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OSCILLATION THEOREMS OF FOURTH ORDER NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

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Abstract: In this paper, we obtain some oscillation theorems for fourth order nonlinear dynamic equations on time scales with and without neutral term. Some examples illustrating the main results are given.

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Key Words: oscillation, fourth order, neutral dynamic equations, time scales

1. Introduction

The theory of time scales which has recently received a lot of attention was introduced by Stefan Hilger in his Ph.D thesis in 1988. Several authors have expounded on various aspects of this new theory and for more details about the theory of time scale calculus, see for example [3, 4]. In recent years there has

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been much research activity concerning the oscillatory behavior of solutions of first,second and third order dynamic equations on time scales, see [1, 2, 3, 4, 5, 6, 10, 12] and the reference cited therein. To best of our knowledge there is no paper dealt with the oscillatory behavior of fourth order dynamic equations on time scales. This motivated us to consider the oscillation of fourth order nonlinear dynamic equations of the form

$$(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta^2} + f(t, x^{\sigma}(t)) = 0, \tag{1.1}$$

and

$$(a(t)(b(t)(x(t) + p(t)x(t-\delta))^{\Delta})^{\Delta})^{\Delta^2} + f(t, x^{\sigma}(t-\delta)) = 0 \tag{1.2}$$

on a time scale interval $[t_0, +\infty)_{\mathbb{T}} = \{t \in \mathbb{T}, t \geq t_0 > 0\}$. Throughout this paper we use the notation $x^{\Delta^{n}}$ in place of n times $\Delta-$derivative of the function $x(t)$. The following conditions are assumed to hold without further mention:

(C_1) a, b and p are positive real valued right-dense continuous functions on $\mathbb{T}$;

(C_2) τ and δ are positive real numbers;

(C_3) $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous with sgn $f(t, u) =$ sgn u, and $f(t, u)$ is nondecreasing in u for each fixed t in $[t_0, \infty)_{\mathbb{T}}$.

By a solution of equation (1.1) (or(1.2)), we mean a nontrivial real valued function x satisfying equation (1.1) (or(1.2)) for $t \geq t_x$ for some $t_x \geq t_0$. A solution x of equation (1.1) (or(1.2)) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Our attention is restricted on those solutions x of equation (1.1) (or(1.2)) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ with $\sup\{|x(t)| : t \in [\bar{t}, \infty)_{\mathbb{T}}\} > 0$ for every $\bar{t} \geq t_x$.

The paper is organized as follows: In the Section 2 we study the oscillatory behavior of strongly sublinear and super linear fourth order dynamic equations (1.1). In Section 3 we establish some oscillation criteria for fourth order neutral dynamic equation (1.2) using Riccati transformation technique. Examples illustrating the main results are given.

2. Oscillation Theorems of Equation (1.1)

In this section, we present some oscillation theorems for the equation (1.1) when the function f is either superlinear or sublinear. We begin with the following formula.

We shall employ the following formula which is a simple consequence of the Keller's Chain rule (page 32 Theorem 1.90 in [3]): For any real $\beta > 0$

$$((x(t))^{\beta})^{\Delta} = \beta x^{\Delta}(t) \int_0^1 [hx^{\sigma} + (1-h)x]^{\beta-1} dh, \tag{2.1}$$

where $x(t)$ is delta differentiable and eventually positive or eventually negative function.

Definition 2.1. The function $f(t,u)$ is said to be strongly *superlinear* if there exists a constant $\alpha > 1$ such that for $u \geq v > 0$ or $u \leq v < 0$,

$$\frac{f(t,u)}{|u|^{\alpha}\mathrm{sgn}u} \geq \frac{f(t,v)}{|v|^{\alpha}\mathrm{sgn}v}, \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

The equation (1.1) is called *strongly superlinear* equation if the function $f(t,u)$ is strongly superlinear.

The function $f(t,u)$ is said to be strongly *sublinear* if there exists a constant β with $0 < \beta < 1$ such that for $u \geq v > 0$ or $u \leq v < 0$,

$$\frac{f(t,u)}{|u|^{\beta}\mathrm{sgn}u} \geq \frac{f(t,v)}{|v|^{\beta}\mathrm{sgn}v}, \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

The equation (1.1) is called *strongly sublinear* equation if the function $f(t,u)$ is strongly sublinear.

