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FIXED AND PERIODIC POINTS FOR MAPPINGS SATISFYING CONTRACTIVE CONDITIONS OF INTEGRAL TYPE

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Abstract: Several fixed and periodic point theorems for mappings satisfying a few contractive conditions of integral type are given and three examples are constructed to illustrate the results presented in this paper.

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1. Introduction and Preliminaries

The famous Banach contraction principle is as follows.

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Theorem 1.1. (see [1]) *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \le cd(x, y), \quad \forall x, y \in X,$$

where $c \in (0, 1)$ is a constant. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

It is well know the Banach contraction principle is a nice and important result in fixed point theory, which has a lot of applications and generalizations in many different directions, see, for example [2-5] and the references cited therein. In 2008, Liu et. al [3] got some nonunique fixed point theorems for four classes of self mappings T in a complete metric space (X, d) as follows, for all $x, y, z \in X$ with $x \ne y \ne z \ne x$,

$$d(Tx, Ty) + d(Ty, Tz) \le \phi(d(x, y) + d(y, z));$$
$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \le \phi(d(x, y) + d(y, z) + d(z, x));$$
$$\max\{d(Tx, Ty), d(Ty, Tz)\} \le \phi(\max\{(d(x, y), d(y, z)\});$$

and

$$\begin{aligned}&\max\{d(Tx, Ty), d(Ty, Tz), d(Tz, Tx)\}\\ &\le \phi(\max\{(d(x, y), d(y, z), d(z, x)\}).\end{aligned}$$

The authors [2,6] and others studied the existence of fixed points for some mappings satisfying contractive conditions of integral type. In 2002, Branciari [2] gave an integral version of the Banach contraction principle and proved the following fixed point theorem:

Theorem 1.2. (see [2]) *Let (X, d) be a complete metric space, $c \in (0, 1)$ and $T : X \to X$ be a mapping satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x, y \in X,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue-integrable mapping which is summable (i.e. with finite integral) on each compact subset of $\mathbb{R}^+$, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to+\infty} T^n(x) = a$ for each $x \in X$,

The aim of this paper is to introduce four classes of contractive mappings of integral type below, for all $x, y, z \in X$ with $x \ne y \ne z \ne x$,

$$\begin{aligned}&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt\\ &\le \phi\Big(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt\Big),\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\int_0^{d(Tx,Ty)}\varphi(t)dt+\int_0^{d(Ty,Tz)}\varphi(t)dt+\int_0^{d(Tz,Tx)}\varphi(t)dt\\&\leq\phi\Big(\int_0^{d(x,y)}\varphi(t)dt+\int_0^{d(y,z)}\varphi(t)dt+\int_0^{d(z,x)}\varphi(t)dt\Big),\end{aligned}\tag{1.2}$$

$$\begin{aligned}&\max\Big\{\int_0^{d(Tx,Ty)}\varphi(t)dt,\int_0^{d(Ty,Tz)}\varphi(t)dt\Big\}\\&\leq\phi\Big(\max\Big\{\int_0^{d(x,y)}\varphi(t)dt,\int_0^{d(y,z)}\varphi(t)dt\Big\}\Big)\end{aligned}\tag{1.3}$$

and

$$\begin{aligned}&\max\Big\{\int_0^{d(Tx,Ty)}\varphi(t)dt,\int_0^{d(Ty,Tz)}\varphi(t)dt,\int_0^{d(Tz,Tx)}\varphi(t)dt\Big\}\\&\leq\phi\Big(\max\Big\{\int_0^{d(x,y)}\varphi(t)dt,\int_0^{d(y,z)}\varphi(t)dt,\int_0^{d(z,x)}\varphi(t)dt\Big\}\Big),\end{aligned}\tag{1.4}$$

and to investigate the existence of fixed and periodic points for the mappings (1.1)-(1.4). The results presented in this paper generalize all results in [3]. Three examples which dwell upon the importance of our results are also included.

