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NEW EDGE FINITE ELEMENTS FOR MAXWELL'S EIGENVALUE PROBLEM

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Abstract: In this paper, we introduce a new edge finite element space which has fewer degrees of freedom than the well known Nedelec space. We prove the unisolvence of degrees of freedom and analyze our space using the discrete de Rham diagram. We give estimates for the rate of convergence of eigenvalues and eigenfunctions.

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1. Model Problem

Maxwell's eigenvalue problem can be written as follows by means of Ampere and Faraday's laws: given a domain $\Omega \in \mathbb{R}^3$, find the resonance frequencies ω with $\omega \neq 0$ and the electromagnetic fields $(\mathbf{E}, \mathbf{H}) \neq (0, 0)$ such that

$$\left\{ \begin{array}{rcll} \mathbf{curl\ E} & = & i\omega\mu\mathbf{H}, & \text{in } \Omega, \\ \mathbf{curl\ H} & = & -i\omega\epsilon\mathbf{E}, & \text{in } \Omega, \\ \mathbf{E} \times \mathbf{n} & = & 0, & \text{on} \partial\Omega, \\ \mathbf{H} \cdot \mathbf{n} & = & 0, & \text{on} \partial\Omega, \end{array} \right. \tag{1}$$

where we assumed perfectly conducting boundary conditions, and ϵ and μ de-

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note the dielectric permittivity and magnetic permeability, respectively. For the sake of simplicity, we consider the material properties ϵ and μ constant and equal to the identity matrix. Let $\mathbf{u}$ be the unknown eigenfunction $\mathbf{E}$ and λ denotes ω^2. Then the classical formulation of the eigenvalue problem is obtained from the Maxwell system by eliminating $\mathbf{H}$: find $\lambda \in \mathbb{R}$ and $\mathbf{u} \neq 0$ such that

$$\begin{cases} \mathbf{curl}\ \mathbf{curl}\ \mathbf{u} &= \lambda \mathbf{u}, \quad \text{in } \Omega, \\ \text{div}\ \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2}$$

In this paper, we shall deal with the two dimensional counterpart of (2) as follows on rectangular grids: for a nonvanishing $\mathbf{u} = (u_1, u_2) : \Omega \to \mathbb{R}^2$, find $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \mathbf{curl}\ \text{curl}\ \mathbf{u} &= \lambda \mathbf{u}, \quad \text{in } \Omega, \\ \text{div}\ \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} \times \mathbf{n} &= 0, \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a polygonal domain. We note that

$$\text{curl}\ \mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}, \quad \mathbf{curl}\ \phi = (\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}).$$

2. Mixed Formulation of Eigenvalue Problem

In order to obtain the mixed formulation of problem (3), we define the Hilbert spaces

$$\mathbf{V} = H(\text{curl}\,, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^2 \mid \text{curl}\,\mathbf{v} \in L^2(\Omega),\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \partial\Omega\},$$
$$W = L^2(\Omega).$$

Multiplying (3) with a smooth test function $\mathbf{v} \in \mathbf{V}$ and integrating over Ω leads to

$$\int_\Omega \mathbf{curl}\ \text{curl}\ \mathbf{u} \cdot \mathbf{v}\ d\mathbf{x} = \int_\Omega \lambda \mathbf{u} \cdot \mathbf{v}\ d\mathbf{x}.$$

Let $-\frac{1}{\lambda}$ curl $\mathbf{u} = p$. From the curl term by parts, we have

$$\begin{aligned} \int_\Omega \mathbf{u} \cdot \mathbf{v} &= \int_\Omega \mathbf{curl}(\frac{1}{\lambda}\ \text{curl}\ \mathbf{u}) \cdot \mathbf{v}\ d\mathbf{x} \\ &= -\int_\Omega \mathbf{curl}\ p \cdot \mathbf{v}\ d\mathbf{x} \end{aligned}$$

