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EMBEDDING THE STRATIFIED SINGULAR QUIVER VARIETY

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Abstract: A stratified singular quiver variety resulting from the action of a complex reductive Lie group on a smooth tangent bundle is embedded in a dual of an infinite dimensional Lie algebra canonically associated to a quiver Q.

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1. Introduction

A quiver $Q = (Q_0, Q_1, s, t : Q_1 \longrightarrow Q_0; h\colon s(h) \longrightarrow t(h))$ consists of a finite set of vertices $Q_0 = \{1, \cdots, n\}$ and a finite set Q_1 of oriented arrows (or edges) together with structural maps $s, t\colon Q_1 \longrightarrow Q_0$ called respectively source and target maps, where n is a positive integer.

A representation of a quiver Q is a collection

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$(V,\varphi) = (V_k,\varphi_h)_{k\in Q_0,h\in Q_1}$ of finite dimensional $\mathbb{C}$-vector spaces together with a collection
$(\varphi_h\colon V_{s(h)} \longrightarrow V_{t(h)}; h \in Q_1)$ of $\mathbb{C}$-linear maps.

A morphism $F : V \longrightarrow W$ between two representation spaces (V,φ) and (W,ψ) of Q is a collection $(F_k\colon V_k \longrightarrow W_k)_{k\in Q_0}$ of $\mathbb{C}$-linear maps such that

$$F_{t(h)} \circ \varphi_h = \psi_h \circ F_{s(h)}, \text{ for all } h \in Q_1.$$

Let $G = \underset{k\in Q_0}{\oplus} GL_{\mathbb{C}}(V_k)$ be a complex reductive Lie group associated to the representation spaces (V,φ), (W,ψ) of the quiver Q of finite type. The families $\{V_k\}_{k\in Q_0}$ and $\{W_l\}_{l\in Q_0}$ of $\mathbb{C}$-vector spaces have dimension vectors

$$v = (dimV_1,\cdots,dimV_n) \in \mathbb{Z}^{Q_0}_{\geq 0},\ w = (dimW_1,\cdots,dimW_n) \in \mathbb{Z}^{Q_0}_{\geq 0},$$

respectively and the collections $\{\varphi_h\}_{h\in Q_1}$, $\psi_l := \{(a_l,b_l)\}_{l\in Q_0}$ of $\mathbb{C}$-linear maps: $V_{s(h)=k} \overset{\varphi_h}{\longrightarrow} V_{t(h)=l} \underset{a_l}{\overset{b_l}{\rightleftarrows}} W_l$. Let $\overline{Q}$ be another quiver obtained from Q by adjoining a reverse arrow $h^\star$ (or $\overline{h} \in Q_1$) to every arrow $h \in Q_1$, where $Q_1 = \Omega\cup\overline{\Omega}$ and $\Omega\cap\overline{\Omega} = \emptyset$ such that $\Omega \subset Q_1$ has arrows with a chosen fixed orientation and $\overline{\Omega}$ its reverse.

The affine algebraic variety [4] and [12]

$$Rep_{\overline{Q}}(v,w) := Rep_{\Omega}(v,w) \oplus Rep_{\overline{\Omega}}(v,w),$$

is endowed with a symplectic 2-form ω such that it is a complex symplectic manifold $(Rep_{\overline{Q}}(v,w),\omega)$.

The Hamiltonian action of the complex reductive Lie group G on $Rep_{\overline{Q}}(v,w)$ is give by,

$$\eta\colon G \times Rep_{\overline{Q}}(v,w) \longrightarrow Rep_{\overline{Q}}(v,w) : (g,\varphi)(h) \mapsto g_{t(h)}.\eta(h).g^{-1}_{s(h)}.$$

The coadjoint action $Ad^* : G \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is defined on the dual of the Lie algebra $\mathcal{G} = \mathrm{Lie}(G)$ of G. These operations together with the equivariant momentum map:

$$\begin{aligned} \Phi : \quad & Rep_{\overline{Q}}(v,w) \longrightarrow \mathcal{G}^* \\ & (\varphi,a,b) \mapsto \Phi(\varphi,a,b) := \epsilon\varphi\varphi + ab = \lambda, \end{aligned}$$

where $\mathcal{G}^*$ is the dual of the Lie algebra $\mathcal{G} := Lie(G)$, and $\lambda \in \mathcal{O}_\lambda \subset \mathcal{G}^*$, a singular value of the momentum map, $\mathcal{O}_\lambda$ a closed coadjoint orbit through λ, give rise via the pre-image $\Phi^{-1}(\mathcal{O}_\lambda)$ to an affine algebraic variety. This results in the construction of the variety:

$$\begin{aligned} Rep^{\lambda}_{\overline{Q}}(v,w) &:= \Phi^{-1}(\mathcal{O}_{\lambda})//_{G} \\ &= Spec(\mathbb{C}[Rep_{\overline{Q}}(v,w)]^{G}/I(\Phi^{-1}(\mathcal{O}_{\lambda}))^{G}), \end{aligned}$$

which is made into a disjoint union of smooth strata or smooth symplectic pieces.