Throughout this paper, we assume that $\mathbb{R}=(-\infty,+\infty)$, $\mathbb{R}^+=[0,+\infty)$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and $\mathbb{N}$ denotes the set of all positive integers. Let (X,d) be a metric space and $T:X\to X$ be a mapping. A point $x_0\in X$ is called an *n-periodic point* of T if there exists $n\in\mathbb{N}$ such that $x_0=T^nx_0$, but $x_0\neq T^kx_0$ for $k\in\{1,2,\cdots,n-1\}$. For each $x\in X$, the set $O_T(x)=\{T^nx\mid n\in\mathbb{N}_0\}$ is said to be an *orbit* of T at x. Define

$$\Psi=\Big\{\varphi\mid\varphi:\mathbb{R}^+\to\mathbb{R}^+\text{is a Lebesgue-integrable mapping which is summable (i.e, with finite integral) on each compact subset of }\mathbb{R}^+\text{ and }\int_0^{\varepsilon}\varphi(t)dt>0\text{ for each }\varepsilon>0\Big\}$$

and

$$\Phi=\Big\{\phi\mid\phi:\mathbb{R}^+\to\mathbb{R}^+\text{ is nondecreasing and right continuous and }\phi(t)<t\text{ for each }t>0\Big\}.$$

The following lemmas play important roles in this paper.

Lemma 1.3. (see [3]) *Let T be a mapping from a metric space (X,d) into itself. If $x_0\in X$ is a n-periodic point of T, then $T^ix_0\neq T^jx_0$ for all $0\leq i<j\leq n-1$.*

Lemma 1.4. (see [4]) *Let $\varphi \in \Psi$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence. Then* $\lim_{n\to\infty}\int_0^{r_n}\varphi(t)dt = 0$ *if and only if* $\lim_{n\to\infty} r_n = 0$.

2. Main Results

In this section, we prove some fixed and periodic point theorems for the mappings (1.1)-(1.4), respectively.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying* (1.1) *for some $(\phi, \varphi) \in \Phi \times \Psi$. Then:*

(a) *T has at most two distinct fixed points in X*;

(b) *if T has 2-periodic points in X, then they are exactly two*;

(c) *T has no any n-periodic points in X for $n \geq 3$*;

(d) *T has a fixed point in X provided that T has an orbit without 2-periodic points.*

Proof. (a) Suppose that T has at least three different fixed points $a, b, c \in X$. According to (1.1) and $(\phi, \varphi) \in \Phi \times \Psi$, we get that

$$\begin{aligned}\int_0^{d(a,c)}\varphi(t)dt + \int_0^{d(c,b)}\varphi(t)dt &= \int_0^{d(Ta,Tc)}\varphi(t)dt + \int_0^{d(Tc,Tb)}\varphi(t)dt \\ &\leq \phi\bigg(\int_0^{d(a,c)}\varphi(t)dt + \int_0^{d(c,b)}\varphi(t)dt\bigg) \\ &< \int_0^{d(a,c)}\varphi(t)dt + \int_0^{d(c,b)}\varphi(t)dt,\end{aligned}$$

which is a contradiction. Hence T has at most two distinct fixed points in X.

(b) Suppose that T has a 2-periodic point $b \in X$. It is clear that

$$T^2(Tb) = T(T^2b) = Tb \neq b = T^2b,$$

which means that Tb is also a 2-periodic point of T different from b. Now we claim that T has the only two 2-periodic points b and Tb. Otherwise T has a 2-periodic point $c \in X \setminus \{b, Tb\}$. It is easy to verify that $Tb \neq Tc \neq T^2b \neq Tb$.

By (1.1) and $(\phi, \varphi) \in \Phi \times \Psi$ we deduce that

$$\begin{aligned}
&\int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,Tb)} \varphi(t)dt \\
&= \int_0^{d(T^2b,T^2c)} \varphi(t)dt + \int_0^{d(T^2c,T^3b)} \varphi(t)dt \\
&\le \phi\Big(\int_0^{d(Tb,Tc)} \varphi(t)dt + \int_0^{d(Tc,T^2b)} \varphi(t)dt \Big) \\
&= \phi\Big(\int_0^{d(T^3b,T^3c)} \varphi(t)dt + \int_0^{d(T^3c,T^2b)} \varphi(t)dt \Big) \\
&\le \phi^2\Big(\int_0^{d(T^2b,T^2c)} \varphi(t)dt + \int_0^{d(T^2c,Tb)} \varphi(t)dt \Big) \\
&= \phi^2\Big(\int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,Tb)} \varphi(t)dt \Big) \\
&< \int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,Tb)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Hence T has only 2-periodic points b and Tb.