$$\begin{aligned}
&= -\int_\Omega \left(\frac{\partial p}{\partial y} v_1 - \frac{\partial p}{\partial x} v_2 \right) \, d\mathbf{x} \\
&= \int_{\partial\Omega} p \, (-v_2, v_1)^t \cdot \mathbf{n} \; d\mathbf{x} - \int_\Omega p \text{ curl } \mathbf{v} \; d\mathbf{x} \\
&= \int_{\partial\Omega} p \, (\mathbf{v} \times \mathbf{n}) \; d\mathbf{x} - \int_\Omega p \text{ curl } \mathbf{v} \; d\mathbf{x} \\
&= -\int_\Omega p \text{ curl } \mathbf{v} \; d\mathbf{x}.
\end{aligned}$$

We also let

$$a(\mathbf{u},\ \mathbf{v}) = \int_\Omega \mathbf{u} \cdot \mathbf{v} \; d\mathbf{x}$$

and

$$b(p,\ \mathbf{v}) = \int_\Omega p \text{ curl} \, \mathbf{v} \; d\mathbf{x}.$$

Then we have the following mixed formulation for problem (3): find $\lambda \in \mathbb{R}$ and $p \in W$ with $p \neq 0$ such that, for some $\mathbf{u} \in \mathbf{V}$,

$$\begin{array}{lcll}
a(\mathbf{u},\ \mathbf{v}) + b(p,\ \mathbf{v}) & = & 0, & \forall \mathbf{v} \in \mathbf{V}, \\
b(\mathbf{u},\ q) & = & -\lambda(p,\ q), & \forall q \in W.
\end{array} \tag{4}$$

We assume that a and b are continuous and that a is symmetric and positive semidefinite. Moreover, we assume that the following source problem associated with (4) has unique solution $(\mathbf{u}, p) \in \mathbf{V} \times W$ to

$$\begin{array}{lcll}
a(\mathbf{u},\ \mathbf{v}) + b(p,\ \mathbf{v}) & = & 0, & \forall \mathbf{v} \in \mathbf{V}, \\
b(\mathbf{u},\ q) & = & -\langle g,\ q \rangle, & \forall q \in W,
\end{array} \tag{5}$$

satisfying the a priori bound

$$\| \mathbf{u} \|_{\mathbf{V}} + \| p \|_W \leq C \| g \|_{W'},$$

where the symbol $\langle \cdot, \cdot \rangle$ in (5) denotes the duality pairing between W and W'.

In order to discretize problem (4), we consider a sequence of regular grids of parallelogram elements. Let K be a parallelogram with vertices $\mathbf{x}_{ij} = (x_i, y_j)$ and let $\widehat{K}$ be a fixed reference element, typically unit square with vertices $\hat{\mathbf{x}}_{ij} = (\hat{x}_i, \hat{y}_j)$ for $i, j = 0, 1$. Then there exists unique bilinear map F_K from $\widehat{K}$ onto K satisfying

$$F_K(\hat{\mathbf{x}}_{ij}) = \mathbf{x}_{ij}, \quad i, j = 0, 1.$$

Vector functions must be transformed in a more careful way to conserve their properties. Suppose $\hat{\mathbf{u}} \in H(\mathrm{curl}\,, \widehat{K})$. Then $\hat{\mathbf{u}}$ is transformed to a function $\mathbf{u}$ defined on K in $H(\mathrm{curl}\,, K)$ via the following formula:

$$\mathbf{u} \circ F_K = (dF_K)^{-T}\hat{\mathbf{u}}, \tag{6}$$

where dF_K is the Jacobian matrix of the mapping F_K. Let $\widehat{\mathbf{V}}(\widehat{K}) \subset H(\mathrm{curl}\,, \widehat{K})$ and $\widehat{W}(\widehat{K})$ be given finite dimensional spaces on $\widehat{K}$. Then we define the following spaces on K using the equation (6),

$$\mathbf{V}_h(K) = \{\mathbf{v} = (dF_K)^{-T}\hat{\mathbf{v}} \mid \ \hat{\mathbf{v}} \in \widehat{\mathbf{V}}(\widehat{K})\} \tag{7}$$

and

$$W_h(K) = \{q = \hat{q} \circ F_K^{-1} \mid \ \hat{q} \in \widehat{W}(\widehat{K})\}. \tag{8}$$

