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COMMON FIXED POINT OF ABSORBING MAPPINGS SATISFYING LIPSCHITZ TYPE CONTRACTIVE CONDITION

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Abstract: In this paper we prove a common fixed point theorem considering a weaker contractive condition and using the new notion called absorbing map, replacing continuity of maps by weaker notion called reciprocal continuity and have used one pair of semi - compatible map instead of compatibility of sequence of mappings. Also we do not assume any condition either on δ or ϕ in our Lipschitz type contractive condition so our result establishes a fixed point theorems under a weaker condition than that of earlier result of [10], [11], [12] and others.

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In the year 1979, Jungck [6] introduced commuting mappings. Commuting

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mapping has been generalized in many ways. It was the turning point in the "fixed point arena" when the notion of weakly commutativity was introduced by Sessa [16] in the year 1982 as a generalization of commutativity and a sharper tool to obtain common fixed points of mappings. A bulk of results were produced and it was the center of vigorous research activity in "Fixed Point Theory and its Application" in various branches of mathematical sciences in the last three decades. Sessa [16] define the concept of weakly commuting mappings by calling self - mappings A and S of a metric spaced (X, d) weakly commuting pair if and only if

$$d(AS(x), SA(x)) \leq d(A(x), S(x)),$$

for all $x \in X$, and other gave some common fixed point theorems of weakly commuting mappings ([1], [2],[7]). Of course, commuting mappings are weakly commuting, but the converse is not true. A major breakthrough was done by Jungck [7] in the year 1986 when he proclaimed the new notion what he called "compatibility" of mapping as a generalization of weak commutativity and its usefulness for obtaining common fixed points of mappings was shown by him. Thereafter, a flood of common fixed point theorems was produced by various researchers by using the improved notion of compatibility of mappings. In fact, every weak commutative pair of mappings is compatible but the converse is not true. Jungck [8] also introduced compatible mappings of type (A) or of type (α). Pathak et. al. [13], [14], [15] introduced compatible mapping of type(B) or of type (C) and type (P) in metric spaces and using these concept, several researchers and mathematicians have proved common fixed point theorems. Recently, weaker notion of compatible mappings called semi - compatible mappings in fuzzy metric space is introduced by Singh et. al. [17] and he proved that the concept of semi - compatible mappings is equivalent to the concept of compatible mapping and compatible mapping of type (α) and of type (β) under some conditions of mappings. In this direction, Ranadive et. al.[3] defined a new notion called absorbing maps, the definition of such mapping is given as

Definition 1. (see [3]) Let A and B $(A \neq S)$ be two self maps of a metric space (X, d) then A will be called S - absorbing if there exists a real number $R > 0$ such that

$$d(Sx, SAx) \leq Rd(Sx, Ax), \text{ for all } x \in X.$$

Similarly S will be called A - absorbing if there exists a real number $R > 0$ (not necessarily the same as above) such that

$$d(Ax, ASx) \leq Rd(Ax, Sx), \text{ for all } x \in X.$$

The map A will be called point wise S - absorbing if for given $x \in X$, there exists $R > 0$ such that

$$d(Sx, SAx) \leq Rd(Sx, Ax).$$

On similar lines we can define point wise A - absorbing map. If we take $S = Ix$ (identity map), then A is trivially I - absorbing.

Definition 2. A pair of self maps (A, S) of a metric space (X, d) is said to weakly compatible if they commute at their coincidence point, i.e. $Ax = Sx$ implies that $ASx = SAx$.

Definition 3. (see [7]) A pair of self maps (A, S) of a metric space (X, d) is said to be compatible or asymptotically commuting if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$. It is clear that a pair of compatible maps is weakly compatible but converse is not true in general.

Definition 4. A pair of self maps (A, S) of a metric space (X, d) is said to be non - compatible if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either non zero or non existent.