(c) Suppose that T has a n-periodic point $a_0 \in X$ with $n \ge 3$. Put

$$\begin{aligned}
&a_k = T^k a_0 \quad \text{and} \\
&d_k = \int_0^{d(a_k,a_{k+1})} \varphi(t)dt + \int_0^{d(a_{k+1},a_{k+2})} \varphi(t)dt, \quad 0 \le k \le n.
\end{aligned} \tag{2.1}$$

Using (1.1), $a_n = a_0$, Lemma 1.3 and $(\phi, \varphi) \in \Phi \times \Psi$, we conclude that

$$\begin{aligned}
d_n &= \int_0^{d(a_n,a_{n+1})} \varphi(t)dt + \int_0^{d(a_{n+1},a_{n+2})} \varphi(t)dt \\
&= \int_0^{d(a_0,Ta_0)} \varphi(t)dt + \int_0^{d(Ta_0,T^2a_0)} \varphi(t)dt = d_0 > 0
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
d_k &= \int_0^{d(a_k,a_{k+1})} \varphi(t)dt + \int_0^{d(a_{k+1},a_{k+2})} \varphi(t)dt \\
&= \int_0^{d(Ta_{k-1},Ta_k)} \varphi(t)dt + \int_0^{d(Ta_k,Ta_{k+1})} \varphi(t)dt \\
&\le \phi\Big(\int_0^{d(a_{k-1},a_k)} \varphi(t)dt + \int_0^{d(a_k,a_{k+1})} \varphi(t)dt \Big) \\
&= \phi(d_{k-1}) < d_{k-1}, \quad 1 \le k \le n.
\end{aligned} \tag{2.3}$$

In light of (2.2) and (2.3), we derive that

$$d_0 = d_n \le \phi(d_{n-1}) < d_{n-1} < \cdots < d_0,$$

which is a contradiction. That is, T has no any n-periodic points in X for $n \ge 3$.

(d) Suppose that there exists a point $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Set

$$\begin{aligned} x_n &= T^n x_0 \quad \text{and} \\ d_n &= \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt, \quad \forall n \in \mathbb{N}_0. \end{aligned}$$

Now we consider two possible cases as follows:

Case 1. Suppose that there exists some $n \in \mathbb{N}_0$ with $x_n = x_{n+1}$. Clearly x_n is a fixed point of T in X.

Case 2. Suppose that $x_n \ne x_{n+1}$ for all $n \in \mathbb{N}_0$. Since T has no 2-periodic points in $O_T(x_0)$, it follows from (c) that $x_n \ne x_m$ for all $n, m \in \mathbb{N}_0$ with $n \ne m$. In view of (2.1) and $(\phi, \varphi) \in \Phi \times \Psi$, we conclude that

$$\begin{aligned} d_n &= \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \\ &= \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt + \int_0^{d(Tx_n,Tx_{n+1})} \varphi(t)dt \\ &\le \phi\Big(\int_0^{d(x_{n-1},x_n)} \varphi(t)dt + \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \Big) \\ &= \phi(d_{n-1}) \le \phi^2(d_{n-2}) \le \cdots \le \phi^n(d_0) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{2.4}$$

Now we assert that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Otherwise there exist an $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}_0} \subseteq \{x_n\}_{n\in\mathbb{N}_0}$ with $d(x_{n_k}, x_{n_k+1}) > \varepsilon_0$ for any $k \in \mathbb{N}_0$. It follows from (2.4) that

$$0 < \int_0^{\varepsilon_0} \varphi(t)dt \le \int_0^{d(x_{n_k},x_{n_k+1})} \varphi(t)dt \le d_{n_k} \to 0 \quad \text{as } k \to \infty,$$

which is a contradiction. Hence (2.5) holds.