Lemma 2.2. *Assume that either*

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\Delta t = \int_{t_0}^{\infty} \frac{1}{b(t)}\Delta t = \infty, \tag{2.2}$$

$$\int_{t_0}^{\infty} \frac{t}{a(t)}\Delta t = \infty \text{ and } 0 < m \leq b(t) \leq M, \tag{2.3}$$

or

$$\int_{t_0}^{\infty} \frac{P(t)}{a(t)}\Delta t = \int_{t_0}^{t} \frac{s}{a(s)}\Delta s = \infty \text{ where } P(t) = \int_{t_0}^{\sigma(t)} \frac{1}{b(s)}\Delta s \to \infty \text{ as } t \to \infty. \tag{2.4}$$

If $x(t)$ is an eventually positive solution of equation (1.1), then there are only the following two cases for t large enough

Case (I) : $x^{\Delta}(t) > 0,\ (b(t)x^{\Delta}(t))^{\Delta} > 0,\ (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0;$

Case (II): $x^{\Delta}(t) > 0,\ (b(t)x^{\Delta}(t))^{\Delta} < 0,\ (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0.$

Proof. Let $x(t)$ be an eventually positive solution of equation (1.1). Then there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. From the equation (1.1) we have

$$(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta^2} < 0$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. So $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta}, a(t)(b(t)x^{\Delta}(t))^{\Delta}$ and $b(t)x^{\Delta}(t)$ are eventually monotonic and of one sign, say for all $t \in [t_2, \infty)_{\mathbb{T}}$.
Suppose that $(a(t_3)(b(t_3)x^{\Delta}(t_3))^{\Delta})^{\Delta} = -c_1 \leq 0$ for some $t_3 \in [t_2, \infty)_{\mathbb{T}}$. Note that $f(t, .) \neq 0$, we assume that $c_1 \neq 0$. It follows that

$$(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \leq -c_1 \tag{2.5}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. Now integrating (2.5) from t_3 to t both sides, we see that there exists $c_2 > 0$ such that

$$(b(t)x^{\Delta}(t))^{\Delta} \leq -c_2 \frac{t}{a(t)}$$

for all $t \in [t_3, \infty)_{\mathbb{T}}$. Again integrating the last inequality, we obtain

$$b(t)x^{\Delta}(t) \leq b(t_3)x^{\Delta}(t_3) - c_2 \int_{t_3}^{t} \frac{s}{a(s)} \Delta s.$$

If (2.2) holds, then there exists $t_4 \in [t_3, \infty)_{\mathbb{T}}$ and $c_3 > 0$ such that $b(t)x^{\Delta}(t) \leq -c_3$ for all $t \in [t_4, \infty)_{\mathbb{T}}$. A final integration yields

$$x(t) \leq x(t_4) - c_3 \int_{t_4}^{t} \frac{1}{b(s)} \Delta s.$$

In view of condition (2.2), we obtain $\lim_{t\to\infty} x(t) = -\infty$. This contradiction implies that $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$.

Now if $a(t)(b(t)x^{\Delta}(t))^{\Delta} > 0$ for $t \in [t_4, \infty)_{\mathbb{T}}$. There exists $t_5 \in \mathbb{T}$ with $t_5 \geq t_4$ such that

$$a(t)(b(t)x^{\Delta}(t))^{\Delta} > 0$$

for $t \in [t_5, \infty)_{\mathbb{T}}$. Since $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$, we have

$$a(t)(b(t)x^{\Delta}(t))^{\Delta} \geq a(t_5)(b(t_5)x^{\Delta}(t_5))^{\Delta} = c_4 > 0 \tag{2.6}$$

for $t \in [t_5, \infty)_{\mathbb{T}}$. If (2.2) holds, divide the last inequality by $a(t)$ and integrate from t_5 to t to obtain

$$b(t)x^{\Delta}(t) - b(t_2)x^{\Delta}(t_2) > c_4 \int_{t_2}^{t} \frac{1}{a(s)} \Delta s \to \infty$$

as $t \to \infty$. Hence $x^{\Delta}(t)$ is eventually positive. If (2.3) holds, we multiply (2.6) by $\frac{t}{a(t)}$ and integrate from t_5 to t to obtain

$$tb(t)x^{\Delta}(t) - t_5 b(t_5)x^{\Delta}(t_5) - \int_{t_5}^{t} (b(s)(x^{\Delta}(s)))^{\sigma} \Delta s > c_4 \int_{t_5}^{t} \frac{s}{a(s)} \Delta s.$$

