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HAMILTONIAN REDUCTION AND MAURER-CARTAN EQUATIONS

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Abstract: A stratified scheme is embedded in the dual of an infinite dimensional Lie subalgebra associated to a Lie super-quiver Q.

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1. Introduction and Background

A quiver $Q = (Q_0, Q_1, s, t : Q_1 \longrightarrow Q_0; h\colon s(h) \longrightarrow t(h))$ consists of a finite set of vertices $Q_0 = \{1, \cdots, n\}$ and a finite set Q_1 of oriented arrows (or edges) together with structural maps $s, t\colon Q_1 \longrightarrow Q_0$ called respectively source and target maps, where n is a positive integer [5] and [18].

Definition 1. A subgraph of a graph (Q_0, Q_1) is the graph $(S(Q_0), S(Q_1))$ such that

$$S(Q_0) \subseteq Q_0, \quad S(Q_1) \subseteq Q_1.$$

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Assume that $(S(Q_0),\ S(Q_1)) := (Q_0^+,\ Q_1^+)$ is the subgraph of the graph (Q_0, Q_1), where Q_0^+ and Q_1^+ are respectively subsets of Q_0 and Q_1, and

$$Q^+ = (Q_0^+,\ Q_1^+,\ s^+,\ t^+ : Q_1^+ \longrightarrow Q_0^+)$$

a subquiver of the quiver Q of finite type, with $s^+(h)$ and $t^+(h)$ respectively the source and the target of the arrow $h \in Q_1^+$.

Definition 2. (see G. Lusztig [17]) A graph is said to be **even** if it satisfies the following **evenness** property:

There exists a partition $Q_0 = Q_0^+ \cup Q_0^-$ such that for any $h \in Q_1$, the vertices $s(h), t(h) \in Q_0$ cannot be both in Q_0^+ or both in Q_0^-.

Suppose Q_0^+, Q_0^-, $\delta = \overset{+}{-}$ is fixed, and the orientation is defined as

$$\Omega^\delta := \{h \in Q_1 :\ s(h) \in Q_0^\delta,\ t(h) \in Q_0^{-\delta}\},$$

where the reverse orientation is defined by $\overline{\Omega}^\delta = \Omega^{-\delta}$. Note (Q_0, Ω^δ) has no cycles other than the trivial ones.

Consider G a finite $\mathbb{C}$-dimensional Lie super-group which admits the following $\mathbb{Z}_2$-graded decomposition: $G := G_+ \oplus G_-$ such that $G_+ \cap G_- = \{e_G\}$, and its Lie super-algebra:

$$\begin{aligned} \mathcal{G} &= \mathrm{Lie}(G) \\ &= T_{e_G} G \\ &= \mathcal{G}_+ \oplus \mathcal{G}_- \ . \end{aligned} \tag{1.1}$$

It is a differential $\mathbb{Z}_2$-graded Lie super-algebra with the differential map

$$d : \mathcal{G}_{\overset{+}{-}} \longrightarrow \mathcal{G}_{\underset{+}{-}}$$

on $\mathcal{G}$, where e_G is the unit element of the Lie super-group G, $\mathcal{G}_+ := \mathrm{Lie}(G_+)$ and $\mathcal{G}_- := \mathrm{Lie}(G_-)$ such that $\mathcal{G}_+ \cap \mathcal{G}_- = \{0\}$.

Definition 3. A matrix C is called decomposable if it is equivalent to a matrix of a block-diagonal form:

$$C = \begin{pmatrix} C_+ & 0 \\ 0 & C_- \end{pmatrix}$$

where the matrices $C_{\underset{+}{-}}$ are the indecomposable associated generalized symmetric Cartan matrices of the Lie subalgebras $\mathcal{G}_{\underset{+}{-}}$.

Let $V = (V_k)_{k\in Q_0}$ and $W = (W_k)_{k\in Q_0}$ be a collection of finite dimensional $\mathbb{C}$-vector spaces with respective dimension vectors

$$v = (dimV_1, \cdots, dimV_n) \in \mathbb{Z}_{\geq 0}^{Q_0}$$

and

$$w = (dimW_1, \cdots, dimW_n) \in \mathbb{Z}_{\geq 0}^{Q_0}.$$

There is one-to-one correspondence between the generalized symmetric Cartan matrices and graphs (Q_0, Q_1) without edge-loops and the $\mathbb{C}$-vector spaces V and W, see [5], [18], and [20].