Then the approximation of the mixed formulation (4) is defined as the solution of the equations: find $\lambda_h \in \mathbb{R}$ and $p_h \in W_h(K)$ with $p_h \neq 0$ such that, for some $\mathbf{u}_h \in \mathbf{V}_h(K)$,

$$\begin{array}{rcll} a(\mathbf{u}_h,\ \mathbf{v}_h) + b(p_h,\ \mathbf{v}_h) & = & 0, & \forall \mathbf{v}_h \in \mathbf{V}_h(K), \\ b(\mathbf{u}_h,\ q_h) & = & -\lambda_h(p_h,\ q_h), & \forall q_h \in W_h(K). \end{array} \tag{9}$$

We define the solution operator $T : L^2(\Omega) \to L^2(\Omega)$ by $Tg = p \in W \subset L^2(\Omega)$, where $p \in W$ is the second component of the solution to (5). It is clear that when g belongs to $L^2(\Omega)$ the duality pairing in the right-hand side of (5) is equivalent to the scalar product $(g,\ q)$.

The discrete counterpart of (5) when g belongs to $L^2(\Omega)$ is as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times W_h$ such that

$$\begin{array}{rcll} a(\mathbf{u}_h,\ \mathbf{v}_h) + b(p_h,\ \mathbf{v}_h) & = & 0, & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\mathbf{u}_h,\ q_h) & = & -(g,\ q_h), & \forall q_h \in W_h. \end{array} \tag{10}$$

We suppose that the second component p_h of the solution of (10) exists and is unique, so we can define the discrete solution operator $T_h : L^2(\Omega) \to L^2(\Omega)$ by $T_h g = p_h \in W_h \subset W \subset L^2(\Omega)$.

It is well known that standard finite elements are not well suited for the problem (9) and (10). The use of edge finite elements provides good results [1], [2], [3], [4]. In the next section, we define a new edge finite element space which has fewer degrees of freedom than the well known Nedelec space [5], [6].

3. New Edge Finite Elements

In this section, we introduce a new edge finite element space on the reference element $\widehat{K}$. We denote $Q_{l,m}(\widehat{K})$ is a polynomial space of maximum degree l in $\hat{x}$ and m in $\hat{y}$.

Definition 1. For $k \geq 1$, we define

$$\widehat{\mathbf{V}}(\widehat{K}) = Q_{k,k+1}(\widehat{K}) \times Q_{k+1,k}(\widehat{K}),$$

where $(\hat{x}^k\hat{y}^{k+1},\ 0)$ and $(0,\ \hat{x}^{k+1}\hat{y}^k)$ are replaced by the single element $(\hat{x}^k\hat{y}^{k+1}, \hat{x}^{k+1}\hat{y}^k)$, and

$$\widehat{W}(\widehat{K}) = Q_{k,k}(\widehat{K}),$$

except constant multiple of the term $\hat{x}^k\hat{y}^k$.

Then we see that the dimensions of $\widehat{\mathbf{V}}(\widehat{K})$ and $\widehat{W}(\widehat{K})$ are $2k^2+6k+3$ and k^2+2k, respectively. From the Definition 1, we give an following example for $k=1$. The space $\widehat{\mathbf{V}}(\widehat{K})$ consists of all vector polynomials $(\hat{u}_1,\ \hat{u}_2)$ where

$$\begin{aligned}\hat{u}_1 &= a_1 + a_2\hat{x} + a_3\hat{y} + a_4\hat{x}\hat{y} + a_5\hat{y}^2 + c\hat{x}\hat{y}^2,\\ \hat{u}_2 &= b_1 + b_2\hat{x} + b_3\hat{y} + b_4\hat{x}\hat{y} + b_5\hat{x}^2 + c\hat{x}^2\hat{y},\end{aligned}$$