If $x^{\Delta}(t) < 0$ for all $t \in [t_5, \infty)_{\mathbb{T}}$, then

$$tb(t)x^{\Delta}(t) - t_5 b(t_5)x^{\Delta}(t_5) - M(x(\sigma(t)) - x(\sigma(t_5))) > c_4 \int_{t_5}^{t} \frac{s}{a(s)} \Delta s.$$

So as $t \to \infty$

$$tb(t)x^{\Delta}(t) - t_5 b(t_5)x^{\Delta}(t_5) + Mx(\sigma(t_5)) > c_4 \int_{t_5}^{t} \frac{s}{a(s)} \Delta s \to \infty.$$

Hence $x^{\Delta}(t)$ is eventually positive.

If (2.4) holds, multiplying (2.6) by $P(t)/a(t)$ and integrating from t_5 to t, we have

$$P(t)b(t)x^{\Delta}(t) - P(t_5)b(t_5)x^{\Delta}(t_5) - \int_{t_5}^{t} (b(s)x^{\Delta}(s))^{\sigma} P^{\Delta}(s) \Delta s > c_4 \int_{t_5}^{t} \frac{P(s)}{a(s)} \Delta s. \tag{2.7}$$

Since $P^{\Delta}(t) = \frac{1}{b(\sigma(t))}$, we have from (2.7) that

$$P(t)b(t)x^{\Delta}(t) - P(t_5)b(t_5)x^{\Delta}(t_5) + x(\sigma(t_5)) > c_4 \int_{t_5}^{t} \frac{P(s)}{a(s)} \Delta s \to \infty$$

as $t \to \infty$. Therefore $x^{\Delta}(t)$ is eventually positive and the proof of Case (I) is complete.

Next if $a(t)(b(t)x^{\Delta}(t))^{\Delta} < 0$ for $t \in [t_6, \infty)_{\mathbb{T}}$ for some $t_6 \geq t_4$, then $b(t)x^{\Delta}(t)$ must be eventually positive since $\int_{t_0}^{\infty} \frac{1}{b(t)} \Delta t = \infty$. Thus case (II) is verified and the proof is complete. □

In the following results we use the notation

$$R(t, t_0) = \int_{t_0}^{t} \frac{1}{b(s)} \Big(\int_{t_0}^{s} \frac{u - t_0}{a(u)} \Delta u \Big) \Delta s,$$

and

$$Q(t, t_0) = \int_{t_0}^{t} \frac{1}{b(s)} \Big(\int_{t_0}^{s} \frac{u}{a(u)} \Delta u \Big) \Delta s.$$

Note that $Q(t, t_0)$ can be written as $Q(t, t_0) = \int_{t_0}^{t} \frac{s}{a(s)} \Big(\int_{t_0}^{t} \frac{1}{b(u)} \Delta u \Big) \Delta s$ which is quite useful in doing calculations.

Lemma 2.3. *Assume that either (2.2), (2.3) or (2.4) hold. If $x(t)$ is an eventually positive solution of equation (1.1), then there exists a positive constants C_1 and C_2 and $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$C_1 \leq x(t) \leq C_2 Q(t, t_0), \tag{2.8}$$

and

$$x(t) \geq R(t, T_0)(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \quad \text{for} \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{2.9}$$

Proof. Given that $x(t)$ is an eventually positive solution of equation (1.1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Now from the Lemma 2.2, we have $b(t)x^{\Delta}(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$ and so $x(t) \geq C_1 > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. We integrate $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta^2} < 0$ twice from t_1 to t we obtain

$$(b(t)x^{\Delta}(t))^{\Delta} \leq \frac{A_0(t - t_1)}{a(t)} + \frac{A_1}{a(t)} \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}},$$

where A_0 andA_1 are constants. Integrating the last inequality again from t_1 to t, we have

$$x^{\Delta}(t) < \frac{A_0}{b(t)} \int_{t_1}^{t} \frac{s}{a(s)} \Delta s + \frac{A_1}{b(t)} \int_{t_1}^{t} \frac{1}{a(s)} \Delta s + \frac{A_2}{b(t)}.$$