At each vertex $k \in Q_0$, we attach a finite dimensional $\mathbb{C}$-vector spaces V_k and W_k. We assume that the decomposition preserves the parity (see [15, 16, 17, 14]), and define $\overset{\delta}{V} = \oplus_{k\in Q_0^\delta} V_k$, where $\delta = \pm$, i.e,

$$V = V^+ \oplus V^-,$$

with each $\mathbb{C}$-vector space component defined as follows:

$$V^+ = (V_k)_{k\in Q_0^+} \text{ and } V^- = (V_k)_{k\in Q_0^-}.$$

A representation of the oriented graph of finite type (Q_+, Ω^+) is a collection of finite dimensional $\mathbb{C}$-vector spaces V_k for any vertex $k \in Q_0^+$ and $\mathbb{C}$-linear maps

$$\varphi_h : V_{s(h)=k\in Q_0^+} \longrightarrow V_{t(h)=l\in Q_0^-},$$

for any oriented edges $h \in Q_1$, which are considered up to non-degenerate endomorphisms of the $\mathbb{C}$-vector spaces V_k. The dimension vector of the $\mathbb{C}$-vector space V is defined as

$$\begin{aligned} v \quad &:= (dimV_1, \cdots, dimV_n) \in \mathbb{Z}_{\geq 0}^{Q_0} \\ &= v^+ \oplus v^-, \text{where} \\ v^+ \quad &= dimV^+ = (dimV_k)_{k\in Q_0^+}, \\ v^- \quad &= dimV^- = (dimV_k)_{k\in Q_0^-}. \end{aligned} \tag{1.2}$$

We suppose the complex reductive Lie group has the following Lie super-group decomposition

$$G := \sqcap_{k\in Q_0} GL_{\mathbb{C}}(V_k) = \sqcap_{k\in Q_0^+} GL(V_k) \oplus \sqcap_{k\in Q_0^-} GL_{\mathbb{C}}(V_k),$$

where $G_\pm = \sqcap_{k\in Q_0^\pm} GL_{\mathbb{C}}(V_k)$ are Lie super-subgroups of G.

Now, following Anna Cannas da Silva [24], C. Bartoni, and U. Bruzzo, D. Hernández-Ruipérez [12], we introduce some definitions that will be needed in the sequel.

Let H be a Lie group and M a smooth manifold.

Definition 4. A Principal H-bundle over M is a smooth manifold $P := (P, \pi, M, H)$, where P is the total space of the bundle, $\pi : P \longrightarrow M$ a smooth projection map, M the base smooth manifold and H the Lie group such that the following properties hold:

(1) the right action $P \times H \longrightarrow P : \ (a, h) \mapsto ah$ is free.

(2) $M := P/H$ and the canonical map $\pi : P \longrightarrow M := P/H$ is smooth.

(3) (**local trivialization**): Let $\{U_k\}_{k \in Q_0} \subset M$ be an open covering of M. For every point $x \in M$ there is a neighborhood U and a diffeomorphism

$$\varphi_U : \pi^{-1}(U) \longrightarrow U \times H : \ u \mapsto \varphi_U(u)$$

such that $\varphi_U(u) := (\pi(u), s_U(u))$, with $s_U : \pi^{-1}(U) \longrightarrow H$ satisfying $(s_U(u \cdot h) = (s_U(u)) \cdot h, \ \forall h \in H, \ u \in \pi^{-1}(U)$.

Let $\pi : P \longrightarrow M$ be a principal fiber bundle over M and $\Gamma(P)$ the space of its smooth sections. Assume that the Lie group H acts transitively on $M := G/H$, i.e, for any $g \in H, m \in M$, g maps m into $gm \in M$, and maps $\pi^{-1}(m)$ into $\pi^{-1}(gm)$.