More precisely, suppose H is a compact subgroup of G and $Rep_{\overline{Q}}(v,w)_{(H)}$ the corresponding orbit type. The partition of $Rep_{\overline{Q}}(v,w)$ into orbit types is locally finite. This implies that the partition of $\Phi^{-1}(\mathcal{O}_{\lambda})$ by $\Phi^{-1}(\mathcal{O}_{\lambda}) \cap Rep_{\overline{Q}}(v,w)_{(H)}$ is also locally finite. Thus, for every compact subgroup $H \subset G$, a partition of the singular reduced space $Rep^{\lambda}_{\overline{Q}}(v,w)$ into smooth symplectic manifolds is given by

$$Rep^{\lambda}_{\overline{Q}}(v,w) = \coprod_{(H)} Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)},$$

where the disjoint union is taken over the set of all conjugacy classes of subgroups H of the complex reductive Lie group $G = \bigoplus_{k \in Q_0} GL_{\mathbb{C}}(V_k)$.

The purpose of this section is to provide the background material needed to facilitate the construction of an embedding of a stratified singular quiver variety into the dual of a special infinite dimensional Lie algebra canonically associated to a quiver of finite type.

The motivation for this construction comes from the paper of Victor Ginzburg [5]. Theorem [5, Theorem 1.2] of the paper deals among other matters with the embedding of a nonsingular algebraic variety called an affine quiver variety obtained from the representation space of a quiver of finite type into the dual space of an appropriate infinite dimensional Lie algebra attached to a quiver ([2], [3], [5], [10], [11], [12], [13], and [14]).

The main objective of this article is to present a generalization of theorem [5, Theorem 1.2] of Victor Ginzburg by constructing in the general case an embedding of a stratified singular quiver variety with strata or symplectic pieces which are smooth manifolds, and gluing the embedded symplectic pieces to realize an embedding of the entire stratified singular quiver variety. To obtain the embedding, we use various techniques used in the papers [18], [12] - [15], and [5].

In what follows, we first prove that for any isotropy subgroup $H \subset G$, the symplectic manifold $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$ can be embedded as a coadjoint orbit in the dual space $\mathcal{L}(Q)^*$ of the Lie algebra $\mathcal{L}(Q)$ associated to the quiver Q. Then, under suitable hypotheses, the singular quiver variety $Rep^{\lambda}_{\overline{Q}}(v,w)$ is finally embedded in $\mathcal{L}(Q)^*$.

The article is organized as follows: Section 1 is a brief introduction stating the motivation and giving the requisite background on the main objects of our study. Section 2 is devoted to the embedding of the smooth strata. The final section 3 contains the statements of the main results of the paper and their proofs.

2. Embedding Smooth Strata

Let $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$ be the set of all points $\bar{q}=[(\varphi,a,b)]\in Rep^{\lambda}_{\overline{Q}}(v,w)$ of G-orbit type (H) with representative element $q=(\varphi,a,b)\in Rep_{\overline{Q}}(v,w)$, where H is a compact subgroup of complex reductive Lie group $G=\sqcap_{k\in Q_0} GL_{\mathbb{C}}(V_k)$. In the theorems that follow, we will occasionally use the following description of the symplectic piece $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$, due to H. Nakajima (see [13]):

$$Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} = Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)\times Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})$$

as a product of two smooth manifolds for every subgroup H of G, where

$$Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w) := \{m=(\varphi,a,b)\in Rep_{\overline{Q}}(v^{(0)},w) : G_m=\{e_G\}\}/_{\sqcap_{k\in Q_0} GL(V_k^{(0)})},$$

and

$$Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)}) := \{(\varphi,a,b)\in Rep_{\overline{Q}}(v,w):\ \varphi$$

satisfies properties (1), (2), (4) and (6) of Lemma 3.27, see [14], the stabilizer

$$G_{(\varphi,a,b)}=\{e_G\};\text{ such that } a=b=0\}/\sqcap_{j=1}^{r} G_{v^{(j)}},$$

with

$$G_{v^{(0)}} := \sqcap_{k\in Q_0} GL_{\mathbb{C}}(V_k^{(0)}),\quad G_{v^{(j)}} := \sqcap_{k\in Q_0} GL_{\mathbb{C}}(V_k^{(j)}),$$

and $\hat{v}_l,\ v_i,\ v^j,\ l,i,j\in Q_0$.