Final integration of the last inequality from t_1 to t yields

$$x(t) < A_0 \int_{t_1}^{t} \frac{1}{b(s)} \Big(\int_{t_1}^{s} \frac{u}{a(u)} \Delta u \Big) \Delta s + A_1 \int_{t_1}^{t} \frac{1}{b(s)} \Big(\int_{t_1}^{s} \frac{1}{a(u)} \Delta u \Big) \Delta s + A_2 \int_{t_1}^{t} \frac{1}{b(s)} \Delta s + A_3.$$

It is easy to see that every term on the right side of the above inequality is less than $Q(t, t_1)$. Therefore there exist a constant $C_2 > 0$ and $T \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x(t) < C_2 Q(t, t_0) \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.10}$$

To prove (2.9), let $T_0 \in [t_0, \infty)_{\mathbb{T}}$ be large enough such that $x(t)$ satisfies Case (I) or Case (II) of Lemma 2.2. Assume first that Case(I) holds. Since $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta}$ is positive and nonincreasing, it follows that

$$\begin{aligned} a(t)(b(t)x^{\Delta}(t))^{\Delta} &\geq a(t)(b(t)x^{\Delta}(t))^{\Delta} - a(T_0)(b(T_0)x^{\Delta}(T_0))^{\Delta} \\ &= \int_{T_0}^{t} (a(s)(b(s)x^{\Delta}(s))^{\Delta})^{\Delta} \Delta s \end{aligned}$$

$$\geq (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta}(t-T_0),$$

or

$$(b(t)x^{\Delta}(t))^{\Delta} \geq \frac{t-T_0}{a(t)}(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \text{ for } t \in [T_0,\infty)_{\mathbb{T}}. \quad (2.11)$$

Now integrating the last inequality from T_0 to t to obtain

$$\begin{aligned} b(t)x^{\Delta}(t) &\geq \int_{T_0}^{t} \frac{s-T_0}{a(s)}(a(s)(b(s)x^{\Delta}(s))^{\Delta})^{\Delta}\Delta s \\ &\geq (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \int_{T_0}^{t} \frac{s-T_0}{a(s)}\Delta s, \end{aligned}$$

or

$$x(t) \geq (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \int_{T_0}^{t} \frac{1}{b(s)}\Big(\int_{T_0}^{s} \frac{u-T_0}{a(u)}\Delta u\Big)\Delta s, \quad (2.12)$$

which prove (2.9). Suppose that Case (II) holds. Multiplying equation (1.1)by $R(\sigma^2(t),T_0)$ and integrating from T_0 to t, we have

$$\int_{T_0}^{t} R(\sigma^2(s),T_0)(a(s)(b(s)x^{\Delta}(s))^{\Delta})^{\Delta^2}\Delta s + \int_{T_0}^{t} R(\sigma^2(s),T_0)f(s,x(s-\delta))\Delta s = 0. \quad (2.13)$$

Now apply integration by parts twice in the first part of the equation (2.13) and then applying Case (II), we obtain (2.9). This completes the proof. □

Theorem 2.4. *Assume that either (2.2), (2.3) or (2.4) hold. Let f be strongly sublinear and*

$$\int_{t_0}^{\infty} f(t, CQ^{\sigma}(t,t_0))\Delta t = \infty \quad (2.14)$$

for all $C \neq 0$. Then all solutions of equation (1.1) are oscillatory.

Proof. Assume that there exists a nonoscillatory solution $x(t)$ of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$) (the case $x(t) < 0$ eventually can be treated similarly and will be omitted). From Lemmas 2.2 and 2.3, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that

$$x^{\Delta}(t) > 0, \text{ and } (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0 \quad (2.15)$$

and

$$R(t,t_1)(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} \le x(t) \le kQ(t,t_1), \tag{2.16}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. Now we define a function v by $v(t) = (a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta}$. Then inequality (2.16) becomes

$$R(t,t_1)v(t) \le x(t) \le kQ(t,t_1). \tag{2.17}$$

By Keller's Chain rule, we have

$$\begin{aligned}
(-v^{1-\beta}(t))^{\Delta} &\ge (1-\beta)v^{-\beta}(\sigma(t))f(t,x^{\sigma}(t)) \\
&= (1-\beta)v^{-\beta}(\sigma(t))x^{\beta}(\sigma(t))\frac{f(t,x^{\sigma}(t))}{x^{\beta}(\sigma(t))} \\
&\ge (1-\beta)v^{-\beta}(\sigma(t))x^{\beta}(\sigma(t))\frac{f(t,kQ^{\sigma}(t,t_1))}{\{kQ^{\sigma}(t,t_1)\}^{\beta}} \\
&\ge (1-\beta)v^{-\beta}(\sigma(t))R^{\beta}(t,t_1)v^{\beta}(t)\frac{f(t,kQ^{\sigma}(t,t_1))}{\{kQ^{\sigma}(t,t_0)\}^{\beta}},
\end{aligned}$$