Definition 5. A representation $H \longrightarrow Diff(P)$ which commutes with the action of H is called a **gauge transformation** if the induced diffeomorphim on the base manifold, i.e. $H \longrightarrow Diff(M)$ is the identity. The **gauge group** of P is the group G_{gau} of all transformations of P.

Define the map $\epsilon : H \longrightarrow \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ such that $\epsilon(h) = 1$ if $h \in \Omega^+$ and $\epsilon(h) = -1$ if $h \in \Omega^-$. Since the Lie super-group G is reductive, only the positive component G_+ is reductive and G_- contains the nilpotent elements of G. The adjoint action $Ad_G : G \times \mathcal{G} \longrightarrow \mathcal{G}$ induces an adjoint action of the complex reductive Lie group G_+ on the Lie algebra

$$\mathcal{G}_+ = \text{Lie}(G_+) = \underset{k \in Q_0^+}{\oplus} \mathcal{G}l(V_k),$$

i.e., $Ad_{G_+} : G_+ \times \mathcal{G}_+ \longrightarrow \mathcal{G}_+$.

Next,we fix a closed Ad_{G_+}-orbit $\mathcal{O} \subset \mathcal{G}_+$ and define the following closed subscheme (see [8], [22], and [25]) in $\mathcal{G}_-$:

$$MC(\mathcal{G}, \mathcal{O}) := \{x \in \mathcal{G}_- : \ dx + \frac{1}{2}[x, x] \in \mathcal{O}\} \subset \mathcal{G}_-.$$

The equation $dx + \frac{1}{2}[x,x] = 0$ is known as the Maurer-Cartan equation. Hence, $MC(\mathcal{G},\mathcal{O})$ is the Maurer-Cartan scheme associated to an orbit $\mathcal{O} \subset \mathcal{G}_+$.

Therefore, for any $a \in \mathcal{G}_+$, let ξ_a be an affine-linear algebraic vector field on $\mathcal{G}_-$ defined for $x \in \mathcal{G}_-$ by

$$\xi_a(x) := [a,x] - da.$$

Lemma 6. *(1) The map*

$$l: \quad \begin{array}{l} \mathcal{G}_+ \longrightarrow \mathcal{G}_+ \\ a \mapsto \xi_a \end{array}$$

which is a Lie algebra homomorphism, i.e., for every $a,b \in \mathcal{G}_+$,

$$l([a,b]) = [l(a), l(b)].$$

(2) For any orbit $\mathcal{O} \subset \mathcal{G}_+$ and any $a \in \mathcal{G}_+$, the vector field ξ_a is tangent to the Maurer-Cartan scheme $MC(\mathcal{G},\mathcal{O}) \subset \mathcal{G}_-$

Proof. By definition $l([a,b]) = \xi_{[a,b]}$ and $[l(a),l(b)] = [\xi_a,\xi_b]$. Hence, for any $x \in \mathcal{G}_-$, the left hand side is
$\xi_{[\xi_a,\xi_b]}(x) = [[a,b],x] - d([a,b])$ and using the Jacobi identity, the right hand side is

$$\begin{aligned} [\xi_a(x),\xi_b(x)] &= \xi_a\xi_b(x) - \xi_b\xi_a(x) \\ &= \xi_a([b,x]-db) + \xi_b([a,x]-da) \\ &= \xi_a([b,x]) - \xi_a(db) - \xi_b([a,x]) + \xi_b(da) \\ &= [a,[b,x]] - da - [a,db] + da \\ &\quad -[b,[a,x]] + db + [b,da] - db \\ &= [a,[b,x]] - [b,[a,x]] - [a,db] + [b,da] \\ &= [[a,b],x] - d([a,b]) = \xi_{[a,b]}(x). \end{aligned}$$