Next we claim that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. If not, then there exists an $\varepsilon > 0$ such that for each $i \in \mathbb{N}_0$, there are $m_i, n_i \in \mathbb{N}_0$ satisfying

$$d(x_{m_i}, x_{n_i}) > \varepsilon \quad \text{and} \quad m_i > n_i > i. \tag{2.6}$$

For each $i \in \mathbb{N}_0$, let m_i denote the least integer exceeding n_i and satisfying (2.6). It follows that

$$d(x_{m_i}, x_{n_i}) > \varepsilon \quad \text{and} \quad d(x_{m_i-1}, x_{n_i}) \le \varepsilon, \quad \forall i \in \mathbb{N}_0. \tag{2.7}$$

Note that

$$\begin{aligned} &d(x_{m_i}, x_{n_i}) \le d(x_{m_i-1}, x_{n_i}) + d(x_{m_i-1}, x_{m_i}), \\ &|d(x_{m_i}, x_{n_i+1}) - d(x_{m_i}, x_{n_i})| \le d(x_{n_i}, x_{n_i+1}), \\ &|d(x_{m_i+1}, x_{n_i+1}) - d(x_{m_i}, x_{n_i+1})| \le d(x_{m_i}, x_{m_i+1}), \quad \forall i \in \mathbb{N}_0. \end{aligned} \tag{2.8}$$

Combining (2.5), (2.7) and (2.8), we deduce that

$$\varepsilon = \lim_{k\to\infty} d(x_{m_i}, x_{n_i}) = \lim_{k\to\infty} d(x_{m_i}, x_{n_i+1}) = \lim_{k\to\infty} d(x_{m_i+1}, x_{n_i+1}). \tag{2.9}$$

Using (1.1), we get that

$$\begin{aligned} &\int_0^{d(x_{m_i+1}, x_{n_i+1})} \varphi(t)dt + \int_0^{d(x_{n_i+1}, x_{n_i})} \varphi(t)dt \\ &\le \phi\bigg(\int_0^{d(x_{m_i}, x_{n_i})} \varphi(t)dt + \int_0^{d(x_{n_i}, x_{n_i-1})} \varphi(t)dt \bigg), \quad \forall i \in \mathbb{N}_0, \end{aligned}$$

letting $i \to \infty$ in the above inequality, we infer that by (2.9), $(\phi, \varphi) \in \Phi \times \Psi$ and Lemma 1.4

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt &= \lim_{i\to\infty} \left(\int_0^{d(x_{m_i+1}, x_{n_i+1})} \varphi(t)dt + \int_0^{d(x_{n_i+1}, x_{n_i})} \varphi(t)dt \right) \\ &\le \lim_{i\to\infty} \phi\bigg(\int_0^{d(x_{m_i}, x_{n_i})} \varphi(t)dt + \int_0^{d(x_{n_i}, x_{n_i-1})} \varphi(t)dt \bigg) \\ &= \phi\bigg(\int_0^{\varepsilon} \varphi(t)dt \bigg) < \int_0^{\varepsilon} \varphi(t)dt, \end{aligned}$$

which is contradiction. Hence $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Thus completeness of (X, d) implies that there exists $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Observe that (1.1), $\phi \in \Phi$ and Lemma 1.4 mean that

$$\begin{aligned} &\int_0^{d(x_{n+1}, Ta)} \varphi(t)dt + \int_0^{d(Ta, Tx_{n+2})} \varphi(t)dt \\ &\le \phi\bigg(\int_0^{d(x_n, a)} \varphi(t)dt + \int_0^{d(a, x_{n+2})} \varphi(t)dt \bigg) \\ &< \int_0^{d(x_n, a)} \varphi(t)dt + \int_0^{d(a, x_{n+2})} \varphi(t)dt \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

which together with Lemma 1.4 yields that $\lim_{n\to\infty} d(x_{n+1}, Ta) = 0$, which gives that $Ta = a$. This completes the proof. □

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying (1.2) for some $(\phi, \varphi) \in \Phi \times \Psi$. Then the conclusions of Theorem 2.1 hold.*