or

$$(-v^{1-\beta}(t))^{\Delta} \ge (1-\beta)k^{-\beta}\frac{R^{\beta}(t,t_1)}{Q^{\beta}(\sigma(t)-\delta,t_1)}f(t,kQ^{\sigma}(t,t_1)). \tag{2.18}$$

Since $\lim_{t\to\infty}\frac{R(t,t_1)}{Q(t,t_1)} = 1$, there is a positive constant M and $T \in [t_1,\infty)_{\mathbb{T}}$ such that $\frac{R^{\beta}(t,t_1)}{Q^{\beta}(\sigma(t),t_1}> M$ for $t \in [T,\infty)_{\mathbb{T}}$. Therefore from (2.18), we have

$$(-((a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta})^{1-\beta})^{\Delta} \ge (1-\beta)Mk^{-\beta}f(t,kQ^{\sigma}(t,t_1)).$$

Now integrating the last inequality from T to t, we have

$$\begin{aligned}
(1-\beta)Mk^{-\beta}\int_T^t f(s,kQ^{\sigma}(s,t_1))\Delta s \le\ & ((a(T)(b(T)x^{\Delta}(T))^{\Delta})^{\Delta})^{1-\beta} \\
& - ((a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta})^{1-\beta}.
\end{aligned}$$

Since $(a(t)(b(t)x^{\Delta}(t))^{\Delta})^{\Delta} > 0$, we have

$$\int_T^t f(s,kQ^{\sigma}(s,t_1))\Delta s < \frac{k^{\beta}}{(1-\beta)M}((a(T)(b(T)x^{\Delta}(T))^{\Delta})^{\Delta})^{1-\beta}.$$

Hence $\int_T^{\infty} f(s,kQ^{\sigma}(s,t_1))\Delta s < \infty$, which contradicts (2.14). This completes the proof of the theorem. □

Theorem 2.5. *Assume that either (2.2), (2.3) or (2.4) hold. Let f be strongly superlinear and*

$$\int_{t_0}^{\infty} R(t,t_0) f(t,C)\Delta t = \infty \tag{2.19}$$

for all $C \neq 0$. Then every solution of equation (1.1) is oscillatory.

Proof. Assume that there is a nonoscillatory solution $x(t)$ of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \in [T,\infty)_{\mathbb{T}}$(the case $x(t) < 0$ eventually can be treated similarly and will be omitted). From the Lemma 2.2 and 2.3, there exists $t_1 \in [T,\infty)_{\mathbb{T}}$ such that (2.15) and

$$x(t) \geq \int_{t_1}^{t} R(s,T) f(s,x^{\sigma}(s))\Delta s, \quad t \geq t_1. \tag{2.20}$$

Since $x(t) > 0$ and $x^{\Delta}(t) > 0$, there exists a constant $k_1 > 0$ such that $x^{\sigma}(t) \geq k_1$ for $t \in [t_1,\infty)_{\mathbb{T}}$. Since f is strongly superlinear, we have for some $\alpha > 1$

$$\frac{f(t,x^{\sigma}(t))}{(x^{\sigma}(t))^{\alpha}} \geq \frac{f(t,k_1)}{k_1^{\alpha}},$$

that is,

$$f(t,x^{\sigma}(t)) \geq k_1^{-\alpha} f(t,k_1)(x^{\sigma}(t))^{\alpha}. \tag{2.21}$$

From equations (2.20) and (2.21), we obtain

$$x(t) \geq \int_{t_1}^{t} k_1^{-\alpha} R(s,t_1)(x^{\sigma}(s))^{\alpha} f(s,k_1)\Delta s, \quad t \geq t_1.$$

Let $\theta(t) = \int_{t_1}^{t} k_1^{-\alpha} R(s,t_1)(x^{\sigma}(s))^{\alpha} f(s,k_1)\Delta s$. Then by Keller's Chain rule,we have

$$(\theta^{1-\alpha}(t))^{\Delta} \leq (1-\alpha)\frac{\theta^{\Delta}(t)}{(\theta^{\sigma}(t))^{\alpha}}. \tag{2.22}$$