Lemma 1. *Let $G=\sqcap_{k\in Q_0} GL_{\mathbb{C}}(V_k)$ be a complex reductive Lie group and $H<G$ an isotropy group. Then the following equality holds:*

$$\mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)\times Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})] = \mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)]\otimes_{\mathbb{C}}\mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})],$$

where $Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)$ and $Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})$ are closed smooth manifolds.

Proof. To prove the claim, we need to show that the algebra homomorphism,

$$\begin{aligned}\psi \quad : \quad & \mathbb{C}[Rep^{\lambda}_{Q,reg}(v^{(0)},\ w)] \otimes_{\mathbb{C}} \mathbb{C}[Rep^{\lambda}_{Q,reg}(\hat{v_1},\ v^{(1)}, \cdots, \hat{v_r},\ v^{(r)})] \\ & \longrightarrow \mathbb{C}[Rep^{\lambda}_{Q,reg}(v^{(0)},\ w) \times Rep^{\lambda}_{Q,reg}(\hat{v_1},\ v^{(1)}; \cdots, \hat{v_r},\ v^{(r)})] \\ & (f \otimes g) \ \mapsto \ \psi(\textstyle\sum_i (f_i \otimes g_i)(x,\ y) = \sum f_i(x) g_i(y),\end{aligned}$$

is an isomorphism. To see this we have to define regular functions on the product of smooth manifolds:

$$Rep^{\lambda}_{Q,reg}(v^{(0)}, w) \times Rep^{\lambda}_{Q,reg}(\hat{v}_1, v^{(1)}, \cdots, \hat{v}_r, v^{(r)}).$$

Let $\mathbb{C}[Rep^{\lambda}_{Q,reg}(v^{(0)}, w)] := \mathbb{C}[T]/I(Rep^{\lambda}_{Q,reg}(v^{(0)}, w))$, where $T = (T_1, \cdots, T_s)$ is an indeterminate and

$$I(Rep^{\lambda}_{Q,reg}(v^{(0)}, w)) := \{F_i \in \mathbb{C}[T] :\ F_i | Rep^{\lambda}_{Q,reg}(v^{(0)}, w) = 0, 1 \leq i \leq s\}$$

the defining ideal of $Rep^{\lambda}_{Q,reg}(v^{(0)}, w)$. The details of the remaining we refer to [17]. □

Theorem 2. *(Embedding the stratum $Rep^{\lambda}_{Q}(v, w)_{(\hat{G})}$)*
Let $[m] = [(\varphi, a, b)] \in Rep^{\lambda}_{Q}(v, w)$ be a singular point and $\hat{G} \subset G$ the isotropy group of its representative m. Then the symplectic piece $Rep^{\lambda}_{Q}(v, w)_{(\hat{G})}$ can be embedded as a coadjoint orbit in the infinite dimensional Lie algebra $\mathcal{L}(Q)^*$, the dual of the Lie algebra $\mathcal{L}(Q)$ canonically associated with the quiver Q of finite type such that the symplectic structure on $Rep^{\lambda}_{Q}(v, w)_{(\hat{G})}$ is mapped under the embedding into the canonical Kirillov-Kostant-Souriau symplectic structure on the coadjoint orbit.

Proof. To prove the claim we will follow closely the constructions and ideas of the proof given by Victor Ginzburg ([5], Theorem 1.2), since the symplectic pieces, just as in V. Ginzburg's case, are smooth affine symplectic varieties. Let $[m] \in Rep^{\lambda}_{Q}(v, w)$ with representative m and $\hat{G} = G_m$ the stabilizer of m, a closed subgroup of G. To show that the stratum or symplectic piece

$$Rep^{\lambda}_{Q}(v, w)_{(\hat{G})} \equiv Rep^{\lambda}_{Q,reg}(v^{(0)}, w) \times Rep^{\lambda}_{Q,reg}(\hat{v}_1, v^{(1)}, \cdots, \hat{v}_r, v^{(r)})$$

satisfies the claims of the theorem, we consider the ring of regular functions defined in Lemma 2.1, i.e.,

$$\begin{aligned}&\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]\\&=\mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)\times Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})]\\&=\mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(v^{(0)},w)]\otimes_{\mathbb{C}}\mathbb{C}[Rep^{\lambda}_{\overline{Q},reg}(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})]\end{aligned}$$

and also we need to construct the following diagram of Lie algebra homomorphisms

$$\mathcal{L}(Q) \quad \overset{\Theta}{\longrightarrow} \quad \mathbb{C}[(Rep_{\overline{Q}}(v,w))_{(\hat{G})}]^G \quad \overset{\Pi}{\longrightarrow} \quad [(Rep^{\lambda}_{\overline{Q}}(v,w))_{(\hat{G})}]$$

where $[(Rep^{\lambda}_{\overline{Q}}(v,w))_{(\hat{G})}] = \mathbb{C}[(Rep_{\overline{Q}}(v,w))_{(\hat{G})}]^G/_{I(\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})})^G}$, the mapping Π is the canonical projection, Θ Lie algebra homomorphism and $\mathcal{L}(Q)$, an infinite dimensional Lie algebra canonically attached to the quiver of finite type Q still to be constructed, $\mathbb{C}[Rep_{\overline{Q}}(v,w)_{(\hat{G})}]^G$, the ring of G-invariant regular functions on