Proof. (a) Assume that T has at least three different fixed points $a, b, c \in X$. In view of (1.2) and $(\phi, \varphi) \in \Phi \times \Psi$, we deduce that

$$\begin{aligned}
&\int_0^{d(a,b)} \varphi(t)dt + \int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,a)} \varphi(t)dt \\
&= \int_0^{d(Ta,Tb)} \varphi(t)dt + \int_0^{d(Tb,Tc)} \varphi(t)dt + \int_0^{d(Tc,Ta)} \varphi(t)dt \\
&\le \phi\Big(\int_0^{d(a,b)} \varphi(t)dt + \int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,a)} \varphi(t)dt \Big) \\
&< \int_0^{d(a,b)} \varphi(t)dt + \int_0^{d(b,c)} \varphi(t)dt + \int_0^{d(c,a)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction.

(b) Assume that T has a 2-periodic point $b \in X$. Obviously, Tb is also a 2-periodic point of T different from b. Now we claim that T has the only two 2-periodic points b and Tb. Otherwise T has a 2-periodic point $c \in X \setminus \{b, Tb\}$. It is easy to show that $Tb \neq Tc \neq T^2b \neq Tb$. It follows from (1.2) that

$$\begin{aligned}
&\int_0^{d(b,Tb)} \varphi(t)dt + \int_0^{d(Tb,c)} \varphi(t)dt + \int_0^{d(c,b)} \varphi(t)dt \\
&= \int_0^{d(T^2b,T^3b)} \varphi(t)dt + \int_0^{d(T^3b,T^2c)} \varphi(t)dt + \int_0^{d(T^2c,T^2b)} \varphi(t)dt \\
&\le \phi\Big(\int_0^{d(Tb,T^2b)} \varphi(t)dt + \int_0^{d(T^2b,Tc)} \varphi(t)dt + \int_0^{d(Tc,Tb)} \varphi(t)dt \Big) \\
&\le \phi^2\Big(\int_0^{d(b,Tb)} \varphi(t)dt + \int_0^{d(Tb,c)} \varphi(t)dt + \int_0^{d(c,b)} \varphi(t)dt \Big) \\
&< \int_0^{d(b,Tb)} \varphi(t)dt + \int_0^{d(Tb,c)} \varphi(t)dt + \int_0^{d(c,b)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Thus T has only 2-periodic points b and Tb.

(c) Suppose that T has an n-periodic point $a_0 \in X$ with $n \ge 3$. Put

$$a_k = T^k a_0, \quad 0 \le k \le n.$$

It follows from Lemma 1.3, (1.2), $a_n = a_0$ and $(\phi, \varphi) \in \Phi \times \Psi$ that

$$\begin{aligned}
&\int_0^{d(a_0,a_1)} \varphi(t)dt + \int_0^{d(a_1,a_2)} \varphi(t)dt + \int_0^{d(a_2,a_0)} \varphi(t)dt \\
&= \phi\Big(\int_0^{d(Ta_{n-1},Ta_n)} \varphi(t)dt + \int_0^{d(Ta_n,Ta_{n+1})} \varphi(t)dt + \int_0^{d(Ta_{n+1},Ta_{n-1})} \varphi(t)dt \Big) \\
&\le \phi\Big(\int_0^{d(a_{n-1},a_n)} \varphi(t)dt + \int_0^{d(a_n,a_{n+1})} \varphi(t)dt + \int_0^{d(a_{n+1},a_{n-1})} \varphi(t)dt \Big) \\
&\le \phi^2\Big(\int_0^{d(a_{n-2},a_{n-1})} \varphi(t)dt + \int_0^{d(a_{n-1},a_n)} \varphi(t)dt + \int_0^{d(a_n,a_{n-2})} \varphi(t)dt \Big) \\
&\le \cdots \\
&\le \phi^n\Big(\int_0^{d(a_0,a_1)} \varphi(t)dt + \int_0^{d(a_1,a_2)} \varphi(t)dt + \int_0^{d(a_2,a_0)} \varphi(t)dt \Big) \\
&< \int_0^{d(a_0,a_1)} \varphi(t)dt + \int_0^{d(a_1,a_2)} \varphi(t)dt + \int_0^{d(a_2,a_0)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction.