and $\widehat{W}(\widehat{K})$ is a space containing polynomial of the following form:

$$\hat{p} = r_1 + r_2\hat{x} + r_3\hat{y}.$$

Now we need to define the degrees of freedom. For this purpose, we define

$$\Psi_k(\widehat{K}) = Q_{k,k-1}(\widehat{K}) \times Q_{k-1,k}(\widehat{K}),$$

where $(\hat{x}^k\hat{y}^{k-1},\ 0)$ and $(0,\ \hat{x}^{k-1}\hat{y}^k)$ are replaced by the element $(\hat{x}^k\hat{y}^{k-1},\ \hat{x}^{k-1}\hat{y}^k)$.

Definition 2. For any $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \widehat{\mathbf{V}}(\widehat{K})$, we consider the following degrees of freedom

$$\int_{\hat{e}} \hat{\mathbf{u}} \times \hat{\mathbf{n}}\ \hat{q}\ d\hat{s}, \qquad q \in P_k(\hat{e}), \quad \text{for each edge } \hat{e} \text{ of } \widehat{K}, \tag{11}$$

$$\int_{\widehat{K}} \hat{\mathbf{u}} \cdot \hat{q}\ d\hat{\mathbf{x}}, \qquad \hat{q} \in \Psi_k(\widehat{K}). \tag{12}$$

In order to see that these choices of $\widehat{\mathbf{V}}(\widehat{K})$ and degrees of freedom determine a finite element subspace of $H(\mathrm{curl}, \widehat{K})$, we need to show that the degrees of freedom are unisolvent.

Theorem 3. (**Unisolvence**) *For any* $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2) \in \widehat{\mathbf{V}}(\widehat{K})$, *the condition* $(11)-(12)$ *uniquely determines* $\hat{\mathbf{u}}$.

Proof. Since the number of conditions, $4(k+1)+[2k(k+1)-1] = 2k^2+6k+3$ equals the dimension of $\widehat{\mathbf{V}}(\widehat{K})$, it suffices to show that if $(11)-(12)$ are all zero then $\hat{\mathbf{u}} = 0$. Since $\hat{\mathbf{u}} \times \hat{\mathbf{n}} \in P_k(\hat{e})$ for each edge $\hat{e}$, $\hat{\mathbf{u}} \times \hat{\mathbf{n}} = 0$ from condition (11). This implies that

$$\hat{u}_1 = \hat{y}(1-\hat{y})\hat{v}_1, \quad \hat{u}_2 = \hat{x}(1-\hat{x})\hat{v}_2,$$

where $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2) \in \Psi_k(\widehat{K})$. From condition (12), we obtain the desired result. □

Remark 4. Our pair has the degrees of freedoms $2k^2+6k+3$ and k^2+2k respectively, hence total of two less than the Nedelec spaces.

Remark 5. Let $\widehat{\mathbf{V}}(\widehat{K})$ and $\widehat{W}(\widehat{K})$ be the space given by Definition 1. Then curl $\widehat{\mathbf{V}}(\widehat{K}) = \widehat{W}(\widehat{K})$. Since we use affine meshes, curl $\mathbf{V}_h(K) = W_h(K)$.

4. Analysis for the New Edge Finite Element Space

To analyze our space, we first define a projection operator $\widehat{R} : \mathbf{H}^{k+1}(\widehat{K}) \to \widehat{\mathbf{V}}(\widehat{K})$ satisfying

$$\int_{\hat{e}} (\hat{\mathbf{u}} - \widehat{R}\hat{\mathbf{u}}) \times \hat{\mathbf{n}}\, \hat{q}\, d\hat{s} = 0, \qquad \hat{q} \in P_k(\hat{e}), \quad \text{for each edge } \hat{e} \text{ of } \widehat{K}, \tag{13}$$

$$\int_{\widehat{K}} (\hat{\mathbf{u}} - \widehat{R}\hat{\mathbf{u}}) \cdot \hat{\mathbf{q}}\, d\hat{\mathbf{x}} = 0, \qquad \hat{\mathbf{q}} \in \Psi_k(\widehat{K}). \tag{14}$$