Definition 5. (see [17]) Self maps A and S of a metric space (X, d) are said to be semi - compatible if $\lim_{n\to\infty} d(ASx_n, Sz) = 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 6. (see [9]) A pair of self maps (A, S) of a metric space (X, d) is said to be reciprocal continuous if $\lim_{n\to\infty} d(ASx_n, Az) = 0$ and $\lim_{n\to\infty} d(SAx_n, Sz) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = z$, $\lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Now we give an example to show that absorbing maps need not commute at their coincidence points. Thus, the notion of absorbing maps is different from other generalizations of commutativity which force the mappings to commute at coincidence points.

Example 1. Let $X = [0, 1]$ and d be the usual metric on X, define $A, S : X \to X$ by $Ax = 1$ for $x \neq 1, A(1) = 0$ and $Sx = 1$ for all $x \in X$. Then the map A is S - absorbing for any $R > 1$ but the pair of maps (A, S) do not commute at their coincidence point $x = 0$.

Following is the theorem proved by Jha (see [5])

Theorem 7. *Let (A, S) and (B, T) be compatible pairs of self mappings of a complete metric space (X, d) such that,*

1. *$A(X) \subset T(X)$, $B(X) \subset S(X)$*
2. *given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$, $\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) < \varepsilon$*
3. *and for $0 \leq k \leq \frac{1}{3}$,*

$$\begin{aligned} d(Ax, By) \;&<\; k\,[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) \\ &\qquad + d(Sx, By) + d(Ax, Ty)], \end{aligned}$$

If one of the mappings A, B, S or T is continuous then A, B, S and T have a unique common fixed point.

To prove our theorem we use the following Lemma of Jachymski (see [4]).

Lemma 1. (see [4]) *Let A, B, S and T be self mappings of a metric space (X, d) such that $A(X) \subset T(X)$, $B(X) \subset S(X)$. Assume further that, given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$*

1. *$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$,*
2. *and $d(Ax, By) < M(x, y)$, whenever $M(x, y) > 0$ where*

$$\begin{aligned} M(x, y) \;&=\; \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \\ &\qquad \frac{[d(Sx, By) + d(Ax, Ty)]}{2}\}. \end{aligned}$$

Then for each $x_0 \in X$, the sequence $\{y_n\}$ in X defined by the rule $y_{2n} = Ax_{2n} = Tx_{2n+1}$; $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence.

Now we prove the following result

Theorem 8. *Let A be point-wise S - absorbing and B be point-wise T - absorbing self mappings of a complete metric space (X, d) such that*

1. *$A(X) \subset T(X)$, $B(X) \subset S(X)$*
2. *given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$, $\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$*
3. *and for $0 \leq k \leq \frac{1}{3}$*

$$\begin{aligned} d(Ax, By) \;&<\; k\,[d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) \\ &\qquad + d(Sx, By) + d(Ax, Ty)], \end{aligned}$$

If one of the mapping pairs (A, S) and (B, T) is semi-compatible and reciprocally continuous maps, then A, B, S and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrarily. By virtue of (1) we can define sequences $\{x_n\}$ and $\{y_n\}$ in X as; $y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$. By using lemma 1 mentioned above we conclude that $\{y_n\}$ is a Cauchy sequence in X. Since X is a complete metric space therefore sequence $\{y_n\}$ converges to the point z (say) in X and hence $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$. Suppose pair of maps (A, S) is reciprocally continuous and semi-compatible then we have,

$$\lim_{n\to\infty} ASx_n = Az, \ \lim_{n\to\infty} SAx_n = Sz \quad \text{and} \quad \lim_{n\to\infty} d(ASx_n, Sz) = 0.$$

Thus we get $Az = Sz$. Now we claim $Az = z$, for if not then by contractive condition (3), we have,

$$\begin{aligned} d(Az, Bx_{2n+1}) &< k\ [d(Sz, Tx_{2n+1}) + d(Az, Sz) + d(Bx_{2n+1}, Tx_{2n+1}) \\ &\qquad + d(Sz, Bx_{2n+1}) + d(Az, Tx_{2n+1})] \end{aligned}$$

