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ON THE SYNDETICALLY HYPERCYCLIC TUPLES

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Abstract: Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional Banach space X. In this paper we want to give necessary and sufficient conditions for $\mathcal{T}$ being syndetically hypercyclic tuple.

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1. Introduction

From now on, let T_1, T_2, ..., T_n be commutative bounded linear operators on an infinite dimensional Banach space X.

Definition 1.1. Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on the Banach space X. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set $Orb(\mathcal{T}, x) = \{{T_1}^{k(1)}{T_2}^{k(2)}...{T_n}^{k(n)}x : k_i \in \mathbb{Z}_+,\ \ i = 1, ..., n\}$. A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

Remember that a sequence of operators $\{T_n\}_{n\geq 0}$ is said to be a hypercyclic sequence on X if there exists some $x \in X$ such that its orbit is dense in X, that is

$$\overline{Orb(\{T_n\}_{n\geq 0}, x)} = \overline{Orb(\{x, T_1x, T_2x, ...\}} = X.$$

In this case the vector x is called hypercyclic vector for the sequence $\{T_n\}_{n\geq 0}$.

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Note that if $\{T_n\}_{n\geq 0}$ is a hypercyclic sequence of operators on X, then X is necessarily separable. Also note that the sequence $\{T^n\}$ is a hypercyclic sequence on X if and only if the operator T is a hypercyclic operator on X.

Definition 1.2. An strictly increasing sequence of positive integers $\{n_k\}_k$ is said to be a syndetic sequence, if $sup_k(n_{k+1} - n_k) < \infty$.

Definition 1.3. A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ on the Banach space X is called syndetically hypercyclic if for any syndetic sequences of positive integers $\{k_{j(i)}\}_{j=1}^{\infty}$ for $i = 1, ..., n$, the sequence $\{T_1^{k_{j(1)}} ... T_2^{k_{j(n)}} : X \to X\}_{j(1),...,j(n)}$ is hypercyclic.

Definition 1.4. We say that the n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $k_i \in \mathbb{Z}_+$, $i = 1, ..., n$ such that $T_1^{k_1} {T_2}^{k_2} ... {T_n}^{k_n}(U) \cap V \neq \emptyset$.

Definition 1.5. A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called weakly mixing if for all open nonempty subsets U, V_1, V_2 of X, there exist $k(1), ..., k(n) \in \mathbb{N}$ such that

$$T_1^{k(1)} ... T_n^{k(n)} U \bigcap V_i \neq \emptyset$$

for i=1,2.

Definition 1.6. A tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open sets U and V, there exist positive integers $M(1)$, ..., $M(n)$ such that

$$T_1^{m(1)} ... T_n^{m(n)}(U) \cap V \neq \emptyset \quad , \quad \forall m(i) \geq M(i) \quad , \quad i = 1, ..., n.$$

Note that, all of the operators in this paper are considered to be continuous and linear on the space X.

2. Main Results

A nice criterion namely the hypercyclicity criterion is used in the proof of our main theorem. It was developed independently by Kitai ([7]), Gethner and Shapiro ([5]). This criterion has been used to show that hypercyclic operators arise within the class of composition operators, weighted shifts and adjoints of multiplication operators. For some source on this topics see $[1 - -18]$.

Note that, if a tuple $\mathcal{T}$ satisfies the hypercyclicity criterion for a syndetic sequence, then $\mathcal{T}$ is a topologically mixing tuple. Also, $\mathcal{T}$ is said to satisfy the Hypercyclicity Criterion if it satisfies the hypothesis of the following theorem.

Theorem 2.1. (Hypercyclicity Criterion) *Suppose that X is a separable infinite dimensional Banach space and $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. If there exist two dense subsets Y and Z in X, and strictly increasing sequences $\{m_{j(i)}\}_j$ for $i = 1, ..., n$ such that:*

1. $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} y \to 0$ for all $y \in Y$;

2. There exist a sequence of functions $\{S_j : Z \to X\}$ such that for every $z \in Z$, $S_j z \to 0$, and $T_1^{m_{j(1)}} ... T_n^{m_{j(n)}} S_j z \to z$, then $\mathcal{T}$ is a hypercyclic tuple.

The idea of the proof in the following theorem follows from [8].

Theorem 2.2. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be the n-tuple of operators T_1, T_2, ..., T_n acting on X. Then the followings are equivalent:*

(i) T is weakly mixing.