Since $\theta^{\Delta}(t) = k_1^{-\alpha} R(t,t_1)(x^{\sigma}(t))^{\alpha} f(t,k_1)$,

$$\begin{aligned}(\theta^{1-\alpha}(t))^{\Delta} &\leq \frac{(1-\alpha)}{(\theta^{\sigma}(t))^{\alpha}} k_1^{-\alpha} R(t,t_1)(x^{\sigma}(t))^{\alpha} f(t,k_1)\\ &\leq \frac{(1-\alpha)}{(x^{\sigma}(t))^{\alpha}} k_1^{-\alpha} R(t,t_1)(x^{\sigma}(t))^{\alpha} f(t,k_1)\end{aligned}$$

or

$$R(t,t_1)f(t,k_1) \leq \frac{k_1^\alpha(\theta^{1-\alpha}(t))^\Delta}{(1-\alpha)}. \tag{2.23}$$

Now integrate the last inequality from t_1 to t and use $\alpha > 1$, we obtain

$$\int_{t_1}^t R(s,T)f(s,k_1)\Delta s \leq \frac{k_1^\alpha}{(\alpha-1)}(\theta^{1-\alpha}(t_1) - \theta^{1-\alpha}(t)). \tag{2.24}$$

Hence $\int_{t_1}^\infty R(s,t_1)f(s,k_1)\Delta s < \infty$, a contradiction to (2.19). The proof is now complete. □

We conclude this section with the following example

Example 2.6. Consider the neutral dynamic equation

$$(tx^{\Delta^2}(t))^{\Delta^2} + 24tx^{1/3}(\sigma(t)) = 0, \qquad t \in [1,\infty)_{\mathbb{T}}. \tag{2.25}$$

It is easy to see that all conditions of Theorem 2.5 are satisfied and hence all solutions of equation (2.25) are oscillatory.

3. Oscillation Theorems of Equation (1.2)

In this section we establish some oscillation criteria for the neutral delay dynamic equation (1.2).

First we define an associated function of $x(t)$ by $z(t) = x(t) + p(t)x(t-\tau)$, and we derive the following lemmas

Lemma 3.1. *Assume that either (2.2), (2.3) or (2.4) hold. If $x(t)$ is an eventually positive solution of equation (1.2), then there are only the following two cases:*

Case (I) $z(t) > 0, z^\Delta(t) > 0, (b(t)z^\Delta(t))^\Delta > 0, (a(t)(b(t)z^\Delta(t))^\Delta)^\Delta > 0,$

Case (II) $z(t) > 0, z^\Delta(t) > 0, (b(t)z^\Delta(t))^\Delta < 0, (a(t)(b(t)z^\Delta(t))^\Delta)^\Delta > 0,$

Lemma 3.2. *Assume that either (2.2), (2.3) or (2.4) hold. If $x(t)$ is an eventually positive solution of equation (1.2), then there exists a positive constants C_1 and C_2 and $T_0 \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$C_1 \leq z(t) \leq C_2 Q(t,t_0) \tag{3.1}$$

and

$$z(t) \geq R(t,T)(a(t)(b(t)z^\Delta(t))^\Delta)^\Delta \quad \textit{for} \quad t \in [T_0,\infty)_{\mathbb{T}}. \tag{3.2}$$

The proofs of Lemmas 3.1 and 3.2 are similar to that of Lemmas 2.2 and 2.3 and hence the details are omitted.

Lemma 3.3. *Assume that either (2.2), (2.3) or (2.4) hold. Let $x(t)$ be an eventually positive solution of equation (1.1), then there exists a $T_0 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$z^{\Delta}(t) \geq (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} R^{\Delta}(t, T_0), \tag{3.3}$$

and

$$z^{\Delta}(t-\delta) \geq (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} R^{\Delta}(t-\delta, T_0) \tag{3.4}$$

for $t \in [T_0, \infty)_{\mathbb{T}}$.