Letting $n \to \infty$ we get,

$$d(Az, z) < k\ [d(Sz, z) + 0 + 0 + d(Sz, z) + d(Az, z)]$$

$$< 3k\ d(Az, z) < d(Az, z)$$

a contradiction, hence $Az = z = Sz$. Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $Tu = z$. Now we assert $Bu = z$, for if not then by contractive condition (3), we have,

$$\begin{aligned} d(Az, Bu) &< k\ [d(Sz, Tu) + d(Sz, Az) + d(Tu, Bu) \\ &\qquad + d(Sz, Bu) + d(Tu, Az)] \end{aligned}$$

i.e. $d(z, Bu) < 2k\ d(z, Bu) < d(z, Bu)$, a contradiction, thus we have $z = Bu$. We now have shown that $Az = Sz = z = Tu = Bu$. Now, since B is point wise T - absorbing, there exists real $R > 0$ such that, $d(Tu, TBu) \leq Rd(Tu, Bu) \Rightarrow Tu = TBu$, i.e. $z = Tz$. By using contractive condition (3), we have,

$$\begin{aligned} d(Az, Bz) &< k\ [d(Sz, Tz) + d(Sz, Az) + d(Tz, Bz) + d(Sz, Bz) \\ &\qquad + d(Tz, Az)]; \\ i.e. d(z, Bz) &< 2k\ d(z, Bz) \Rightarrow Bz = z. \end{aligned}$$

Thus $z = Az = Bz = Sz = Tz$ and hence z is a common fixed point of A, B, S and T.

Uniqueness. Let z' be another fixed point of mappings A, B, S and T, by contractive condition we have,

$$\begin{aligned} d(Az, Bz') &< k\ [d(Sz, Tz') + d(Az, Sz) + d(Bz', Tz') \\ &\qquad + d(Sz, Bz') + d(Az, Tz')] \\ \text{i.e., } d(z, z') &< k\ [d(z, z') + d(z, z) + d(z', z') \\ &\qquad + d(z, z') + d(z, z')] = 3k\ d(z, z') \end{aligned}$$

a contradiction. Thus z is a unique common fixed point of A, B, S and T. □

Example 2. Let $X = [2, 20]$ and d be the usual metric on X. Define A, B, S and T self maps on X as,

$$\begin{aligned} &Ax = 2 \text{ if } x = 2, Ax = 3 \text{ if } x > 2; \\ &Bx = 2 \text{ if } x = 2 \text{ or } \geq 5, Bx = 6 \text{ if } 2 < x < 5; \\ &Sx = 2 \text{ if } x = 2, Sx = 6 \text{ if } x > 2; \\ &Tx = 2 \text{ if } x = 2, Tx = x + 4 \text{ if } 2 < x < 5, Tx = \frac{x+1}{2} \text{ if } x \geq 5. \end{aligned}$$

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point $x = 2$. It can be verified in this example that A, B, S and T satisfy the contractive condition (2) and (3) and that $\delta(\epsilon) = 1$ if $\epsilon \geq 3$ and $\delta(\epsilon) = 3 - \epsilon$ if $\epsilon < 3$. Also $\liminf_{\epsilon \to 3} \delta(\epsilon) = 0 \neq \delta(3)$ and δ fails to be non-decreasing or lower semi continuous. It can also be verified that A and S are reciprocally continuous and compatible mappings and mapping B is point wise T-absorbing. However, all the mappings involved in the example are discontinuous at the common fixed point.

In our next result we relax the condition of completeness of metric space (X, d) and require only one of range space to be complete.

