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AN EASY METHOD TO PRICE QUANTO FORWARD CONTRACTS IN THE HJM MODEL WITH STOCHASTIC INTEREST RATES

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Abstract: Recently, a spot martingale measure pricing method to derive pricing formulas of quanto forward contracts within the Heath, Jarrow and Morton (1992) interest rate model was published. Although the spot martingale measure pricing method is interesting, it requires that one knows something about the dependence between the discount factor and the payoff of the derivative security. Thus, more procedures are required for computations. This paper proposes another simple approach. In particular, we demonstrate the forward measure pricing methodology to the valuation of quanto forward contracts within the HJM interest rate model and provide great ease in deriving closed form solutions.

AMS Subject Classification: 91B28, 60G48 , 60J75
Key Words: quanto forward contracts, Heath-Jarrow-Morton model, forward measure pricing method

1. Introduction

Recently, a spot martingale measure pricing method to derive pricing formulas of quanto forward contracts within the Heath, Jarrow and Morton (1992, hereafter HJM) interest rate model under deterministic volatilities was pub-

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lished. Although the spot martingale measure pricing method is interesting, it requires that one knows something about the dependence between the discount factor and the payoff of the derivative security. Thus, more procedures are required for computations. The idea of term structure modelling with the direct use of forward measures was previously exploited by Hansen (1994), who used the forward induction to produce an arbitrage-free diffusion-type model for prices of a finite family of zero-coupon bond with different maturities. It is well known that the forward measure pricing methodology (Jamshidian, 1987, 1989; and Geman et al., 1995) has been widely used in pricing securities when interest rates are stochastic. This paper proposes another simple method which uses the forward measure pricing methodology to the valuation of quanto forward contracts within the HJM interest rate model. The technique provides great ease in deriving closed form solutions for various derivatives contract with European-style payoffs under stochastic interest rates. An outline for this paper is as follows: Section 2 introduces the terminology, notation, and assumptions of the model along the same lines as in Amin and Jarrow (1991). Section 3 applies the forward measure pricing methodology to the valuation of quanto forward contracts within the HJM interest rate model, and Section 4 concludes the paper.

2. The Economy

Let W be a n-dimensional standard Brownian motion given on a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is assumed to be the right-continuous and P-completed version of the natural filtration of W. We consider a continuous-time trading economy with a trading interval $[0, T^*]$ for the fixed horizon date T^*. As there will be considerable notation, for easy reference, we list it all in one place.

$f^i(t,T)$ = the ith foreign market instantaneous forward rate contracted at time t for borrowing and lending at time T with $0 \leq t \leq T \leq T^*$.
$f(t,T)$ = the domestic market instantaneous forward rate contracted at time t for borrowing and lending at time T with $0 \leq t \leq T \leq T^*$.
$B^i(t,T)$ = the time t price in the ith foreign currency of a foreign zero-coupon bond with maturity T and unit face value.
$B(t,T)$ = the time t price in the domestic currency of a domestic zero-coupon bond with maturity T and unit face value.
$Q^i(t)$ = the spot exchange rate at time t (denominated in domestic currency

per unit of the ith foreign currency).
$\bar{Q}^i$ = the prescribed exchange rate (denominated in domestic currency per unit of the ith foreign currency).
K^i = the strike price denominated in the ith foreign currency.
$r^i(t)$ = the spot interest rate at time t in the ith foreign market.
$r(t)$ = the spot interest rate at time t in the domestic market.
$B^i(t)$ = the foreign savings account at time t in the ith foreign market, $B^i(t) = \exp[\int_0^t r^i(u)du]$.
$B(t)$ = the domestic savings account at time t, $B(t) = \exp[\int_0^t r(u)du]$.
$Z^i(t)$ = the price of a given asset at time t, expressed in units of the ith foreign currency.
P^* = the domestic (spot) martingale probability measure.
P_T = the domestic forward martingale probability measure.
Here the superscrip i indicates that a given process represents a quantity related to the ith foreign market.
As in Amin and Jarrow (1991), Heath et al. (1992), and Wu (2008), the following assumptions and relations are also used:

Assumption 1. (Domestic Forward Interest Rate Dynamics)

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T) \cdot dW(t), \quad \forall t \in [0,T], \quad T \in [0,T^*], \tag{1}$$

where $\alpha(t,T)$ and $\sigma(t,T)$ are subject to some regularity conditions (see Amin and Jarrow, 1991). Using (1), we see that the dynamics of domestic bond price processes under the market probability measure P are

$$dB(t,T) = B(t,T)\left[\left(r(t) - \alpha^*(t,T) + \frac{1}{2}|\sigma^*(t,T)|^2\right)dt - \sigma^*(t,T) \cdot dW(t)\right], \tag{2}$$

where for any $t \in [0,T]$ we have $\alpha^*(t,T) = \int_t^T \alpha(t,u)du$, $\sigma^*(t,T) = \int_t^T \sigma(t,u)du$ (see Heath et al., 1992, for a proof). As usual, the dot "·" stands for the Euclidean inner product in $\Re^n$. Also, we write $|\cdot|$ to denote the Euclidean norm in $\Re^n$.