$Rep_{\overline{Q}}(v,w)_{(\hat{G})}$ and $I(\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})})^G \subset \mathbb{C}[Rep_{\overline{Q}}(v,w)_{(\hat{G})}]^G$ the ideal of G-invariant regular functions on the affine algebraic variety

$\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})}$, i.e.,

$$I(\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})}) := \{f \in \mathbb{C}[Rep_{\overline{Q}}(v,w)_{(\hat{G})}] : \ f|\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})} = 0\}.$$

It was shown in [4], [13], [16], and [18], that for any isotropy group $\hat{G} \subset G$, the symplectic piece or stratum $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$ is a symplectic manifold with a symplectic structure $(\omega_\lambda)_{(\hat{G})}$ satisfying the property that

$$\Pi^*_R(\omega_\lambda)_{(\hat{G})} = \omega|_{\Phi^{-1}(\mathcal{O}_\lambda)\cap Rep_{\overline{Q}}(v,w)_{(\hat{G})}} = \omega|_{\Phi^{-1}(\mathcal{O}_\lambda)_{(\hat{G})}},$$

using similar notations as in [4].

This symplectic structure makes the coordinate ring $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$ into an infinite dimensional Lie algebra with respect to the Poisson bracket $\{\cdot,\cdot\}_{(\omega_\lambda)_{(\hat{G})}}$ defined by the symplectic form $(\omega_\lambda)_{(\hat{G})}$. Define an evaluation map on $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$ as follows:

$$\begin{aligned}ev:\quad &Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})} \longrightarrow \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*\\&x\mapsto ev(f)(x)=f(x),\end{aligned}$$

where $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$ is an affine symplectic manifold and f an element of the infinite dimensional $\mathbb{C}$-vector space $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$, which is a Poisson manifold endowed with the Kirillov-Kostant-Souriau bracket. This implies that the evaluation map

$$ev: \quad Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})} \longrightarrow \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$$

is a morphism of Poisson manifolds and hence a Poisson map. Moreover, because $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$, the symplectic piece, is a smooth manifold, any regular function $f \in \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$ separates points of $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$. It then follows that for any point $x \in Rep_{\overline{Q}}(v,w)_{(\hat{G})}$, the tangent space to $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$ at the orbit $\hat{x} \in Rep^{\lambda}_{\overline{Q}}(v,w)$, with x a representative of $\hat{x}$, $T_x Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$, is spanned by the differentials of the regular functions in $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$. This shows that the evaluation map, ev, is injective. Furthermore, the image of ev, $ev(Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}) \subset \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$ and the coadjoint infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$ on $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$ restricts to an action on $ev(Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})})$ such that the following diagram

$$\begin{array}{ccc}
\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}] \times \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}]^* & \xrightarrow{ad^*} & \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}]^* \\
1_{\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}]} \times proj \downarrow & & \downarrow proj \\
\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}] \times ev(Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})}) & \xrightarrow{\mu = ad^*|} & ev(Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(\hat{G})})
\end{array}$$

commutes, where ad^* is the coadjoint infinitesimal Hamiltonian action of the Lie algebra $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$ on its dual Lie algebra, $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$, i.e., for any
$\xi \in \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]$ and $\alpha \in \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$, define

$$ad^*_{\xi}\alpha \text{ by } ad^*_{\xi}(\alpha)\eta = - < \alpha, [\xi,\eta] >= -\alpha([\xi,\eta]),\ \forall \eta \in \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}],$$

the mapping $proj$ is the Lie algebra projection map and $\mu := ad^*|$, the restriction of the coadjoint infinitesimal Hamiltonian action is an infinitesimally transitive action. It is then clear that via the evaluation embedding $Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}$ becomes a coadjoint orbit in $\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^*$. If we now dualize the Lie algebras and their Lie algebra morphisms:

$$\mathcal{L}(Q) \quad \xrightarrow{\Theta} \quad \mathbb{C}[(Rep_{\overline{Q}}(v,w))_{(\hat{G})}]^G \quad \xrightarrow{\Pi} \mathbb{C}[(Rep^{\lambda}_{\overline{Q}}(v,w))_{(\hat{G})}]$$

we obtain the following diagram of Poisson morphisms:

$$\begin{array}{ccccc}
Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})} & \overset{ev}{\hookrightarrow} & \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(\hat{G})}]^* & \xrightarrow{\Pi^*} & (\mathbb{C}[Rep_{\overline{Q}}(v,w)_{(\hat{G})}]^G)^* \\
& \searrow_{v} & & \swarrow \Theta^* & \\
& & \mathcal{L}(Q)^*, & &
\end{array}$$

where $\upsilon := \Theta^* \circ \Pi^* \circ ev$. We then need to do two things in order to be able to prove the theorem:

