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# TUPLE OF OPERATORS AND HYPERCYCLICITY CRITERION

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**Abstract:** In this paper we characterize the equivalent conditions for a tuple of commutative bounded linear operators, satisfying the hypercyclicity criterion.

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## 1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space $X$.

**Definition 1.1.** Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space $X$. We will let $\mathcal{F} = \{{T_1}^{k_1}{T_2}^{k_2}, ..., {T_n}^{k_n} : k_i \in \mathbb{Z}_+, \;\; i = 1, ..., n\}$ be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of $x$ under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}$. A vector $x$ is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in $X$ and in this case the tuple $\mathcal{T}$ is called hypercyclic. Also, by $\mathcal{T}_d^{(k)}$ we will refer to the set of all $k$ copies of an element of $\mathcal{F}$, i.e. $\mathcal{T}_d^{(k)} = \{S_1 \oplus ... \oplus S_k : S_1 = ... = S_k \in \mathcal{F}\}$. We say that $\mathcal{T}_d^{(k)}$ is hypercyclic provided there exist $x_1, ..., x_k \in X$ such that $\{W(x_1 \oplus ... \oplus x_k) : W \in \mathcal{T}_d^{(k)}\}$ is dense in the $k$ copies of $X$, $X \oplus ... \oplus X$.

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For simplicity we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if $T_1$, $T_2$ are commutative bounded linear operators on a Banach space $X$, and $\{m_j\}$, $\{n_j\}$ are two sequences of natural numbers, then we say $\{{T_1}^{m_j}{T_2}^{n_j} : j \geq 0\}$ is hypercyclic if there exists $x \in X$ such that $\{{T_1}^{m_j}{T_2}^{n_j}x : j \geq 0\}$ is dense in $X$.

**Definition 1.2.** A pair $(T_1, T_2)$ is called topologically mixing if for any given open sets $U$ and $V$, there exist two positive integers $M$ and $N$ such that $T_1^m T_2^n(U) \cap V \neq \emptyset$ for all $m \geq M$ and $n \geq N$.

**Definition 1.3.** We say that a pair $\mathcal{T} = (T_1, T_2)$ is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences $(\{m_k\}, \{n_k\})$ of integers provided for all pair of subsequences $(\{m_{k_j}\}, \{n_{k_j}\})$ of $(\{m_k\}, \{n_k\})$, the sequence $\{{T_1}^{m_{k_j}}{T_2}^{n_{k_j}} : j \geq 1\}$ is hypercyclic. We say that a pair $\mathcal{T}$ is hereditarily hypercyclic, if it is hereditarily hypercyclic with respect to a pair of nonnegative increasing sequences.

**Definition 1.4.** An strictly increasing sequence of positive integers $\{n_k\}$ is said to be syndetic if $\sup_n\{n_{k+1} - n_k\} < \infty$.

The formulation of the hypercyclicty criterion in the next section was given by N. S. Feldman (see [4]). Here, we want to extend some properties of hypercyclic operators to a pair of commuting operators, and although the techniques work for any n-tuple of operators but for simplicity we prove our results only for the case $n = 2$. For some other topics we refer to [1–20].

## 2. Main Results

In this section we characterize the equivalent conditions for a pair of operators, satisfying the hypercyclicity criterion.

**Theorem 2.1.** (The Hypercyclicity Criterion for a Tuple) *Suppose $X$ is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2)$ is a pair of continuous linear mappings on $X$. If there exist two dense subsets $Y$ and $Z$ in $X$, and a pair of strictly increasing sequences $\{m_j\}$ and $\{n_j\}$ such that:*

*1. $T_1^{m_j} T_2^{n_j} \to 0$ on $Y$ as $j \to \infty$,*

*2. There exists a sequence of function $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_j} T_2^{n_j} S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.*

**Theorem 2.2.** (see [18]) *Let $\mathcal{T} = (T_1, T_2)$ be a pair of operators acting on a separable infinite dimensional Banach space $X$. Then the followings are equivalent:*