Then this operator has the following property:

Lemma 6. *We have*

$$(\widehat{\text{curl}}(\hat{\mathbf{u}} - \widehat{R}\hat{\mathbf{u}}), \hat{q})_{\widehat{K}} = 0, \quad \forall\, \hat{\mathbf{u}} \in \mathbf{H}^{k+1}(\widehat{K}), \ \ \forall\, \hat{q} \in \widehat{W}(\widehat{K}).$$

Proof. First, we note that $\hat{q}\mid_{\hat{e}} \in P_k(\hat{e})$ for all $\hat{q} \in \widehat{W}(\widehat{K})$. Since $\mathbf{curl}\ \hat{q} \in \Psi_k(\widehat{K})$, we have by the definition of $\widehat{R}$,

$$\begin{aligned}
(\widehat{\text{curl}}\ \widehat{R}\hat{\mathbf{u}}, \hat{q})_{\widehat{K}} &= -\int_{\partial\widehat{K}} (\widehat{R}\hat{\mathbf{u}} \times \hat{\mathbf{n}})\hat{q}\ d\hat{s} + \int_{\widehat{K}} \widehat{R}\hat{\mathbf{u}} \cdot \widehat{\mathbf{curl}}\ \hat{q}\ d\hat{\mathbf{x}} \\
&= -\int_{\partial\widehat{K}} (\hat{\mathbf{u}} \times \hat{\mathbf{n}})\hat{q}\ d\hat{s} + \int_{\widehat{K}} \hat{\mathbf{u}} \cdot \widehat{\mathbf{curl}}\ \hat{q}\ d\hat{\mathbf{x}} \\
&= (\widehat{\text{curl}}\ \hat{\mathbf{u}}, \hat{q})_{\widehat{K}}.
\end{aligned}$$

□

For an arbitrary element $K = F_K(\widehat{K})$, we can define the corresponding projection $R_K : \mathbf{H}^{k+1}(K) \to \mathbf{V}_h(K)$ by $R_K = F_K \circ \widehat{R} \circ F_K^{-1}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbf{H}^{k+1}(\widehat{K}) & \xrightarrow{\widehat{R}} & \widehat{\mathbf{V}}(\widehat{K}) \\ {\scriptstyle F_K}\downarrow & & \downarrow{\scriptstyle F_K} \\ \mathbf{H}^{k+1}(K) & \xrightarrow[R_K]{} & \mathbf{V}_h(K) \end{array}$$

Lemma 7. *We have*

$$(\mathrm{curl}(\mathbf{u} - R_K\mathbf{u}), q)_K = 0, \quad \forall\ \mathbf{u} \in \mathbf{H}^{k+1}(K), \ \ \forall\ q \in W_h(K).$$

Proof. From Lemma 6, we know that

$$\begin{aligned}(\mathrm{curl}\ R_K\mathbf{u},\ q)_K &= \int_K J_K\ \mathrm{curl}\ R_K\mathbf{u}\ q\ J_K^{-1}\ d\mathbf{x} \\ &= \int_{\widehat{K}} \widehat{\mathrm{curl}}\ \widehat{R}\hat{\mathbf{u}}\ \hat{q}\ d\hat{\mathbf{x}} \\ &= \int_{\widehat{K}} \widehat{\mathrm{curl}}\ \hat{\mathbf{u}}\ \hat{q}\ d\hat{\mathbf{x}} \\ &= \int_{\widehat{K}} J_K^{-1}\ \widehat{\mathrm{curl}}\ \hat{\mathbf{u}}\ \hat{q} J_K\ d\hat{\mathbf{x}} \\ &= (\mathrm{curl}\ \mathbf{u},\ q)_K\end{aligned}$$

where J_K is a determinant of dF_K. □

Finally, a global projection operator $R_h : \mathbf{H}^{k+1}(\Omega) \to \mathbf{V}_h$ is defined piecewise:

$$(R_h\mathbf{v})|_K = R_K(\mathbf{v}|_K).$$

Since $\mathbf{V}_h(K)$ contains full set of polynomials of degree k, we have the following approximation property of R_K.