(1) Again following V. Ginzburg [5] and R. Bocklandt and L. Le Bruyn [1], we should construct the Lie algebra $\mathcal{L}(Q)$ and the morphism Θ, and

(2) Show that the composite morphism $\upsilon = \Theta^* \circ \Pi^* \circ ev$ is a monomorphism. The proof will be completed after the construction of the Lie algebra $\mathcal{L}(Q)$ and the series of Lemmas, Properties and Theorem 2.3 below. □

The proof is the same line as the one given by V. Ginzburg [5, Theorem 1.2].

However, fix the following notation $\otimes = \otimes_{\mathbb{C}}$ to be used in the sequel, and a unital $\mathbb{C}$-algebra [5]

$$B := \mathbb{C}^{Q_0} = \underbrace{\mathbb{C} \times \cdots \times \mathbb{C}}_{\text{n times}} = \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\text{n-summands}}.$$

For each vertex $l \in Q_0$, $1_l \in \mathbb{C}$ denotes the idempotent corresponding to the l^{th} direct summands of $\mathbb{C}$. It is then immediate that B is a semi-simple $\mathbb{C}$-algebra generated by the vertex idempotents. Define $\Gamma_{\overline{Q}}$ [5] as the $\mathbb{C}$-vector space generated by the set of arrows $\{h^\star \in Q_1\}$. Then clearly $\Gamma_{\overline{Q}}$ has a B-bimodule structure.

Define a B-bimodule $T^k_B(\Gamma_{\overline{Q}}) := \underbrace{\Gamma_{\overline{Q}} \otimes_B \Gamma_{\overline{Q}} \otimes_B \cdots \otimes_B \Gamma_{\overline{Q}}}_{\text{k copies}}$.

For the finite B-bimodule $\Gamma_{\overline{Q}}$, define $A = T_B(\Gamma_{\overline{Q}}) := \bigoplus_{k \geq 0} T^k_B(\Gamma_{\overline{Q}})$ to be the graded associative tensor algebra such that $T^0_B(\Gamma_{\overline{Q}}) = B$. According to Ginzburg [5, Theorem 2.5], $\mathcal{L}(Q) := A/[A, A]$ is a Lie algebra, and the coordinate ring denoted by

$$\begin{aligned} \mathcal{D}em(A, v, w) \quad &= \mathbb{C}[Hom_{B-alg}(A, EndV) \\ &\oplus Hom_{B-alg}(A, Hom(W, V)) \\ &\oplus Hom_{B-alg}(A, Hom(V, W))] \end{aligned}$$

admits a G-invariant Poisson structure $\{\cdot,\cdot\}_{\omega_{Rep}}$. Therefore, the G-invariant regular functions $\mathcal{D}em(A, v, w)^G \subset \mathcal{D}em(A, v, w)$ form a Poisson subalgebra. Now, for any isotropy group $H < G$, we are interested in embedding the stratum $Rep^{\lambda}_{\overline{Q}}(v, w)_{(H)}$, which is a smooth affine subvariety of $Spec(\mathcal{D}em(A, v, w)^G)$, in the dual $\mathcal{L}(Q)^*$ of the infinite dimensional Lie algebra $\mathcal{L}(Q) := A/[A, A]$ equipped with the Lie-Poisson structure $\{\cdot,\cdot\}_\omega$.

Theorem 3. (Embedding of Smooth Strata) *The composite map*

$$\begin{array}{llll} Emb: & Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} & \hookrightarrow & Spec(\mathcal{D}em(A,v,w)^{G}) \\ & & & \downarrow evaluation \\ & & & (\mathcal{D}em(A,v,w)^{G})^{*} \\ & & \mathcal{L}(Q)^{*} \underset{trace^{*}}{\longleftarrow} & \end{array}$$

is injective and makes $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$ a coadjoint orbit of $\mathcal{L}(Q)^{*}$.