Theorem 9. *Let A be point wise S-absorbing and B be point wise T-absorbing self mappings of a complete metric space (X, d) such that*

1. $A(X) \subset T(X)$, $B(X) \subset S(X)$
2. *given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for all* $x, y \in X$,$\varepsilon < M(x, y) < \varepsilon + \delta \Rightarrow d(Ax, By) \leq \varepsilon$ *and*

3. for $0 \leq k \leq 1/3$

$$\begin{aligned} d(Ax, By) &< k\ [d(Sx, Ty) + d(Ax, Sx) + d(By, Ty) \\ &\qquad + d(Sx, By) + d(Ax, Ty)] \end{aligned}$$

If one of $A(X)$, $B(X)$, $S(X)$ *and* $T(X)$ *is complete subspace of* X *then* A, B, S *and* T *have a unique common fixed point.*

Proof. Let $x_0 \in X$ be arbitrarily. By virtue of (1) we can define sequences $\{x_n\}$ and $\{y_n\}$ in X as; $y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$. Using lemma 1 above, we conclude that $\{y_n\}$ is a Cauchy sequence in X. Suppose $T(X)$ is a complete subspace of X, the the subsequence $y_{2n} = Tx_{2n+1}$ is a Cauchy sequence in $T(X)$ and converges to a point z in $T(X)$ (say). Let $v \in T^{-1}z$, then $Tv = z$. Since $\{y_{2n}\} \to z$ hence subsequence $\{y_{2n+1}\}$ also converges to the same point z. Now by contractive condition (3) we have,

$$\begin{aligned} d(Ax_{2n}, Bv) &< k\ [d(Sx_{2n}, Tv) + d(Ax_{2n}, Sx_{2n}) + d(Bv, Tv) \\ &\qquad + d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv)] \end{aligned}$$

so letting $n \to \infty$, we have, $d(z, Bv) \leq 2k\ d(z, Bv)$, a contradiction therefore $z = Bv$. Now $z = Bv \in B(X) \subseteq S(X)$ so there exist $w \in X$ such that $Sw = z$. Now using contractive condition (3) we get,

$$\begin{aligned} d(Aw, Bx_{2n+1}) &< k\ [d(Sw, Tx_{2n+1}) + d(Aw, Sw) + d(Bx_{2n+1}, Tx_{2n+1}) \\ &\qquad + d(Sw, Bx_{2n+1}) + d(Aw, Tx_{2n+1})], \end{aligned}$$

so letting $n \to \infty$, we have,

$$d(Aw, z) < k\ [d(Sw, z) + d(Aw, z) + d(z, z) + d(z, z) + d(Aw, z)].$$

Therefore, $d(Aw, z) < 2k\ d(z, Aw)$ & so $z = Aw$. Thus we have $z = Bv = Tv = Aw = Sw$.

Since B is point wise T-absorbing, there exists $R > 0$ such that, $d(Tv, TBv) \leq R\ d(Tv, Bv)$, i.e., $Tv = TBv = z \Rightarrow Tz = z$, and S is point wise A-absorbing, there exists $R > 0$ (not necessarily the same as above), such that $d(Aw, SAw) \leq R\ d(Aw, Sw)$, i.e. $z = Aw = SAw \Rightarrow Sz = z$. Now by contractive condition (3), we have

$$\begin{aligned} d(Az, Bx_{2n+1}) &< k\ [d(Sz, Tx_{2n+1}) + d(Az, Sz) + d(Bx_{2n+1}, Tx_{2n+1}) \\ &\qquad + d(Sz, Bx_{2n+1}) + (Az, Tx_{2n+1})] \end{aligned}$$

$$\text{i.e.} \quad d(Az,z) \;<\; k\,[d(z,z)+d(Az,z)+d(z,z)+d(z,z) +d(Az,z)] \le 2k\; d(Az,z),$$

a contradiction hence $Az = z$. Similarly, by putting $x = x_{2n}$ and $y = z$ in contractive condition we can get $z = Bz$. Thus we have shown that $z = Az = Bz = Sz = Tz$. Therefore z is a common fixed point of A, B, S and T. Uniqueness can be proved easily using contractive condition. □

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