(d) Suppose that there exists a point $x_0 \in X$ such that T has no 2-periodic points in $O_T(x_0)$. Set

$$\begin{aligned}
&x_n = T^n x_0 \quad \text{and} \\
&d_n = \int_0^{d(x_n,x_{n+1})} \varphi(t)dt + \int_0^{d(x_{n+1},x_{n+2})} \varphi(t)dt \\
&\qquad + \int_0^{d(x_{n+2},x_n)} \varphi(t)dt, \quad \forall n \in \mathbb{N}_0.
\end{aligned}$$

Now we consider two possible cases as follows:

Case 1. Suppose that there exists some $n \in \mathbb{N}_0$ with $x_n = x_{n+1}$. Clearly x_n is a fixed point of T in X.

Case 2. Suppose that $x_n \neq x_{n+1}$ for all $n \in N_0$. Since T has no 2-periodic points in $O_T(x_0)$, it follows from (c) that $x_n \neq x_m$ for all $n, m \in \mathbb{N}_0$ with $n \neq m$. By virtue of (1.2) and $\phi \in \Phi$, we have

$$\begin{aligned}
0 \le \int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le d_n \le \phi(d_{n-1}) \le \phi^2(d_{n-2}) \le \cdots \le \phi^n(d_0) \\
\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which together with Lemma 1.4 yields that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Next we claim that $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. If not, then there exists an $\varepsilon > 0$ such that for each $i \in \mathbb{N}_0$, there are $m_i, n_i \in \mathbb{N}_0$ satisfying (2.6). Similarly we infer that (2.7)-(2.9) hold. Using (1.2), we get that, $\forall i \in \mathbb{N}_0$,

$$\begin{aligned}
&\int_0^{d(x_{m_i+1},x_{n_i+1})}\varphi(t)dt + \int_0^{d(x_{n_i+1},x_{n_i})}\varphi(t)dt + \int_0^{d(T^{m_i+1}x,T^{m_i}x)}\varphi(t)dt \\
&\le \phi\Big(\int_0^{d(x_{m_i},x_{n_i})}\varphi(t)dt + \int_0^{d(x_{n_i},x_{n_i-1})}\varphi(t)dt + \int_0^{d(T^{m_i}x,T^{m_i-1}x)}\varphi(t)dt\Big),
\end{aligned}$$

letting $i \to \infty$ in the above inequality, we infer that by (2.9) and $(\phi, \varphi) \in \Phi \times \Psi$

$$\begin{aligned}
&\int_0^{\varepsilon}\varphi(t)dt \\
&= \lim_{i\to\infty}\Big(\int_0^{d(x_{m_i+1},x_{n_i+1})}\varphi(t)dt + \int_0^{d(x_{n_i+1},x_{n_i})}\varphi(t)dt + \int_0^{d(T^{m_i+1}x,T^{m_i}x)}\varphi(t)dt\Big) \\
&\le \lim_{i\to\infty}\phi\Big(\int_0^{d(x_{m_i},x_{n_i})}\varphi(t)dt + \int_0^{d(x_{n_i},x_{n_i-1})}\varphi(t)dt + \int_0^{d(T^{m_i}x,T^{m_i-1}x)}\varphi(t)dt\Big) \\
&= \phi\Big(\int_0^{\varepsilon}\varphi(t)dt\Big) < \int_0^{\varepsilon}\varphi(t)dt,
\end{aligned}$$

which is contradiction. Hence $\{x_n\}_{n\in\mathbb{N}_0}$ is a Cauchy sequence. Thus completeness of (X, d) implies that there exists a point $a \in X$ such that $\lim_{n\to\infty} x_n = a$. Using (1.2), $(\phi, \varphi) \in \Phi \times \Psi$ and Lemma 1.4, we conclude that

$$\begin{aligned}
&\int_0^{d(x_{n+1},Ta)}\varphi(t)dt + \int_0^{d(Ta,Tx_{n+1})}\varphi(t)dt + \int_0^{d(Ta,Tx_{n+2})}\varphi(t)dt \\
&\le \phi\Big(\int_0^{d(x_n,a)}\varphi(t)dt + \int_0^{d(a,x_{n+1})}\varphi(t)dt + \int_0^{d(a,x_{n+2})}\varphi(t)dt\Big) \\
&< \int_0^{d(x_n,a)}\varphi(t)dt + \int_0^{d(a,x_{n+1})}\varphi(t)dt + \int_0^{d(a,x_{n+2})}\varphi(t)dt \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