(ii) For any pair of non-empty open subsets U,V in X, and for any syndetic sequences $\{k_{j(i)}\}_j$ for $i = 1, ..., n$, there exist $j_0(i)$ in $\mathbb{N}$ such that $T_1^{k_{j_0(1)}} ... T_2^{k_{j_0(n)}} U$ $\cap V \neq \emptyset$ for $i = 1, ..., n$.

(iii) It suffices in (ii) to consider only those sequences $\{k_{j(i)}\}_j$ for which there is some $k(i) \geq 1$ with $k_{j(i)} \in \{k(i), 2k(i)\}$ for all k and $i = 1, ..., n$.

Proof. $(i) \to (ii)$: Given $\{k_{j(i)}\}_j$ for $i = 1, ..., n$, and U, V satisfying the hypothesis of condition (ii), put $k(i) = sup\{k_{(j+1)(i)} - k_{j(i)}\}_k$ for $i = 1, ..., n$. Since $\mathcal{T}$ is weakly mixing, one can see that for $i = 1, ..., n$ there exist $p(i) \in \mathbb{N}$ such that

$$(T_1^{p(1)} ... T_n^{p(n)}(U)) \bigcap ((T_1^{j(1)})^{-1} ... (T_n^{j(n)})^{-1}(V)) \neq \emptyset$$

for all $j(i) = 1, 2, ..., k(i)$ for $i = 1, 2, ..., n$. This implies that

$$T_1^{p(1)+j(1)} ... T_n^{p(n)+j(n)}(U) \bigcap V \neq \emptyset$$

for all $i = 1, 2, ..., n$. By the assumption on $\{k_{j(i)}\}_j$ we have that

$$\{k_{j(i)} : j \in \mathbb{N}\} \cap \{p(i) + 1, p(i) + 2, ..., p(i) + k(i)\} \neq \emptyset$$

for $i = 1, ..., n$.

If we select $k_{j_0(i)}$ $(i = 1, ..., n)$, in those intersections, then we get

$$T_1^{k_{j_0(1)}} ... T_n^{k_{j_0(n)}} U \cap V \neq \emptyset.$$

$(ii) \to (iii)$. This case is trivial.

$(iii) \to (i)$: Suppose that U, V_1, V_2 are non-empty open subsets of X. We will show that there exist $k(i) \in \mathbb{N}$ $(i = 1, ..., n)$ such that

$$T_1^{k(1)} ... T_n^{k(n)} U \cap V_i \neq \emptyset,$$

for $i = 1, 2$. This will imply that $\mathcal{T}$ is weakly mixing. Since (iii) is satisfied, then we can take $k(i) \in \mathbb{N}$ for $i = 1, ..., n$ such that

$$T_1^{k(1)} ... T_n^{k(n)} V_1 \cap V_2 \neq \emptyset.$$

By continuity, we can find $\widetilde{V_1} \subset V_1$ open and non-empty such that

$$T_1^{k(1)} ... T_n^{k(n)} \widetilde{V_1} \subset V_2.$$

Also, there exist some $p(i) \in \mathbb{N}$ such that

$$T_1^{p(1)+m(1)} ... T_n^{p(n)+m(n)} U \bigcap \widetilde{V_1} \neq \emptyset$$

for $m(i) = 0, k(i)$ for $i = 1, ..., n$. Otherwise we would find strictly increasing sequences of positive integers $\{k_{j(i)}\}_j$ such that $k_{(j+1)(i)} - k_{j(i)} \in \{k(i), 2k(i)\}$ for $i = 1, ..., n$, satisfying

$$T_1^{k_{\mathrm{j}(1)}} ... T_n^{k_{\mathrm{j}(\mathrm{n})}} U \cap \widetilde{V_1} = \emptyset$$

for all $j \in \mathbb{N}$. Now we have

$$T_1^{p(1)+k(1)} ... T_n^{p(n)+k(n)} U \bigcap \widetilde{V_1} \neq \emptyset,$$

and

$$T_1^{k(1)} ... T_2^{k(n)} (T_1^{p(1)} ... T_n^{p(n)} U \cap \widetilde{V_1}) \subset (T_1^{p(1)+k(1)} ... T_n^{p(n)+k(n)} U) \cap (T_1^{k(1)} ... T_n^{k(n)} \widetilde{V_1}).$$

If we fix $l_i = p(i) + m(i)$ for $i = 1, ..., n$, we conclude that

$$T_1^{l_1} ... T_n^{l_{\mathrm{n}}} U \cap V_i \neq \emptyset$$

for $i = 1, 2$. This completes the proof. $\square$

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