Lemma 8. *There is a constant C independent of h such that*

$$\| \mathbf{u} - R_K\mathbf{u} \|_0 \leq Ch^{k+1} \| \mathbf{u} \|_{k+1},$$

for all $\mathbf{u} \in \mathbf{H}^{k+1}(K)$.

We also need to define an operator $\mathbf{\Pi}_h : L^2(\Omega) \to W_h$. First, let $\widehat{\mathbf{\Pi}}$ be the local L^2-projection onto $\widehat{W}(\widehat{K})$. Then define $\mathbf{\Pi}_K : L^2(K) \to W_h(K)$ by

$$\mathbf{\Pi}_K p = (\widehat{\mathbf{\Pi}}\hat{p}) \circ F_K^{-1}$$

with $\hat{p} = p \circ F_K$. Finally, we let

$$(\mathbf{\Pi}_h p) \mid_K = \mathbf{\Pi}_K (p \mid_K).$$

Now we will prove that the following de Rham diagram commutes [7]:

$$\begin{array}{ccc} \mathbf{V} & \xrightarrow{\nabla\times} & W \\ {\scriptstyle R_h}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{\Pi}_h} \\ \mathbf{V}_h & \xrightarrow[\nabla\times]{} & W_h \end{array}$$

Theorem 9. *If $\mathbf{u}$ is smooth enough such tat $R_h\mathbf{u}$ and $\mathbf{\Pi}_h(\nabla \times \mathbf{u})$ are defined, then $\nabla \times R_h\mathbf{u} = \mathbf{\Pi}_h(\nabla \times \mathbf{u})$.*

Proof. We perform the analysis on the reference element. Using the definition of projection operator $\widehat{\mathbf{\Pi}}$ and Green's theorem of the following form:

$$\int_\Omega \text{curl } \mathbf{u} \cdot q \, d\mathbf{x} \; = \int_\Omega \mathbf{u} \cdot \mathbf{curl} \; q \, d\mathbf{x} + \int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \cdot q \, dA,$$

we have

$$\begin{aligned} &\int_{\widehat{K}} (\text{curl } \widehat{R}\hat{\mathbf{u}} - \widehat{\mathbf{\Pi}}(\text{curl } \hat{\mathbf{u}})) \cdot \hat{q} \, d\hat{\mathbf{x}} \\ &= \int_{\widehat{K}} (\text{curl } \widehat{R}\hat{\mathbf{u}} - \text{curl } \hat{\mathbf{u}}) \cdot \hat{q} \, d\hat{\mathbf{x}} \\ &= \int_{\widehat{K}} (\widehat{R}\hat{\mathbf{u}} - \hat{\mathbf{u}}) \cdot \mathbf{curl} \; \hat{q} \, d\hat{\mathbf{x}} + \int_{\partial\widehat{K}} (\hat{\mathbf{n}} \times (\widehat{R}\hat{\mathbf{u}} - \hat{\mathbf{u}})) \cdot \hat{q} \, d\hat{A}. \end{aligned}$$

Since $\hat{q} \in P_k(\hat{e})$ for each edge $\hat{e}$ and $\mathbf{curl} \; \hat{q} \in \Psi_k(\widehat{K})$, the right-hand side of above formula vanishes by (13) and (14). $\square$

From this diagram, one can easily derive the inf-sup condition [8], [9].

5. Estimates for the Rate of Convergence

In order to obtain good approximation of the eigenvalue problem (4), we need to show that the weak approximability of W with respect to a, the strong approximability of W and the Fortid condition are satisfied [10], [11].

