = \frac{M}{N}$. This implies that $K + V = M$. Since M is a CGAWS-module, there exists a generalized weak supplement $V^{'}$of K with $V^{'} \leq V$. Therefore $\frac{M}{N} = \frac{K}{N} + \frac{\left(V^{'}+N\right)}{N}$. Let $f : M \to \frac{M}{N}$ be a canonical epimorphism. Since $K \cap V^{'} \leq Rad(M)$, we have

$$\frac{K}{N} \cap \frac{(V^{'} + N)}{N} = \frac{K \cap (V^{'} + N)}{N} = \frac{((K \cap V^{'}) + N)}{N}$$

$$= \quad f\left(K \cap V'\right) \leq f(Rad(M)) \leq Rad(\frac{M}{N}).$$

If one uses $\frac{(V'+N)}{N} \leq \frac{V}{N}$, then he can see that $\frac{K}{N}$ has a generalized ample weak supplement in $\frac{M}{N}$ which completes the proof. □

Corollary 17. *(1) Every direct summand of a CGAWS-module is cofinitely generalized amply weak supplemented.*

(2)Every homomorphic image of a CGAWS-module is a CGAWS-module.

Proposition 18. *Let M be a cofinitely generalized weak supplemented and $\pi-$projective module. Then M is cofinitely generalized amply weak supplemented.*

Proof. Let U be a cofinite submodule of M and $M = U + V$ for $V \leq M$. Suppose that X be a generalized weak supplement of U in M. Since M is $\pi-$projective, there exists a homomorphism $f : M \to M$ such that $f(M) \leq V$ and $(1 - f)(M) \leq U$. In this case, we get

$$M = f(M) + (1 - f)(M) \leq f(U + X) + U \leq U + f(X) \leq M$$

and so $M = U + f(X)$. Let $u \in U \cap f(X)$. Then there exists $x \in X$ with $u = f(x)$. If we write $x - u = x - f(x) = (1 - f)(x)$, then we get $x - u \in U$ and $x \in U$. Hence $x \in U \cap X$ and

$$U \cap f(X) \leq f(U \cap X) \leq f(Rad(M)) = Rad(f(M)) \leq Rad(M).$$

As a result, $f(X)$ is a generalized weak supplement of U in M and so M is a CGAWS-module. □

Corollary 19. *Every projective and CGWS-module is cofinitely generalized amply weak supplemented.*

Corollary 20. *Let $(M_i)_{i\in I}$ be a family of projective modules. Then $\oplus_{i\in I} M_i$ is cofinitely generalized amply weak supplemented if and only if for every $i \in I$, M_i is cofinitely generalized amply weak supplemented.*

Proof. The necessity is obvious by Corollary 17(1). Since for every $i \in I$, M_i is cofinitely generalized amply weak supplemented, M_i is cofinitely generalized weak supplemented for all $i \in I$. Then by Proposition 3, $\oplus_{i\in I} M_i$ is also cofinitely generalized weak supplemented. Since for every $i \in I$, M_i is projective, $\oplus_{i\in I} M_i$ is also projective. Therefore $\oplus_{i\in I} M_i$ is cofinitely generalized amply weak supplemented by Corollary 19. □

Corollary 21. *Let R be a ring. Then the following statements are equivalent:*
(i) R is semilocal.
(ii) Every left $R-$module is generalized weakly supplemented.
(iii) Every left $R-$module is generalized amply weak supplemented.
(iv) Every left $R-$module is cofinitely generalized weak supplemented.
(v) Every left $R-$module is cofinitely generalized amply weak supplemented.

Proof. The implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ can be seen in [5]. The others, $(ii) \Leftrightarrow (iii)$, $(ii) \Longleftrightarrow (iv)$ and $(ii) \Leftrightarrow (v)$ are obvious. □

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