*(i)* $\mathcal{T}$ *satisfies the hypercyclicity criterion.*

*(ii)* $\mathcal{T}$ *is hereditarily hypercyclic.*

*(iii)* $\mathcal{T}_d^{(2)}$ *is hypercyclic.*

**Theorem 2.3.** (see [19]) *Let* $\mathcal{T}$ *be a pair of operators* $T_1$ *and* $T_2$ *on the an infinite dimensional Banach space* $X$*. Also, let* $T_i^{\square}$ *has no eigenvalues for* $i = 1, 2$*. Then the followings are equivalent:*

*(i)* $\mathcal{T}_d^{(2)}$ *is hypercyclic.*

*(ii) for every nonempty open subsets* $U$, $V$ *of* $X$*, there exists a pair of integers* $(m, n)$ *such that* $T_1^m T_2^n(U) \cap V \neq \emptyset$ *and* $T_1^{m+1} T_2^{n+1}(U) \cap V \neq \emptyset$*.*

*(iii) there exists a positive integer* $p$ *such that for any nonempty open subsets* $U$, $V$ *of* $X$*, there exists a pair of integers* $(m, n)$ *such that* $T_1^m T_2^n(U) \cap V \neq \emptyset$ *and* $T_1^{m+p} T_2^{n+p}(U) \cap V \neq \emptyset$*.*

**Lemma 2.4.** (see [20]) *Let* $X$ *be a separable infinite dimensional Banach space and* $\mathcal{T} = (T_1, T_2)$ *be the pair of operators* $T_1$ *and* $T_2$*. The followings are equivalent:*

*(i) for every pair of nonnegative sequence of integers* $(\{m_k\}, \{n_k\})$ *with* $m_{k+1} - m_k \leq 2$ *and* $n_{k+1} - n_k \leq 2$ *for every* $k$*, the sequence* $\{{T_1}^{m_k} {T_2}^{n_k} : k \geq 0\}$ *is hypercyclic.*

*(ii) for every nonempty open subsets* $U$, $V$ *of* $X$*, there exists a pair of integers* $(m, n)$ *such that* $T_1^m T_2^n(U) \cap V \neq \emptyset$ *and* $T_1^{m+1} T_2^{n+1}(U) \cap V \neq \emptyset$*.*

**Proposition 2.5.** (see [20]) *Let* $\mathcal{T}$ *be a pair of operators* $T_1$ *and* $T_2$ *on the an infinite dimensional Banach space* $X$*. If* $\mathcal{T}$ *satisfies the hypercyclicity criterion, then* $\mathcal{T}_d^{(MN)}$ *is hypercyclic for all* $M, N \geq 2$*. The converse is also true.*

**Theorem 2.6.** *Let* $\mathcal{T}$ *be a pair of operators* $T_1$ *and* $T_2$ *on the an infinite dimensional Banach space* $X$*. If* $\mathcal{T}$ *satisfies the hypercyclicity criterion for syndetic sequences* $\{m_k\}_k$ *and* $\{n_k\}_k$*, then* $\mathcal{T}$ *is topologically mixing.*

*Proof.* Since the sequences $\{m_k\}_k$ and $\{n_k\}_k$ in the hypercyclicity criterion are syndetic, there are some positive integers $m$ and $n$ such that for all $k \geq 0$ we have $m_{k+1} - m_k \leq m$ and $n_{k+1} - n_k \leq n$. Let $U$ and $V$ be any open sets in $X$. For $i = 0, 1, 2, ..., m$ and $j = 0, 1, 2, ..., n$ consider open sets $V_{i,j}$ such that $T_1^i T_2^j(V_{i,j}) = V$.