which together with Lemma 1.4 ensures that

$$\lim_{n\to\infty} d(x_{n+1}, Ta) = 0,$$

which gives that $Ta = a$. This completes the proof. □

Similar to the proofs of Theorems 2.1 and 2.2, we have the following results.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying* (1.3) *for some* $(\phi, \varphi) \in \Phi \times \Psi$. *Then the conclusions of Theorem 2.1 hold.*

Theorem 2.4. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying* (1.4) *for some* $(\phi, \varphi) \in \Phi \times \Psi$. *Then the conclusions of Theorem 2.1 hold.*

3. Remarks and Examples

Remark 3.1. If $\varphi(t) = 1$ for all $t \geq 0$, then Theorems 2.1-2.4 reduce to Theorems 1-4 in [3], respectively. The following examples illustrate Theorems 2.1-2.4.

Example 3.1. Let $X = \mathbb{N} \setminus \{3\}$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ by

$$Tx = \begin{cases} 1, & x = 1, \\ 2, & \forall x \in X \setminus \{1\} \end{cases} \quad \text{and} \quad \phi(t) = \frac{2}{7}t, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

It is obvious that $(\phi, \varphi) \in \Phi \times \Psi$.

Let $x, y, z \in X$ with $x \neq y \neq z \neq x$. In order to verify that T satisfies (1.1), we have to consider eight possible cases as follows:

Case 1. Let $x = 1$, $y = 2$ and $z \in X \setminus \{1, 2\}$. It is clear that

$$\begin{aligned}
&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt \\
&= \int_0^{d(1,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt \\
&= 1 < \frac{10}{7} \leq \phi\Big(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt \Big);
\end{aligned}$$

Case 2. Let $x = 1$, $y \in X \setminus \{1, 2\}$ and $z = 2$. Notes that

$$\begin{aligned}
&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt \\
&= \int_0^{d(1,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt \\
&= 1 < \frac{26}{7} \leq \phi\Big(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt \Big);
\end{aligned}$$

Case 3. Let $x = 1$, $y, z \in X \setminus \{1,2\}$. It follows that

$$\begin{aligned}&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt\\ &= \int_0^{d(1,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt\\ &= 1 < \frac{20}{7} \le \phi\bigg(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt\bigg);\end{aligned}$$

Case 4. Let $x = 2$, $y = 1$, $z \in X \setminus \{1,2\}$. Notes that

$$\begin{aligned}&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt\\ &= \int_0^{d(2,1)} \varphi(t)dt + \int_0^{d(1,2)} \varphi(t)dt\\ &= 2 < \frac{20}{7} \le \phi\bigg(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt\bigg);\end{aligned}$$

Case 5. Let $x = 2$ and $y, z \in X \setminus \{1,2\}$. It is easy to get that

$$\begin{aligned}&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt\\ &= \int_0^{d(2,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt\\ &= 0 \le \phi\bigg(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt\bigg);\end{aligned}$$

Case 6. Let $x \in X \setminus \{1,2\}$, $y = 1$ and $z \in X \setminus \{1,2\}$. It is obvious that

$$\begin{aligned}&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt\\ &= \int_0^{d(2,1)} \varphi(t)dt + \int_0^{d(1,2)} \varphi(t)dt\\ &= 2 < \frac{50}{7} \le \phi\bigg(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt\bigg);\end{aligned}$$

Case 7. Let $x \in X \setminus \{1, 2\}$, $y = 2$ and $z \in X \setminus \{1, 2\}$. It follows that

$$\begin{aligned}
&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt \\
&= \int_0^{d(2,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt \\
&= 0 \leq \phi\Big(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt \Big);
\end{aligned}$$

Case 8. Let $x, y, z \in X \setminus \{1, 2\}$. Notes that

$$\begin{aligned}
&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt \\
&= \int_0^{d(2,2)} \varphi(t)dt + \int_0^{d(2,2)} \varphi(t)dt \\
&= 0 \leq \phi\Big(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt \Big),
\end{aligned}$$

that is, (1.1) holds. Similarly, it is easy to verify that (1.4) holds. Thus the conditions of Theorems 2.1 and 2.4 are fulfilled. Clearly T has two fixed points 1 and 2 in X.