Proof. It is enough to show that any regular functions on $Spec(\mathcal{D}em(A,v,w)^{G})$ of the form $trace(\hat{a})$, for every $a \in \mathcal{L}(Q)$, separate points and are tangent to the smooth affine subvariety or stratum

$$Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} \subset Spec(\mathcal{D}em(A,v,w)^{G}).$$

The coordinate ring

$$\mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}]^{G} \subset \mathcal{D}em(A,v,w)^{G}$$

and hence it is true for the invariant regular functions on $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$ which are obtained by restricting the invariant regular functions of $\mathcal{D}em(A,v,w)^{G}$ and the evaluation map

$$\begin{array}{ll} ev: & Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} \to \mathbb{C}[Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}]^{*} \\ & x \longrightarrow ev(f)(x) = f(x) \end{array}$$

is injective, similar to the case of the previously defined evaluation map. V. Ginzburg [5] gives a detailed proof of this theorem, where we can refer the readers. □

However, the theorem 2.2 shows clearly that the stratum $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$ can be embedded in $\mathcal{L}(Q)^{*}$ as a coadjoint orbit. For each isotropy group $H < G$ [6, 16] we have the following decompositions:

$$Rep_{\overline{Q}}(v,w) = \coprod_{(H)} Rep_{\overline{Q}}(v,w)_{(H)}$$

and

$$Rep^{\lambda}_{\overline{Q}}(v,\ w) = \coprod_{(H)} Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)}$$

are stratified manifolds, where,

$$Rep_{\overline{Q}}(v,w)_{(H)} = \{(\varphi,\ a,\ b) \in Rep_{\overline{Q}}(v,w):\ G_{(\phi,\ a,b)} \text{ is conjugate to } H\}$$

and

$$Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} = (\Phi^{-1}(\mathcal{O}_{\lambda}) \cap Rep_{\overline{Q}}(v,w)_{(H)})//G,$$

with $\Phi^{-1}(\mathcal{O}_{\lambda})$ the pre-image of the coadjoint orbit $\mathcal{O}_{\lambda}$.

The conjugate class of the isotropy group H [13] and [16] and the set of conjugacy classes of closed subgroups of a Lie group G is ordered, with order the reverse to the order of conjugacy class. The class $(H) < (K)$ is equivalent to K being conjugate to a subgroup of H. Hence, for any isotropy groups $H, K < G$ with $(H) < (K)$, the frontier property [20] and [16], implies that $Rep^{\lambda}_{\overline{Q}}(v,w)_{(H)} \cap \overline{Rep^{\lambda}_{\overline{Q}}(v,w)}_{(K)} \neq \varnothing$, is open and dense in $Rep^{\lambda}_{\overline{Q}}(v,\ w))_{(H)}$.

Consider conjugate subgroups K of H subgroup of G, then there is a finite collection $\{H_k\}_{k\in R_0\subseteq Q_0}$ of subgroups which are isotropy subgroups satisfying the frontier property. So, for all $H_k < G, \forall k = 0, \cdots, n-1$ we have

$$(H_{n-1}) < (H_{n-2}) < \cdots < (H_0),$$

where

$$H_0 < H_1 < \cdots < H_{n-2} < H_{n-1} < G = H_n,$$

where the set of indices $R_0 \subseteq Q_0$. The smooth space [19], and [16]

$$Rep^{\lambda}_{\overline{Q}}(v,w))_{(H)_{princ.}} := Rep^{\lambda}_{\overline{Q}}(v,w)_{(H_{max})},$$

is a principal stratum and it is defined by the following conditions:

$$(H_0) = (H_{max}) > (H_1), \quad \text{with} \quad \overline{Rep^{\lambda}_{\overline{Q}}(v,\ w)}_{(H_{max})} \cap Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_1)}$$

open and dense in $Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_1)}$, similarly

$$(H_2) < (H_1), \quad \text{with} \quad Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_2)} \cap \overline{Rep^{\lambda}_{\overline{Q}}(v,\ w)}_{(H_1)}$$

open and dense in $Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_2)}$. So continuing this process, we obtain a sequence ending at

$$(H_{n-1}) < (H_{n-2}), \quad \text{with} \quad Rep^{\lambda}_{\overline{Q}}(v,\ w))_{(H_{n-1})} \cap \overline{Rep^{\lambda}_{\overline{Q}}(v,\ w)}_{(H_{n-2})},$$

where

$$Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_i)} \cap \overline{Rep^{\lambda}_{\overline{Q}}(v,\ w)}_{(H_i)} = Rep^{\lambda}_{\overline{Q}}(v,\ w)_{(H_i)}.$$

Then the principal stratum is defined as

$$\begin{aligned}
&Rep^{\lambda}_{Q}(v,\ w)_{(H_{max})}\\
&= [\overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_{max})} \cap Rep^{\lambda}_{Q}(v,\ w)_{(H_1)}] \cap \cdots\\
&\cdots \cap [Rep^{\lambda}_{Q}(v,\ w)_{(H_{n-1})} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_{n-2})}],
\end{aligned}$$

However, if the collection $\{H_i\}_{i \in R_0}$ does not satisfy the frontier property, then the principal stratum does not exist.