Let $Y$ and $Z$ be the dense sets that are given in the hypercyclicity criterion. Choose $x \in U \cap Y$, and take $\epsilon > 0$ such that $B(x, \epsilon) \subset U$. Also, for each

$i = 0, 1, ..., m$ and $j = 0, 1, ..., n$ take $z_{i,j} \in V_{i,j} \cap Z$, and note that we may assume that $\epsilon$ is small enough such that $B(z_{i,j}, 2\epsilon) \subset V_{i,j}$. Let $k_0$ be large enough such that for all $k \geq k_0$, $i = 0, 1, 2, ..., m$ and $j = 0, 1, 2, ..., n$: $\|T_1^{m_k} T_2^{n_k}(x)\| \leq \epsilon$, $\|S_k(z_{i,j})\| < \epsilon$, and $\|T_1^{m_k} T_2^{n_k} S_k(z_{i,j}) - z_{i,j}\| < \epsilon$. Set $M = m_{k_0}$ and $N = n_{k_0}$. Let $p \geq M$ and $q \geq N$. Then there are some $m_k$ and $n_k$ with $k \geq k_0$ and $0 \leq r \leq m$ and $0 \leq s \leq n$ such that $p = m_k + r$ and $q = n_k + s$. Define $x_{p,q} = x + S_k(z_{r,s})$. Then $T_1^p T_2^q(x_{p,q}) = T_1^r T_2^s(T_1^{m_k} T_2^{n_k}(x + S_k(z_{r,s}))$. Note that $T_1^{m_k} T_2^{n_k}(x + S_k(z_{r,s})) \in B(z_{r,s}, 2\epsilon) \subset V_{r,s}$ and so $T_1^p T_2^q(x_{p,q}) \in T_1^r T_2^s(V_{r,s}) = V$. This completes the proof. □

**Theorem 2.7.** *Let $\mathcal{T}$ be a pair of operators $T_1$ and $T_2$ on an infinite dimensional Banach space $X$. Also, let $T_i^{\square}$ has no eigenvalues for $i = 1, 2$. Then the followings are equivalent:*

*(i) $\mathcal{T}$ satisfies the hypercyclicity criterion.*

*(ii) for any integers $M, N \geq 2$ and every sequences $\{m_k\}$ and $\{n_k\}$ such that $m_{k+1} - m_k \leq M$ and $n_{k+1} - n_k \leq N$ for every $k$, $\{T_1^{m_k} T_2^{n_k} : k \geq 0\}$ is hypercyclic.*

*(iii) for every sequences $\{m_k\}$ and $\{n_k\}$ such that $m_{k+1} - m_k \leq 2$ and $n_{k+1} - n_k \leq 2$ for every $k$, $\{T_1^{m_k} T_2^{n_k} : k \geq 0\}$ is hypercyclic.*

*Proof.* (i) implies (ii): Let $\{m_k\}$ and $\{n_k\}$ be sequences such that for every $k$, $m_{k+1} - m_k \leq M$ and $n_{k+1} - n_k \leq N$ with $M, N \geq 2$. Since $\mathcal{T}$ satisfies the hypercyclicity criterion, there exist sequences $\{p_k\}$ and $\{q_k\}$ such that $\mathcal{T}$ is hereditarily hypercyclic with respect to $(\{p_k\}, \{q_k\})$. Let $U$ and $V$ be nonempty open subsets of $X$. Put $U_{ij} = T_1^{-(i-1)} T_2^{-(j-1}(U)$ and $V_{ij} = V$ for $i = 1, ..., M$ and $j = 1, ..., N$. By Proposition 2.6, $\mathcal{T}_d^{(MN)}$ is hypercyclic, thus there exist $k_1, k_2 \geq MN$ such that for $i = 1, ..., M$ and $j = 1, ..., N$ we have $T_1^{k_1} T_2^{k_2}(U_{ij}) \cap V \neq \emptyset$, which means that $T_1^{k_1 - i + 1} T_2^{k_2 - j + 1}(U) \cap V \neq \emptyset$. Set $A_1 = \{k_1 - (M-1),\ k_1 - (M-2),\ ..., k_1 - 1,\ k_1\}$ and $A_2 = \{k_2 - (N-1),\ k_2 - (N-2),\ ..., k_2 - 1,\ k_2\}$. Then $A_1$ and $A_2$ contain respectively $M$ and $N$, so at least one of the elements of $A_1$ is equal to an element $m_{k_0}$ of the sequence $\{m_k\}$, and one of the elements of $A_2$ is equal to an element $n_{k_0}$ of the sequence $\{n_k\}$, and $T_1^{m_{k_0}} T_2^{n_{k_0}}(U) \cap V \neq \emptyset$. This implies that the sequence $\{T_1^{m_k} T_2^{n_k} : k \geq 0\}$ is hypercyclic and so (ii) holds. The assertion (iii) follows from assertion (ii) by taking $M = N = 2$. Finally, by Lemma 2.4 with Theorems 2.2 and 2.3, the assertion (iii) implies (i) and so the proof is complete. □

**References**

[1] F. Bayart, E. Matheron, *Dynamics of Linear Operators*, Cambridge University Press (2009).