(2) For any Ad_G-orbit $\mathcal{O} \subset \mathcal{G}_+$ and $a \in \mathcal{G}_+$, the vector field ξ_a is tangent to the Maurer-Cartan scheme $MC(\mathcal{G}_+,\mathcal{O}) \subset \mathcal{G}_-$. From (1) the vector field $\xi_a : \mathcal{G}_- \longrightarrow \mathcal{G}_-$ is a Lie algebra homomorphism and since $\xi_a \in \mathcal{G}_+$, the exponential map $exp : \mathcal{G}_+ \longrightarrow G_+$ is well-defined. Furthermore, the following commutative diagram

$$\begin{array}{ccc} \mathcal{G}_+ \times \mathcal{G}_- & \longrightarrow & \mathcal{G}_- \\ expx \times id_{\mathcal{G}_-} \downarrow & & \downarrow id_{\mathcal{G}_-} \\ G_+ \times \mathcal{G}_- & \xrightarrow{\Psi_{gau}} & \mathcal{G}_-, \end{array} \tag{1.3}$$

induces the gauge action of G_+ on the Lie algebra $\mathcal{G}_-$, i.e, $\Psi_{gau} : G_+ \times \mathcal{G}_- \longrightarrow \mathcal{G}_-$. Since $MC(\mathcal{G}, \mathcal{O}) \subset \mathcal{G}_-$, the vector field ξ_a is tangent to the Maurer-Cartan scheme $MC(\mathcal{G}, \mathcal{O})$. □

The gauge action induces a set of orbits on $\mathcal{G}_-$.

Definition 7. The quotient $\mathcal{M}(\mathcal{G}, \mathcal{O})(v, w) = MC(\mathcal{G}, \mathcal{O})/_{G_+}$ is called a subscheme attached to the closed Ad_{G_+}-orbit $\mathcal{O} \subset \mathcal{G}_+$.

Let $\beta \in \wedge^2\mathcal{G}^*$, i.e., $\beta : \mathcal{G} \times \mathcal{G} \longrightarrow \mathbb{C}$ be an even nondegenerate invariant bilinear 2-form on $\mathcal{G}$ with the following restrictions:

(1) $\beta|\mathcal{G}_+$ is symmetric;

(2) $\beta|\mathcal{G}_- =: \omega$ is skew symmetric

such that $\mathcal{G}_+ \perp \mathcal{G}_-$ and $(dx, y) + (-1)^{|x|}(x, dy) = 0$ holds, where $|x|$ denotes $\mathbb{Z}_2$-degree or parity of x. For any $x, y, z \in \mathcal{G}_-$ one has

$$\beta([x, y], z) = \beta(x, [y, z]), \text{ i.e,} \beta(ad_x(y), z) = \beta(x, ad_y(z)). \tag{1.4}$$

It is then clear that $\mathcal{G}_-$ equipped with the 2-form ω is a symplectic vector space. Hence the gauge action $\Psi_{gau} : G_+ \times \mathcal{G}_- \longrightarrow \mathcal{G}_-$ is a symplectic action. This action leaves invariant the symplectic structure ω on $\mathcal{G}_-$ and it induces the momentum map

$$\begin{array}{rl} \Phi : & \mathcal{G}_- \longrightarrow {\mathcal{G}_+}^* \overset{\beta}{\simeq} \mathcal{G}_+ \\ & x \mapsto \Phi(x) = dx + \frac{1}{2}[x, x] \end{array} \tag{1.5}$$

which is G_+ equivariant with respect to the coadjoint action on $\mathcal{G}_+$, i.e.,

$$\begin{array}{lcl} G_+ \times \mathcal{G}_- & \overset{1_{G_+} \times \Phi}{\longrightarrow} & G_+ \times {\mathcal{G}_+}^* \\ \Psi_{gau} \downarrow & & \downarrow Ad^*_{G_+} \\ \mathcal{G}_- & \overset{\Phi}{\longrightarrow} & {\mathcal{G}_+}^* \end{array} \tag{1.6}$$