Proof. By Lemma 3.1 we have

$$z(t) > 0, \;\; z^{\Delta}(t) > 0, \;\; (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta^2} \leq 0. \tag{3.5}$$

Then

$$\begin{aligned} b(t)z^{\Delta}(t) &\geq \int_{t_1}^{t} (b(s)z^{\Delta}(s))^{\Delta} \Delta s \\ &\geq \int_{t_1}^{t} \frac{1}{a(s)} \Big(\int_{t_1}^{s} (a(u)(b(u)z^{\Delta}(u))^{\Delta})^{\Delta} \Delta u \Big) \Delta s \\ &\geq (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} \int_{t_1}^{t} \frac{s-t_1}{a(s)} \Delta s \end{aligned}$$

or

$$z^{\Delta}(t) \geq (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} R^{\Delta}(t, t_1). \tag{3.6}$$

Since $\delta > 0$, we have from the last inequality that

$$z^{\Delta}(t-\delta) \geq (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} R^{\Delta}(t-\delta, t_1). \tag{3.7}$$

This completes the proof. □

Lemma 3.4. *Assume that either (2.2), (2.3) or (2.4) hold. If $x(t)$ is an eventually positive solution of equation (1.2), then there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$(1-p(t))z(t) \leq x(t) \leq z(t) \;\; \text{for} \;\; t \in [T, \infty)_{\mathbb{T}}. \tag{3.8}$$

Proof. From the definition of $z(t)$, $z(t) \geq x(t)$ for $t \in [T, \infty)_{\mathbb{T}}$. By Lemma 3.1,we have $z(t) > 0$ and $z^{\Delta}(t) > 0$ for $t \in [T, \infty)_{\mathbb{T}}$. Hence $x(t) = z(t) - p(t)x(t-\tau) \geq z(t) - p(t)z(t-\tau) \geq (1-p(t))z(t)$ for $t \in [T, \infty)_{\mathbb{T}}$. □

Theorem 3.5. *Assume that either (2.2), (2.3) or (2.4) hold. Furthermore assume that there exists a real valued rd-continuous function $q(t)$ such that*

$$\frac{f(t,u)}{u} \geq Mq(t) > 0 \quad \text{for } u \neq 0, t \in [t_0, \infty)_{\mathbb{T}}. \tag{3.9}$$

If there exists a positive Δ- differentiable function $\alpha(t)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \alpha(s)\Big[M(1-p(s-\delta))q(s) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha^2(s)R^{\Delta}(s-\delta,t_0)}\Big]\Delta s = \infty, \tag{3.10}$$

then every solution of equation (1.2) is oscillatory.

Proof. Suppose to the contrary that there is a nonoscillatory solution $x(t)$ of equation (1.2). Without loss of generality, we may assume that $x(t)$ is eventually positive(the case $x(t)$ is eventually negative can be treated similarly and will be omitted). Then from the definition of $z(t)$, we can find a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $z(t) > 0$. By Lemmas 3.1 and 3.2, we see that there exists a $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that

$$z^{\Delta}(t) > 0 \text{ and } (a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta} > 0.$$

Now define $w(t) = \alpha(t)\frac{(a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta}}{z(t-\delta)}$. Then $w(t) > 0$ and

$$\begin{aligned} w^{\Delta}(t) &= \frac{\alpha(t)}{z(t-\delta)}(a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta^2} \\ &\quad +((a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta})^{\sigma}\Big(\frac{\alpha(t)}{z(t-\delta)}\Big)^{\Delta} \\ &\leq -M\alpha(t)q(t)(1-(p(t-\delta)) \\ &\quad +((a(t)(b(t)z^{\Delta}(t))^{\Delta})^{\Delta})^{\sigma}\frac{\alpha^{\Delta}(t)z(t-\delta)-\alpha(t)z^{\Delta}(t-\delta)}{z(t-\delta)z^{\sigma}(t-\delta)} \\ &\leq -M\alpha(t)q(t)(1-p(t-\delta)) + \frac{\alpha^{\Delta}(t)}{\alpha^{\sigma}(t)}w(\sigma(t)) \\ &\quad -\frac{\alpha(t)R^{\Delta}(t-\delta,t_1)}{(\alpha^{\sigma}(t))^2}w^2(\sigma(t)) \\ &\leq -M\alpha(t)q(t)(1-p(t-\delta)) + \frac{(\alpha^{\Delta}(t))^2}{4\alpha(t)R^{\Delta}(t-\delta,t_0)}. \end{aligned}$$

Integrating the last inequality from t_2 to t, we obtain

$$\int_{t_2}^{t} \alpha(s)\Big[Mq(s)(1-p(s)) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha^2(s)R^{\Delta}(s-\delta,t_0)}\Big]\Delta s \leq w(t_2)$$

and this contradicts (3.10).This completes the proof. ∎

The following corollaries are immediate .