Example 3.2. Let $X = \left[\frac{1}{2}, 1\right]$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define $T : X \to X$ by

$$Tx = \frac{1}{1+x}, \quad \forall x \in X \quad \text{and} \quad \phi(t) = \frac{20}{81}t, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

It is clear that $(\phi, \varphi) \in \Phi \times \Psi$.

Let $x, y, z \in X$ with $x \neq y \neq z \neq x$. Obviously,

$$\begin{aligned}
&\int_0^{d(Tx,Ty)} \varphi(t)dt + \int_0^{d(Ty,Tz)} \varphi(t)dt + \int_0^{d(Tz,Tx)} \varphi(t)dt \\
&= \left(\frac{1}{1+x} - \frac{1}{1+y}\right)^2 + \left(\frac{1}{1+y} - \frac{1}{1+z}\right)^2 + \left(\frac{1}{1+z} - \frac{1}{1+x}\right)^2 \\
&= \frac{(x-y)^2}{(1+x)^2(1+y)^2} + \frac{(y-z)^2}{(1+y)^2(1+z)^2} + \frac{(x-z)^2}{(1+z)^2(1+x)^2} \\
&\leq \frac{16}{81}[(x-y)^2 + (y-z)^2 + (z-x)^2] \\
&< \frac{20}{81}[(x-y)^2 + (y-z)^2 + (z-x)^2] \\
&= \phi\left(\int_0^{d(x,y)} \varphi(t)dt + \int_0^{d(y,z)} \varphi(t)dt + \int_0^{d(z,x)} \varphi(t)dt\right),
\end{aligned}$$

that is, (1.2) holds. Thus the conditions of Theorem 2.2 are fulfilled. It is easy to verify that for each $x_0 \in X \setminus \{\frac{\sqrt{5}-1}{2}\}$, the order $O_T(x_0)$ of T at x_0 has no 2-periodic point in X. Thus Theorem 2.2 ensures that T has a fixed point in X. Clearly $\frac{\sqrt{5}-1}{2} \in X$ is a fixed point of T.

Example 3.3. Let (X, d) and (ϕ, φ) be the same as in Example 3.1. Define $T : X \to X$ by

$$Tx = \begin{cases} 2, & x = 1, \\ 1, & \forall x \in X \setminus \{1\}. \end{cases}$$

It is easy to check that the conditions of Theorem 2.3 are satisfied and T has two 2-periodic points 1 and 2 in X.

Remark 3.2. In Theorem 2.3, the presence of 2-periodic points excludes the presence of fixed points and vice verse. Otherwise there exist two points $a, b \in X$ such that $a = Ta$, $b = T^2b$ with $a \neq Tb \neq b$. In view of (1.3), we obtain that

$$\begin{aligned}
&\max\left\{\int_0^{d(a,b)}\varphi(t)dt,\int_0^{d(b,Tb)}\varphi(t)dt\right\}\\
&=\max\left\{\int_0^{d(T^2a,T^2b)}\varphi(t)dt,\int_0^{d(T^2b,T^3b)}\varphi(t)dt\right\}\\
&\le\phi\left(\max\left\{\int_0^{d(Ta,Tb)}\varphi(t)dt,\int_0^{d(Tb,T^2b)}\varphi(t)dt\right\}\right)\\
&\le\phi^2\left(\max\left\{\int_0^{d(a,b)}\varphi(t)dt,\int_0^{d(b,Tb)}\varphi(t)dt\right\}\right)\\
&<\max\left\{\int_0^{d(a,b)}\varphi(t)dt,\int_0^{d(b,Tb)}\varphi(t)dt\right\},
\end{aligned}$$

which is a contradiction.

Remark 3.3. Examples 3.1 and 3.3 show that Theorems 2.1, 2.3 and 2.4 are different from Theorem 2.1 in [2].

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