Let $\lambda = \Phi(x)$ fixed by the $Ad^*_{G_+}$-action on $\mathcal{G}_-$ be a singular value of Φ at x in $\mathcal{G}_-$, and $\mathcal{O}_\lambda$ the coadjoint orbit passing through λ in the dual ${\mathcal{G}_+}^*$ of $\mathcal{G}_+$. Then the pre-image $\Phi^{-1}(\mathcal{O}_\lambda)$ is not a smooth scheme. We define the quotient scheme by

$$\begin{array}{rl} \mathcal{M}_\lambda(\mathcal{G}, \mathcal{O})(v, w) & := \Phi^{-1}(\mathcal{O}_\lambda)//_{G_+} \\ & = Spec(\mathbb{C}[MC(\mathcal{G}, \mathcal{O})]^{G_+}/I(\Phi^{-1}(\mathcal{O}_\lambda))^{G_+}), \end{array} \tag{1.7}$$

where $\mathbb{C}[MC(\mathcal{G},\mathcal{O})]$ is the ring of polynomial functions on $MC(\mathcal{G},\mathcal{O})$ and

$$I(\Phi^{-1}(\mathcal{O}_\lambda)) = \{f \in \mathbb{C}[MC(\mathcal{G},\mathcal{O})] :\ f|\Phi^{-1}(\mathcal{O}_\lambda) = 0\}$$

the defining ideal of $\Phi^{-1}(\mathcal{O}_\lambda)$. Let $\bar{x} \in M_\lambda(\mathcal{G},\mathcal{O})(v,w)$ and $x \in MC(\mathcal{G},\mathcal{O})$ be its representative.

Define the isotropy group of x by $G_{+,x} = \{g \in G_+ :\ gx = x\}$. Since $v = \dim_{\mathbb{C}} G$ is finite, we know that the group G_+ is a finite dimensional complex reductive Lie group. Indeed, to control the behavior of the G_+-action at infinity and to guarantee the separation property of the singular scheme $M_\lambda(\mathcal{G},\mathcal{O})(v,w)$, we introduce the notions of saturation and orbital convexity, see [26], [7]. Let G_+ be a complex reductive Lie group, the even component of the complex reductive Lie super-group G contains the maximal compact real Lie group denoted by K_+ such that

$$G_+ = K_+^{\mathbb{C}} = K_+ \otimes_{\mathbb{R}} \mathbb{C}.$$

For any $x \in MC(\mathcal{G},\mathcal{O})$, define the isotropy group of G_+ at x denoted by

$$\hat{G_+} := G_{+,x} = \{g \in G_+ :\ gx = x\}$$

and the smooth stratum

$$MC(\mathcal{G},\mathcal{O})_{(\hat{G_+})} := \{y \in MC(\mathcal{G},\mathcal{O}) :\ gG_{+,y}g^{-1} = \hat{G_+},\ \forall g \in G_+\},$$

where $G_{+,y}$ is the isotropy group of G_+ at y, and $(\hat{G}_+)$ the conjugacy class of the isotropy group $\hat{G}_+$ at x.

Set $\bar{x} \in \mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)$ with its representative x. Define the stratum

$$\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(\hat{G_+})} := (MC(\mathcal{G},\mathcal{O}))_{(\hat{G})} \cap \Phi^{-1}(\mathcal{O}_\lambda)//_{G_+}.$$

The embedding of a smooth manifold into the dual of an infinite dimensional Lie algebra associated with the quiver Q was realized by V. Ginzburg [5, Theorem 1.2]. However, using H. Nakajima's quiver variety and following V. Ginzburg, the authors [4] embedded a stratified singular quiver variety into a similar dual of an infinite dimensional Lie algebra attached to a quiver.

The purpose of this paper is to construct a similar imbedding into the Lie algebra associated with a super-quiver Q introduced by G. Lustig [17]. The decomposition of the super reductive Lie group G shows that the embedding can only be achieved into the Lie algebra associated with the Lie subgroup G_+; since the other component of the Lie subgroup G_- carries nilpotent elements of G.

The paper is organized as follows. There are two sections of which this introduction and background is the first one. Then Section 2 are the statements and proofs of our main results.

2. Results

The singular affine scheme $\mathcal{M}_\lambda(\mathcal{G}, \mathcal{O}(v,w))$ is stratified by orbit types, see [19, Lemma 6.5], and [20, Lemma 3.27].