Corollary 3.6. *If $b(t) \equiv 1$ and the condition (3.10) in Theorem 3.5 is replaced by*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \alpha(s)\Big[(1-p(s-\delta))q(s) - \frac{(\alpha^{\Delta}(s))^2}{4\alpha^2(s)R^{\Delta}(s-\delta)}\Big]\Delta s = \infty,$$

where $R(t) = \int_{t_0}^{t}\int_{t_0}^{s} \frac{u}{a(u)}\Delta u \Delta s$, then all solutions of the equation

$$(a(t)(x(t)+p(t)x(t-\tau))^{\Delta^2})^{\Delta^2} + q(t)x^{\sigma}(t-\delta) = 0$$

are oscillatory.

Corollary 3.7. *If $a(t) \equiv 1, b(t) \equiv 1$, and the condition (3.10) in Theorem 3.5 is replaced by*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \alpha(s)\Big[(1-p(s-\delta))q(s) - \frac{1}{2}\Big(\frac{\alpha^{\Delta}(s)}{\alpha(s)s}\Big)^2\Big]\Delta s = \infty,$$

then all solutions of the equation

$$(x(t)+p(t)x(t-\tau))^{\Delta^4} + q(t)x^{\sigma}(t-\delta) = 0$$

are oscillatory.

We conclude this paper with the following example.

Example 3.8. Consider the neutral dynamic equation

$$\Big(x(t) + \frac{1}{t+3}x(t-1)\Big)^{\Delta^4}(t) + \frac{\lambda}{t^2}x(\sigma(t)) = 0, \qquad t \in [1,\infty)_{\mathbb{T}}. \tag{3.11}$$

Here $q(t) = \frac{\lambda}{t^2}$. If we take $\alpha(t) = t$, then

$$\limsup_{t\to\infty} \int_{1}^{t} \Big(s(1-\frac{1}{s})\frac{\lambda}{s^2} - \frac{1}{s^4}\Big)\Delta s = \infty$$

if $\lambda > 0$. Hence by Theorem 3.5, all solutions of (3.11) are oscillatory if $\lambda > 0$.

References

[1] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Second Order Dynamic Equations*, Taylor and Francis, London (2003).

[2] R.P. Agarwal, M. Bohner, A. Perterson, Dynamic equations on time scales: A survey, *J. Comp. Appl. Math.*, Special Issue on Dynamic Equations on Time Scales (Ed-s: R.P. Agarwal, M. Bohner, D. O'Regan), Preprint in Ulmer Seminare 5, **141**, No-s: 1-2 (2002), 1-26.

[3] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston (2001).

[4] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales, Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhauser, Boston (2001).

[5] L. Erbe, A. Perterson, S.H. Saker, Oscillation criteria for second order nonlinear delay dynamic equations, *J. Math. Anal. Appl.*, **333** (2007), 505-522.

[6] M. Huang, W. Feng, Oscillation for forced second order nonlinear dynamic equations on time scales, *Elect. J. Diff. Eqn.*, **145** (2005), 1-8.

[7] S. Hilger, Analysis on measure chains - A unified approach to continuous and discrete, *Results Math.*, **8** (1990), 18-56.

[8] J. Jiang, Oscillation of second order nonlinear neutral delay difference equations, *Appl. Math. Comput.*, **146** (2003), 791-801.

[9] H.J. Li, W.L. Liu, Oscillation criteria for second order neutral differnetial equation, *Canad. J. Math.*, **48** (2006), 871-886.

[10] S.H. Samir, D. O'Regan, R.P. Agarwal, Oscillation theorems for second order nonlinear neutral delay dynamic equations on time scales, *Acta. Math. Sinica*, English Series, **24** (2007), 1409-1432.

[11] E. Thandapani, I.M. Arockiasamy, Oscillatory and asymptotic behaviour of fourth order nonlinear neutral delay difference equations, *Indian. J. Pure. Appl. Math.*, **32**, No. 1 (2001), 109-123.

[12] H. Wu, R.K. Zhuang, R.M. Mathsen, Oscillation criteria for second order nonlinear neutral variable delay dynamic equations, *Appl. Math. Comp.*, **178** (2006), 321-331.