For any $k \in R_0$, the coordinate ring $\mathbb{C}[Rep^{\lambda}_{Q}(v,\ w)_{(H_k)})]$ is a sheaf of smooth functions on $Rep^{\lambda}_{Q}(v,\ w)_{(H_k)}$. Hence, each stratum is a ringed space.

For all $H_i < G$, $\forall\ i \in R_0$, the map :

$$\begin{array}{ccc}
Rep_{\overline{Q}}(v,\ w)_{(H_i)} & \xrightarrow{\pi_{H_i}} & Rep^{\lambda}_{Q}(v,\ w)_{(H_i)}\\
& & \swarrow \exists\ \beta_{H_i}\\
\cup & & \\
\pi^{-1}_{H_i} Rep^{\lambda}_{Q}(v,w)_{(H_i)} & &
\end{array}$$

is a projection map onto the i^{th} summand such that the above diagram commutes. The map

$$\beta_{H_i}\ :\ Rep^{\lambda}_{Q}(v,\ w)_{(H_i)} \longrightarrow \pi^{-1}_{H_i} Rep^{\lambda}_{Q}(v,\ w)_{(H_i)}$$

is an isomorphism for every $i \in R_0$. We can then construct a family $\{\beta_{H_i}\}_{H_i < G}$ of isomorphisms. Let

$$Y_i := \pi^{-1}_{H_i} Rep^{\lambda}_{Q}(v,\ w)_{(H_i)},\ \ X_i := Rep^{\lambda}_{Q}(v,\ w)_{(H_i)}$$

be two ringed spaces.

The open subsets $Y_{ij} := \beta_{H_i}[(Rep^{\lambda}_{Q}(v,\ w)_{(H_i)} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_j)})]$, of Y_i which can be regarded as ringed spaces since each stratum has coordinate ring as a ring of smooth functions on to it . For all $H_i,\ H_j < G$, define the isomorphism:

$$\begin{aligned}
\chi_{ij}\ &:= \beta_{H_j} \circ \beta^{-1}_{H_i}|_{Y_{ij}} : \beta_{H_i}[Rep^{\lambda}_{Q}(v,\ w)_{(H_i)} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_j)}]\\
\longrightarrow\ &\beta_{H_j}[Rep^{\lambda}_{Q}(v,\ w)_{(H_i)} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_j)}]
\end{aligned}$$

satisfying the following conditions:

(1) $Y_i = Y_{ii}$ and $\chi_{ii} = id_{Y_i}$

(2) $\chi_{ij} \circ \chi_{ji} = id_{Y_{ij}}$

(3) $\chi_{ij} \circ \chi_{jk}|_{Y_{ijk}} = \chi_{ik}$, where $Y_{ijk} = Y_{kj} \cap \chi_{jk}^{-1}(Y_{ji})$
with $Y_{kj} := \beta_{H_k}[Rep_{Q}^{\lambda}(v,\ w)_{(H_k)} \cap \overline{Rep_{Q}^{\lambda}(v,\ w)}_{(H_j)}]$, and
$(H_i) < (H_j) < (H_k)$, $for\ H_k < H_j < H_i < G$.

Lemma 4. (Gluing Lemma, [8, Lemma 1.32]) *Given a collection* $\{Y_i := \pi_{H_i}^{-1} Rep_{Q}^{\lambda}(v,\ w)_{(H_i)}\}_{H_i<G}$ *of ringed spaces, a collection*

$$\{Y_{ij} := Rep_{Q}^{\lambda}(v,\ w)_{(H_i)} \cap \overline{Rep_{Q}^{\lambda}(v,\ w)}_{(H_j)}\ :\ H_i,\ H_j < G\}$$

of open subsets

$$Y_{ij} := \beta_{H_i}[Rep_{Q}^{\lambda}(v,\ w)_{(H_i)} \cap \overline{Rep_{Q}^{\lambda}(v,\ w)}_{(H_j)}];$$

of $Y_i := \pi_{H_i}^{-1}(Rep_{Q}^{\lambda}(v,\ w)_{(H_i)})$ *and a collection*

$$\{\chi_{ij}\ :\ (H_i) < (H_j),\ for\ H_j < H_i < G\}$$

of isomorphisms $\chi_{ij}\ :\ Y_{ij} \longrightarrow Y_{ji}$ *satisfying the above three conditions, there exists a ringed space* $Rep_{Q}^{\lambda}(v,\ w)$*, an open cover*

$$\{X_i := Rep_{Q}^{\lambda}(v,\ w)_{(H_i)}\}_{i\in R_0}$$

of $Rep_{Q}^{\lambda}(v,\ w)$*, and a collection*

$$\{\beta_{H_i}\ :\ Rep_{Q}^{\lambda}(v,\ w)_{(H_i)} \longrightarrow \pi_{H_i}^{-1} Rep_{Q}^{\lambda}(v,\ w)_{(H_i)}\}$$

of isomorphisms such that

$$\chi_{ij} = \beta_{H_i} \circ \beta_{H_j}^{-1}|_{Y_{ij}}\ for\ H_i,\ H_j < G\ with\ (H_i) < (H_j)\ ;\ \forall i,j \in R_0.$$