Theorem 8. *Let $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ be such that $\mathcal{G}_- \perp \mathcal{G}_+$ is a differential graded Lie super-algebra with an* **even** *nondegenerate invariant bilinear 2-form satisfying*

$$\beta([x,y],z) = \beta(x,[y,z]),\ \forall x,y,z \in \mathcal{G} = Lie(G)$$

with the following restrictions:

(1) $\beta|\mathcal{G}_+$ is symmetric;

(2) $\beta|\mathcal{G}_- =: \omega$ is skew-symmetric making the smooth vector space $\mathcal{G}_-$ a symplectic vector space. Suppose $\mathcal{O}$ is an Ad_{G_+}-orbit, and $\mathcal{O}_\lambda$ a closed coadjoint orbit of ${\mathcal{G}_+}^* \overset{\beta}{\simeq} \mathcal{G}_+$ passing through the singular point $\lambda = \Phi(x) \in \Phi^{-1}(\mathcal{O}_\lambda)$, the pre-image of $\mathcal{O}_\lambda$ in

$$MC(\mathcal{G}, \mathcal{O}) := \{x \in \mathcal{G}_- :\ \Phi(x) = dx + \frac{1}{2}[x,x] \in \mathcal{O}\} \subset \mathcal{G}_-$$

by the G_+-equivariant momentum map $\Phi : \mathcal{G}_- \longrightarrow {\mathcal{G}_+}^*$.

Let $\bar{x} \in \mathcal{M}_\lambda(\mathcal{G}, \mathcal{O})(v,w)_{(H)}$ be an element of the smooth stratum, with (H) the conjugacy class of the isotropy group of G_+ at the representative x of $\bar{x}$. Then

(1) The form β induces a nondegenerate 2-form on the tangent space $T_{\bar{x}}\mathcal{M}_\lambda (\mathcal{G}, \mathcal{O})(v,w)_{(H)}$ at $\bar{x}$. The 2-forms $\beta|\mathcal{G}_+$ and $\omega := \beta|\mathcal{G}_-$ give rise to a symplectic structure on $\mathcal{M}_\lambda(\mathcal{G}, \mathcal{O})(v,w)_{(H)}$.

(2) The smooth affine scheme $\mathcal{M}_\lambda(\mathcal{G}, \mathcal{O})(v,w)_{(H)}$ can be embedded as a coadjoint orbit in $\mathcal{L}(Q^+)^*$ the dual of the infinite dimensional Lie algebra $\mathcal{L}(Q^+)$ canonically attached to the quiver Q^+.

Proof. (1) To prove (1), we follow [13]. Let $\bar{x} \in \mathcal{M}_\lambda(\mathcal{G}, \mathcal{O})_{(H)}$, $x \in MC(\mathcal{G}, \mathcal{O})$ its representative, and define the isotropy group of G_+ at $x \in MC(\mathcal{G}, \mathcal{O}) \subset \mathcal{G}_-$, by $H = \{g \in G_+ :\ \ gx = x,\ \forall x \in MC(\mathcal{G}, \mathcal{O})\}$. Since the stratum

$\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H)}$ has the properties of the Lemma 6.5 [19], it implies that it is the product of two smooth vector spaces, i.e.,

$\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H)}$
$\simeq \mathcal{M}_{\lambda,reg}(\mathcal{G},\mathcal{O})(v^{(0)},w) \times \mathcal{M}_{\lambda,reg}(\mathcal{G},\mathcal{O})(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)})$
where

$$\begin{aligned}&\mathcal{M}_{\lambda,reg}(\mathcal{G},\mathcal{O})(\hat{v}_1,v^{(1)},\cdots,\hat{v}_r,v^{(r)}) =\\ &= \{x \in MC(\mathcal{G},\mathcal{O}):\ x \text{ as in properties}\\ &(1),(2),(4),(6) \text{ of } [19, \text{Lemma } 6.5]\}/_{\sqcap_{i=1}^{r} G_{v^{(i)}}}\end{aligned} \tag{2.1}$$

with $v^{(i)} := (dimV_1^{(i)},\cdots,dimV_r^{(i)})$, and $G_{v^{(i)}} = \sqcap_{k\in Q_0^+} GL(V_k^{(i)})$.