Definition 10. The weak approximability of W with respect to a means that there exists $\rho_w(h)$, tending to 0 as h tends to 0, such that

$$b(\mathbf{v}_h,\ q)\ \le\ \rho_w(h)\,|\,\mathbf{v}_h\,|_a\|\,q\,\|_W,\quad \forall\, q\in W,\ \forall\, \mathbf{v}_h\in \mathbb{K}_h,$$

where $\mathbb{K}_h=\{\mathbf{v}_h\in \mathbf{V}_h|\ b(\mathbf{v}_h,\ q_h)=0,\ \forall\, q_h\in W_h\}$.

Definition 11. The strong approximability of W means that there exists $\rho_s(h)$, tending to 0 as h tends to 0, such that, for every $q\in W$, there exists $q_h\in W_h$ with

$$\|\,q-q_h\,\|_W\ \le\ \rho_s(h)\,\|\,q\,\|_W\ .$$

Definition 12. The Fortid condition is satisfied if there exists $\rho_F(h)$, tending to 0 as h tends to 0, such that

$$|\,\mathbf{v}-P_h\mathbf{v}\,|_a\le\ \rho_F(h)\,\|\,\mathbf{v}\,\|_V,\quad \forall\,\mathbf{v}\in\mathbf{V},$$

where P_h is a Fortin operator.

We choose $\mathbf{V}_h$ and W_h given by Definition 1, (7) and (8) as discrete spaces for the approximation of $\mathbf{V}$ and W. Then the weak approximability is satisfied, from the equality curl $\mathbf{V}_h(K)=W_h(K)$ in Remark 5 and inf-sup condition. Also, the strong approximability is a consequence of standard approximation properties in W. If we use the projection operator R_K in Section 4, then it is not difficult to see that this operator is indeed a Fortin operator by Lemma 7. From Lemma 8, the Fortid condition is satisfied.

Let λ be an eigenvalue of (4) of multiplicity m and let $E\subset W$ be the corresponding eigenspace. We denote by $\lambda_{1,h},\ \lambda_{2,h},\dots,\lambda_{m,h}$ the discrete eigenvalues converging to λ and by E_h the direct sum of the corresponding eigenspaces. The gap between Hilbert spaces is defined by

$$\delta(E,\ F)=\sup_{p\in E,\ \|p\|_0=1}\ \inf_{q\in F}\,\|\,p-q\,\|_0,$$

$$\hat{\delta}(E,\ F)=\max(\delta(E,\ F),\ \delta(F,\ E)).$$

Since the weak approximability, the strong approximability and the Fortid condition are satisfied, the eigenfunction convergence follows directly from the result of Mercier et al. [12].

Theorem 13. *There is a constant C such that*

$$\hat{\delta}(E,\ E_h) \leq C \parallel T - T_h \parallel_{\mathcal{L}(L^2,L^2)},$$

where the gap is evaluated in the L^2-norm.

We assume that the source problems (5) and (10) are well-posed with $g \in L^2$, so that we can define the operator $A : L^2 \rightarrow \mathbf{V}$ associated with the first component of the solution as $Ag = \mathbf{u}$. Analogously, $A_h : L^2 \rightarrow \mathbf{V}_h$ denotes the discrete operator associated to the first component of the solution as $A_h g = \mathbf{u}_h$.

Theorem 14. *Assume that the operators A and A_h are well-defined. Then there is a constant C such that, for sufficiently small h,*

$$\begin{aligned} \mid \lambda - \lambda_{i,h} \mid \leq\ & C(\parallel (T - T_h)_{|E} \parallel^2_{\mathcal{L}(L^2,W)} \\ & + \parallel (T - T_h)_{|E} \parallel_{\mathcal{L}(L^2,W)} \parallel (A - A_h)_{|E} \parallel_{\mathcal{L}(L^2,\mathbf{V})} \\ & + \parallel (A - A_h)_{|E} \parallel^2_{\mathcal{L}(L^2,L^2)}), \end{aligned}$$

for $i = 1, \ldots, m$.

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