Proof. The proof follows the same lines of reasoning as that given by S. Iitaka ([8, Lemma 1.32]). □

3. Main Results

The statements of our main results and their proofs, which to a large extent follow directly from the constructions on Section 2 are presented here.

Theorem 5. *For any finite collection of isotropy groups* $\{H_i\}_{i\in R_0}$ *of* G*, where* R_0 *is the set of indices in reverse order satisfying the conditions:*

(1) $\{e_G\} < H_{max} = H_0 < H_1 < \cdots < H_{n-1} < G$

(2) $(H_0) = (H_{max}) > (H_1) > \cdots > (H_{n-1})$

(3) $(H_i) < (H_j), Rep^{\lambda}_{Q}(v,\ w)_{(H_i)} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_j)})$ *is open and dense in* $Rep^{\lambda}_{Q}(v,\ w)_{(H_i)}$*, for any* $i,\ j \in R_0$*, with* (H_i) *the conjugacy class of the isotropy group* $H_i < G$ *for any* $i \in Q_0$*.*

Then the principal stratum

$$\begin{aligned}&Rep^{\lambda}_{Q}(v,\ w)_{(H_{max})} =\\&= [\overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_{max})} \cap Rep^{\lambda}_{Q}(v,\ w)_{(H_1)}] \cap \cdots\\&\cdots \cap [Rep^{\lambda}_{Q}(v,\ w)_{(H_{n-1})} \cap \overline{Rep^{\lambda}_{Q}(v,\ w)}_{(H_{n-2})}],\end{aligned}$$

can be embedded as a coadjoint orbit in $\mathcal{L}(Q)^*$*, the dual of the infinite dimensional Lie algebra* $\mathcal{L}(Q)$ *canonically associated with the quiver* Q*.*

Proof. The proof is the same as the proof of Theorem 2.1. Thus, $\forall k \in R_0$ each $Rep^{\lambda}_{Q}(v,\ w)_{(H_i)}$ is embedded in $\mathcal{L}(Q)^*$. Theorem 2.1 along with Lemma 2.2(Glueing Lemma) show, the principal stratum $Rep^{\lambda}_{Q}(v,\ w)_{(H_{max})}$ is embedded as a coadjoint orbit in $\mathcal{L}(Q)^*$. □

In Sections 2 and 3 above, it was shown that $\forall k \in R_0$, the strata

$$Rep^{\lambda}_{Q}(v, w)_{(H_k)},$$

and the principal stratum

$$Rep^{\lambda}_{Q}(v,\ w)_{princ.} := Rep^{\lambda}_{Q}(v,\ w)_{(H_{max})}$$

are embedded as coadjoint orbits in the dual $\mathcal{L}(Q)^*$ of the infinite dimensional Lie algebra $\mathcal{L}(Q)$. In this section, we prove that the singular quiver variety $Rep^{\lambda}_{Q}(v,\ w)$ under the conditions of sections (2) and (3) can be embedded in $\mathcal{L}(Q)^*$ as a coadjoint orbit.

Theorem 6. *Under hypotheses of Theorem 2.1 and Theorem 3.1,*

(1) For any isotropy group $H_k < G,\ \forall\ k = 1, \cdots, n$*, the strata* $Rep^{\lambda}_{Q}(v,\ w)_{(H_k)}$ *can be embedded in* $\mathcal{L}(Q)^*$*.*

(2) The principal stratum denoted by $Rep_Q^\lambda(v,\ w)_{(H_{max})}$ *can also be embedded in* $\mathcal{L}(Q)^*$.

The stratified singular quiver variety $Rep_Q^\lambda(v,\ w) = \coprod_{(H)} Rep_Q^\lambda(v,\ w)_{(H)}$ *can be embedded as a coadjoint orbit in* $\mathcal{L}(Q)^*$*, where* $\mathcal{L}(Q)^*$ *is the dual of the infinite dimensional Lie algebra* $\mathcal{L}(Q)$.

Proof. To prove the conditions (1) and (2) of the theorem, we follow the proofs of Theorem 2.1 and Theorem 3.1 so that the strata $Rep_Q^\lambda(v,w)_{(H)}$ are individually embedded making use of the Gluing Lemma and the frontier property. □

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