Assumption 2. (The ith Foreign Forward Interest Rate Dynamics)

$$df^i(t,T) = \alpha^i(t,T)dt + \sigma^i(t,T) \cdot dW(t), \quad \forall t \in [0,T], \quad T \in [0,T^*], \tag{3}$$

where the drift and volatilities are assumed to satisfy the same conditions as for the corresponding terms in Assumption 1. Similarly, one can show that the dynamics of $B^i(t,T)$ under the market measure P are

$$dB^i(t,T) = B^i(t,T) \left[\left(r^i(t) - \alpha^{i^*}(t,T) + \frac{1}{2}\left|\sigma^{i^*}(t,T)\right|^2\right)dt - \sigma^{i^*}(t,T)\cdot dW(t)\right], \quad (4)$$

where for any $t \in [0,T]$ we have

$$\alpha^{i^*}(t,T) = \int_t^T \alpha^i(t,u)du, \quad \sigma^{i^*}(t,T) = \int_t^T \sigma^i(t,u)du.$$

Assumption 3. (The ith Spot Exchange Rate Dynamics)

$$dQ^i(t) = Q^i(t)[\mu_{Q^i}(t)dt + \sigma_{Q^i}(t)\cdot dW(t)], \quad (5)$$

where $\mu_{Q^i}(t)$ and $\sigma_{Q^i}(t)$ are subject to some regularity restrictions (see Amin and Jarrow, 1991).

Assumption 4. The price of an arbitrary foreign asset Z^i that pays no dividends satisfies

$$dZ^i(t) = Z^i(t)[\mu_{Z^i}(t)dt + \sigma_{Z^i}(t)\cdot dW(t)], \quad (6)$$

where the drift and volatilities are assumed to satisfy the same conditions as for the corresponding terms in Assumption 3.

It follows from Proposition 1 in Heath et al. (1992) that we can readily derive the dynamics of $B(t,T)$, $B^i(t,T)$ and $Z^i(t)$ under the domestic martingale measure P^* as follows:

$$dB(t,T) = B(t,T)[r(t)dt - \sigma^*(t,T)\cdot dW^*(t)], \quad (7)$$

$$dB^i(t,T) = B^i(t,T)\left[\left(r^i(t) + \sigma_{Q^i}(t)\cdot\sigma^{i^*}(t,T)\right)dt - \sigma^{i^*}(t,T)\cdot dW^*(t)\right], \quad (8)$$

and

$$dZ^i(t) = Z^i(t)[(r^i(t) - \sigma_{Z^i}(t)\cdot\sigma_{Q^i}(t))dt + \sigma_{Z^i}(t)\cdot dW^*(t)]. \quad (9)$$

3. Pricing Formulas for Quanto Forward Contracts

This section demonstrates how to value quanto forward contracts in the preceding economy by the forward measure pricing method. Recently, pricing formulas of quanto forward contracts within the Heath, Jarrow and Morton

(1992) interest rate model by the spot martingale measure pricing method was published (see Wu, 2008; Musiela and Rutkowski, 2005 and the references therein), in which the computation of the risk-neutral pricing formula, $B^{-1}(t)E_{P_T^*}(B(T)X|\mathcal{F}_t)$, requires that we know something about the dependence between the discount factor $B(T)$ and the payoff X of the derivative security. Thus, more procedures are required for computations. It is well known that the forward measure pricing methodology (Jamshidian, 1987 and 1989; and Geman et al., 1995) has been widely used in pricing securities when interest rate are stochastic. Accordingly, we focus on the forward measure pricing methodology to the valuation of quanto forward contracts within the HJM interest rate model.

Definition 1. A probability measure P_T on $(\Omega, \mathcal{F}_T)$ equivalent to P^* with the Radon-Nikodým density given by the formula

$$\frac{dP_T}{dP^*} = \frac{1}{B(T)B(0,T)} \quad P^* - a.s. \tag{10}$$

is called the forward martingale measure for the settlement date T.