Note that the derivative of the momentum map $\Phi : \mathcal{G}_- \longrightarrow {\mathcal{G}_+}^*$ at the point x is given by

$$\begin{aligned}T_x\Phi = \Phi'_x:\quad & T_x(\mathcal{G}_-) \simeq \mathcal{G}_- \longrightarrow T_{\Phi(x)}({\mathcal{G}_+}^*)\\ & y \mapsto \Phi'_x(y) = dy + [x,y].\end{aligned} \tag{2.2}$$

Lemma 1.1 (2) shows ξ_a is tangent to $MC(\mathcal{G},\mathcal{O})$ and for any $x \in MC(\mathcal{G},\mathcal{O})$, $\xi_a(x) := [a,x] - da$. We have

$$\begin{aligned}\Phi'_x(\xi_a(x)) &= d(\xi_a(x)) + [x,\xi_a(x)]\\ &= d([a,x]-da) + [x,[a,x]-da]\\ &= [da,x] + [a,dx] + [x,-da] + [x,[a,x]]\\ &= [a,dx] + [x,[a,x]]\\ &= [a,dx] + \tfrac{1}{2}[a,[x,x]] \text{ by Jacobi identity}\\ &= [a,dx + \tfrac{1}{2}[a,[x,x]]\\ &= [a,\Phi(x)],\ \forall a \in \mathcal{G}_+.\end{aligned} \tag{2.3}$$

This is an equation of motion. Hence $\Phi(x)$ is G_+-equivariant. Moreover, $\forall a \in \mathcal{G}_+,\ x,y \in \mathcal{G}_-$ we have

$$\begin{aligned}\omega(\xi_a(x),y) &= \omega([a,x]-da,y),\\ &= \beta([a,x],y) - \omega(da,y) \text{ by bilinearity}\\ &= \beta(ad_a(x),y) + \omega(y,da) \text{ by skew-symmetry}\\ &= \beta(a,ad_x(y)) + \omega(y,da) \text{ by } (7.4)\\ &= \beta(a,[x,y]) + \beta(a,dy)\\ &= \beta(a,dy+[x,y])\\ &= \beta(a,\Phi'_x(y)).\end{aligned} \tag{2.4}$$

The G_+-equivariance of Φ and the equation $\mathcal{G} = \mathcal{G}_+ \oplus \mathcal{G}_-$ show that the G_+ action on the Lie algebra $\mathcal{G}_-$ is Hamiltonian. Hence, the tangent space

$T_{\bar{x}}\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H)}$ is smooth at $\bar{x}$. Thus $\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H)}$ has a symplectic structure denoted $\omega_{(H)}$.

(2) The prove of (2) follows with a slight modification the prove given by V. Ginzburg [5, **?**]. □

For any two isotropy groups H and K for which the strata satisfy the frontier property, we know that the principal stratum is smooth. So the frontier property along with the gluing Lemma [8, **?**], give the following theorem.

Theorem 9. *Under the hypotheses of Theorem 2.1 and for any collection of isotropy groups $\{H_k\}_{k\in Q_0^+}$ of G, which satisfy the conditions (1)-(3) of Theorem 6.4 [4], the following hold:*

(1) The principal stratum

$$\begin{aligned}&\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_0)} =: \overline{\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_{max})}}\\ &= [\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_1)} \cap \overline{\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_{max})}}] \cap \cdots\\ &\cdots \cap [\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_{n-1})} \cap \overline{\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_{n-2})}}],\end{aligned} \tag{2.5}$$

satisfies the condition (1) of Theorem 1.1, with the conjugacy class (H_0) the same as the conjugacy class (H_{max}).

(2) The principal stratum $\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_0)}$ satisfies the condition (2) of Theorem 1.1.

Proof. The proof is the same as the proof of the Theorem 2.1. □

Next since the singular affine scheme

$$\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w) = \bigcup_{(H_k)} \mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_k)}.$$

Theorem 10. *Under the hypotheses of Theorem 2.1 and Theorem 2.2, the affine quiver variety $\mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w) = \bigcup_{(H_k)} \mathcal{M}_\lambda(\mathcal{G},\mathcal{O})(v,w)_{(H_k)}$ satisfies conditions (1) and (2) of the Theorem 2.1.*

Proof. See proofs of Theorem 2.1 and Theorem 2.2. □

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