[2] J. Bes, A. Peris, Hereditarily hypercyclic operators, *J. Func. Anal.*, **167** (1) (1999), 94-112.

[3] N.S. Feldman, Hypercyclic pairs of coanalytic Toeplitz operators, *Integral Equations Operator Theory*, **58**, No. 20 (2007), 153-173.

[4] N.S. Feldman, Hypercyclic tuples of operators and somewhere dense orbits, *J. Math. Appl.*, **346** (2008), 82-98.

[5] G. Godefroy, J.H. Shapiro, Operators with dense invariant cyclic manifolds, *J. Func. Anal.*, **98** (1991), 229-269.

[6] S. Grivaux, Hypercyclic operators, mixing operators and the bounded steps problem, *J. Operator Theory*, **54** (2005), 147-168.

[7] S. Yarmahmoodi, K. Hedayatian, B. Yousefi, Supercyclicity and hypercyclicity of an isometry plus a nilpotent, *Abstract and Applied Analysis*, **2011**, doi 10.1155/2011/686832 (2011).

[8] B. Yousefi, H. Rezaei, Hypercyclicity on the algebra of Hilbert-Schmidt operators, *Results in Mathematics*, **46** (2004), 174-180.

[9] B. Yousefi, H. Rezaei, Some necessary and sufficient conditions for hypercyclicity criterion, *Proc. Indian Acad. Sci. (Math. Sci.)*, **115**, No. 2 (2005), 209-216.

[10] B. Yousefi, A. Farrokhinia, On the hereditarily hypercyclic vectors, *Journal of the Korean Mathematical Society*, **43**, No. 6 (2006), 1219-1229.

[11] B. Yousefi, H. Rezaei, J. Doroodgar, Supercyclicity in the operator algebra using Hilbert-Schmidt operators, *Rendiconti Del Circolo Matematico di Palermo*, Serie II, **LVI** (2007), 33-42.

[12] B. Yousefi, H. Rezaei, Hypercyclic property of weighted composition operators, *Proc. Amer. Math. Soc.*, **135**, No. 10 (2007), 3263-3271.

[13] B. Yousefi, S. Haghkhah, Hypercyclicity of special operators on Hilbert function spaces, *Czech. Math. Journal*, **57**, No. 132 (2007), 1035-1041.

[14] B. Yousefi, H. Rezaei, On the supercyclicity and hypercyclicity of the operator algebra, *Acta Mathematica Sinica*, **24**, No. 7 (2008), 1221-1232.

[15] B. Yousefi, R. Soltani, Hypercyclicity of the adjoint of weighted composition operators, *Proc. Indian Acad. Sci. (Math. Sci.)*, **119**, No. 3 (2009), 513-519.

[16] B. Yousefi, J. Izadi, Weighted composition operators and supercyclicity criterion, *International Journal of Mathematics and Mathematical Sciences*, **2011**, doi 10.1155/2011/514370 (2011).

[17] B. Yousefi, Hereditarily transitive tuples, *Rend. Circ. Mat. Palermo*, **2011**, doi 10.1007/S12215-011-0066-y (2011).

[18] B. Yousefi, On the hypercyclicity criterion for a tuple of operators, *International Journal of Applied Mathematics*, To Appear.

[19] B. Yousefi, Tuple of operators and hypercyclicity, *International Journal of Pure and Applied Mathematics*, To Appear.

[20] B. Yousefi, On the hypercyclicity for a tuple of operators , *International Journal of Pure and Applied Mathematics*, To Appear.