When the bond price is governed by (7), in view of (10) and Girsanov's theorem, the Radon-Nikodým density of P_T with respect to P^* equals

$$\frac{dP_T}{dP^*} = \varepsilon_T\left(\int_0^{\cdot} -\sigma^*(u,T)\cdot dW^*(u)\right) \quad P^* - a.s., \tag{11}$$

where the member on the right-hand side of (11) is the Doléans-Dade exponential, which is given by the following expression

$$\varepsilon_t\left(\int_0^{\cdot} -\sigma^*(u,T)\cdot dW^*(u)\right) = \exp\left(\int_0^t -\sigma^*(u,T)\cdot dW^*(u) - \frac{1}{2}\int_0^t |\sigma^*(u,T)|^2\, du\right).$$

Furthermore, the process W^T given by the formula

$$W^T(t) = W^*(t) + \int_0^t \sigma^*(u,T)du, \quad \forall t \in [0,T], \tag{12}$$

follows a standard n-dimensional Brownian motion under the forward measure P_T. Geman (1989) observed that the forward price of any financial asset follows a (local) martingale under the forward neutral probability associated with the settlement date of a forward contract.

Lemma 1. *The forward price at time t for the delivery date T of an attainable contingent claim X, settling at time T, equals*

$$F_X(t,T) = E_{P_T}\left(X|\,\mathcal{F}_t\right), \quad \forall t \in [0,T], \tag{13}$$

provided that X is P_T-integrable. In particular, the forward price process $F_X(t,T), \quad t \in [0,T]$, is a martingale under the forward measure P_T.

The following well known lemma provides a version of the risk neutral valuation formula which is tailored to the stochastic interest rate framework.

Lemma 2. *The arbitrage price of an attainable contingent claim X which settles at time T is given by the formula*

$$\Pi_t(X) = B(t,T)E_{P_T}\left(X|\,\mathcal{F}_t\right), \quad \forall t \in [0,T]. \tag{14}$$

Next we deal with the four common quanto forward contracts as follows:

Theorem 1. *A quanto forward contract with the payoff at expiry equals*

$$f_1(T) = Q^i(T)(Z^i(T) - K^i).$$

Then for $0 \le t \le T$, the risk-neutral price at time t of the quanto forward contract is

$$f_1(t) = Q^i(t)(Z^i(t) - K^iB^i(t,T)). \tag{15}$$

Proof. By applying Lemmas 1, 2, we obtain

$$\begin{aligned} f_1(t) &= B(t,T)E_{P_T}\left(Q^i(T)(Z^i(T) - K^i)\middle|\,\mathcal{F}_t\right) \\ &= B(t,T)E_{P_T}\left(\frac{Q^i(T)Z^i(T)}{B(T,T)}\middle|\,\mathcal{F}_t\right) \\ &\quad -B(t,T)K^iE_{P_T}\left(\frac{Q^i(T)B^i(T,T)}{B(T,T)}\middle|\,\mathcal{F}_t\right) \\ &= Q^i(t)(Z^i(t) - K^iB^i(t,T)). \end{aligned}$$

Theorem 2. *A quanto forward contract with the payoff at expiry equals*

$$f_2(T) = \bar{Q}^i(Z^i(T) - K^i)$$

(such a contract is also known as a guaranteed exchange rate forawrd contract (a GER forward contract for short)). Then for $0 \le t \le T$, the risk-neutral price at time t of the quanto forward contract is

$$f_2(t) = \bar{Q}^i(Z^i(t)QA(t,T) - K^iB(t,T)), \tag{16}$$

where

$$QA(t,T) = \frac{B(t,T)}{B^i(t,T)}\rho(t,T), \tag{17}$$

and

$$\rho(t,T) = \\ \exp\left\{-\int_t^T \Big(\sigma_{Z^i}(u) + \sigma^{i^*}(u,T)\Big)\cdot\Big(\sigma_{Q^i}(u) - \sigma^{i^*}(u,T) + \sigma^*(u,T)\Big)\,du\right\}. \tag{18}$$

Proof. It follows immediately from (8), (9) and (12) that

$$\begin{aligned} d\left(\frac{Z^i(t)}{B^i(t,T)}\right) &= \frac{Z^i(t)}{B^i(t,T)}\Big[\Big(\sigma_{Z^i}(t) + \sigma^{i^*}(t,T)\Big) \\ &\quad \cdot\Big(dW^T(t) - \Big(\sigma_{Q^i}(t) - \sigma^{i^*}(t,T) + \sigma^*(t,T)\Big)\,dt\Big)\Big]. \end{aligned} \tag{19}$$

In view of the general valuation (14), it is clear that we need to evaluate the conditional expectations

$$\begin{aligned} f_2(t) &= B(t,T)E_{P_T}\left(\bar{Q}^i(Z^i(T) - K^i)\middle|\,\mathcal{F}_t\right) \\ &= B(t,T)\bar{Q}^iE_{P_T}\left(\frac{Z^i(T)}{B^i(T,T)}\middle|\,\mathcal{F}_t\right) - B(t,T)K^i\bar{Q}^i \\ &= B(t,T)\bar{Q}^iI - B(t,T)K^i\bar{Q}^i. \end{aligned}$$

We know from (19) that

$$\begin{aligned} \frac{Z^i(T)}{B^i(T,T)} &= \frac{Z^i(t)}{B^i(t,T)}\rho(t,T)\exp\left\{\int_t^T\Big(\sigma_{Z^i}(u) + \sigma^{i^*}(u,T)\Big)\right. \\ &\quad \left.\cdot dW^T(u) - \frac{1}{2}\int_t^T\Big|\sigma_{Z^i}(u) + \sigma^{i^*}(u,T)\Big|^2du\right\}. \end{aligned} \tag{20}$$

To evaluate I, we introduce an auxiliary probability measure P_R by setting

$$\frac{dP_R}{dP_T} = \varepsilon_T\left(\int_0^\cdot\Big(\sigma_{Z^i}(u) + \sigma^{i^*}(u,T)\Big)\cdot dW^T(u)\right) = \eta_T, \quad P_T - a.s.$$

By virtue of the Bayes rule, it is easily seen that I equals

$$I = \frac{Z^i(t)}{B^i(t,T)}\rho(t,T)E_{P_T}\left(\frac{\eta_T}{\eta_t}\middle|\,\mathcal{F}_t\right) = \frac{Z^i(t)}{B^i(t,T)}\rho(t,T)E_{P_R}\left(1\middle|\,\mathcal{F}_t\right)$$

$$= \frac{Z^i(t)}{B^i(t,T)}\rho(t,T)$$

and thus

$$f_2(t) = \bar{Q}^i(Z^i(t)QA(t,T) - K^iB(t,T)).$$

Theorem 3. *A quanto forward contract with the payoff at expiry equals*

$$f_3(T) = \bar{Q}^i Z^i(T) - Q^i(T)K^i.$$

Then for $0 \le t \le T$, the risk-neutral price at time t of the quanto forward contract is

$$f_3(t) = \bar{Q}^i Z^i(t)QA(t,T) - K^iQ^i(t)B^i(t,T), \tag{21}$$

where $QA(t,T)$ is given by (17).

Proof.

$$\begin{aligned} f_3(t) &= B(t,T)E_{P_T}\left(\bar{Q}^iZ^i(T) - Q^i(T)K^i \middle| \mathcal{F}_t\right) \\ &= B(t,T)\bar{Q}^iE_{P_T}\left(Z^i(T)\middle| \mathcal{F}_t\right) - B(t,T)K^iE_{P_T}\left(Q^i(T)\middle| \mathcal{F}_t\right) \\ &= B(t,T)\bar{Q}^i\frac{Z^i(t)}{B^i(t,T)}\rho(t,T) - B(t,T)K^iE_{P_T}\left(\frac{Q^i(T)B^i(T,T)}{B(T,T)}\middle| \mathcal{F}_t\right) \\ &= \bar{Q}^iZ^i(t)QA(t,T) - B(t,T)K^i\frac{Q^i(t)B^i(t,T)}{B(t,T)} \\ &= \bar{Q}^iZ^i(t)QA(t,T) - K^iQ^i(t)B^i(t,T). \end{aligned}$$

Theorem 4. *A quanto forward contract with the payoff at expiry equals*

$$f_4(T) = Q^i(T)Z^i(T) - \bar{Q}^iK^i.$$

Then for $0 \le t \le T$, the risk-neutral price at time t of the quanto forward contract is

$$f_4(t) = Q^i(t)Z^i(t) - K^i\bar{Q}^iB(t,T). \tag{22}$$

Proof.

$$\begin{aligned} f_4(t) &= B(t,T)E_{P_T}\left(Q^i(T)Z^i(T) - \bar{Q}^iK^i\middle| \mathcal{F}_t\right) \\ &= B(t,T)E_{P_T}\left(\frac{Q^i(T)Z^i(T)}{B(T,T)}\middle| \mathcal{F}_t\right) - B(t,T)\bar{Q}^iK^i \\ &= B(t,T)\frac{Q^i(t)Z^i(t)}{B(t,T)} - B(t,T)\bar{Q}^iK^i = Q^i(t)Z^i(t) - K^i\bar{Q}^iB(t,T). \end{aligned}$$

4. Concluding Remarks

The forward measure is convenient in calculating various contingent claim prices under stochastic interest rates. This study proposes a simple method which uses the forward measure pricing methodology to derive the valuation formulas for quanto forward contracts within the Heath, Jarrow and Morton (1992) interest rate model. This technique provides great ease in deriving closed form solutions for various derivatives contract with European-style payoffs under stochastic interest rates. For instance, it has been applied to the valuation of quanto options under stochastic interest.

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