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MATHEMATICAL MODELING OF OIL RECOVERY

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Abstract: This paper covers the subject of immiscible, incompressible displacement. The message here is that there is but one displacement theory, that of Buckley and Leverett.

Solution of the quasi-linear first-order hyperbolic equation has been reached using the finite-difference method applied on an example which is of interest in its own right, and the concepts of discontinuous solution have been introduced.

Several graphs have plotted using the data we obtained applying the finite-difference method for the purpose of illustrating the water saturation distribution during different period of time, and to study the effect of changing the ratio between the oil and the water viscosities on the frontal advance curve at fixed time.

AMS Subject Classification: 65M06
Key Words: percolation, porous, media, oil recovery

1. Introduction

At the present time the role of mathematical modeling of oil and gas recovery processes considerably increases, and that is because of: (1) Difficulty in scaling up laboratory experiments. (2) High cost of large-scale oil fields experiments. This work is devoted to the mathematical modeling of water-flooding, which is the most important and widely used method of oil recovery. The subject of the present paper is the Buckley-Leverett equation, which describes two-phase displacement of immiscible liquids through porous media, and the simultaneous

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flow of oil and water, in particular. Water-flooding of reservoirs, which is the basic process involved in the development of oil fields, can be described in a large-scale approximation by a quasi-linear first-order hyperbolic equation. The remainder of the paper concentrates on applying a simple numerical simulation method, which is more convenient for simplicity, to provide the required solution particularly for displacement problems(Barenblatt G.I et al, 1991; Smith, 1984).

2. Equations of Two-Phase Flow Through a Porous Medium

Water-flooding of oil reservoirs will be described by the differential equations of the simultaneous flow of two immiscible, incompressible liquids through a porous medium, derived below. We use the macroscopic approximation of the mechanics of a continuous medium situated at the same point(in a volume element).The fraction of the volume element occupied by the pore space is the porosity m (Bedrikovetsky, 1993). The ratio of the volume that a fluid occupies to the pore volume is called the saturation of that fluid. We will consider the part of the pore space occupied by water as σ will be known as the water-saturation or, more simply, the saturation. The percolation velocities of the phases are defined as their volume rates of flow per unit surface area. The velocities of water and oil are denoted by W_H and W_p respectively. We shall also use the total flow velocity W

$$W = W_H + W_p \tag{1}$$

In 1942 Buckley and Leverett presented what is recognized as the basic equation for describing immiscible displacement in one dimension(Buckley and Leverett, 1942). To describe the simultaneous flow of oil and water in the reservoir, applying Darcy's law, the absolute permeability k used earlier must be replaced by the effective permeabilities $k_p(\sigma)$ and $k_H(\sigma)$ respectively, therefore, the law may by generalized for each of the phases in the form

$$W_H = \frac{-k_H(\sigma)}{\mu_H}\frac{\partial p}{\partial x}, \tag{2}$$

$$W_p = \frac{-k_p(\sigma)}{\mu_p}\frac{\partial p}{\partial x}. \tag{3}$$

Here k_H: the effective permeability of water
μ_H : water viscosity
K_p : the effective permeability of oil

μ_p : oil viscosity
K: absolute permeability
k_H^*: relative permeability of water $k_H^* = \frac{K_H}{k}$
k_p^*: relative permeability of oil $k_p^* = \frac{K_p}{k}$
The effective permeability of water and oil are dependent on the saturations of each fluid and the sum of the effective pemeabilties is always less than the absolute permeability.

For displacement in a horizontal reservoir, the fractional flow curve or the Buckley-Leverett function is defined as:

$$f(\sigma) = \frac{\frac{k_H^*(\sigma)}{\mu_H}}{\frac{k_H^*(\sigma)}{\mu_H} + \frac{k_p^*(\sigma)}{\mu_p}} = \frac{1}{1 + \frac{\mu_H k_p^*(\sigma)}{\mu_p k_H^*(\sigma)}}, \mu_0 = \frac{\mu_p}{\mu_H} \tag{4}$$

And so, $f(\sigma)$ is equal to the ratio of the mobility of water to the sum of the mobilities of the two phases.

$$W f'(\sigma)\frac{\partial\sigma}{\partial x} + m\frac{\partial\sigma}{\partial t} = 0 \tag{5}$$

The resulting equation which describes the two-phase flow through a porous medium is a quasi-linear first-order partial differential equation in just one unknown(the saturation $\sigma(x,t)$), and can be solved using numerical methods.

3. The Numerical Simulation of Immiscible, Incompressible Displacement

In this section, we consider the numerical solution to the quasi-linear hyperbolic first-order partial differential equation (equation (5)), by using the finite-difference method.

In this method, knowing that σ is a function of the independent variables x and t, we subdivide the x-t plane into sets of equal rectangles of side $\delta x = h, \delta t = k$, by equally spaced grid lines parallel to Oy, defined by $x_i = ih, i = 0, 1, 2, .$ and equally grid lines parallel to Ox, defined by $y_j = jk, j = 0, 1, 2, .$ (Smith, 1984).

Denote the value of σ at the representative mesh point $D(ih, jk)$ by

$$D = \sigma(ih, jk) = \sigma_{i,j}$$

Using the forward-difference approximating for
$\frac{\partial\sigma}{\partial t}$ at D is

$$\frac{\partial\sigma}{\partial t} \cong \frac{\sigma_{i,j+1} - \sigma_{i,j}}{k} \tag{6}$$

And the backward-difference approximation for $\frac{\partial f(\sigma)}{\partial x}$ at D is

$$\frac{\partial f(\sigma)}{\partial x} \cong \frac{f(\sigma_{i,j}) - f(\sigma_{i-1,j})}{h} \tag{7}$$

With leading error of O(h).

Applying this method for equation (5), which can be expressed as:

$$\frac{\partial \sigma}{\partial t} + \frac{W}{m} f'(\sigma) \frac{\partial \sigma}{\partial x} = 0 \tag{8}$$

Or

$$\frac{\partial \sigma}{\partial t} + \frac{W}{m} \frac{\partial f(\sigma)}{\partial x} = 0$$

Where

$$f'(\sigma) = \frac{df(\sigma)}{d\sigma}, \frac{\partial f(\sigma)}{\partial x} = \frac{df(\sigma)}{d\sigma} \frac{\partial \sigma}{\partial x}$$

We introduce dimensionless variables of length and time:

$$\xi = \frac{x}{L}, \quad \tau = \frac{Wt}{mL} \quad (\tau \geq 0, 0 \leq \xi \leq 1) \tag{9}$$

Where L is the length of the reservoir , m L is the pore volume for a reservoir of unit cross-sectional area.

The dimensionless time τ can be interpreted as the ratio of the volume of liquid injected into the reservoir by time t to the pore volume of the reservoir.

From equation (9)

$$x = L\xi, \quad t = \frac{mL}{W}\tau$$

And

$$\partial x = L\partial\xi, \quad \partial t = \frac{mL}{W}\partial\tau$$

Substituting these in equation (8) gives

$$\frac{W}{mL} \frac{\partial \sigma}{\partial \tau} + \frac{W}{mL} \frac{\partial f(\sigma)}{\partial \xi} = 0$$

or

$$\frac{\partial \sigma}{\partial \tau} + \frac{\partial f(\sigma)}{\partial \xi} = 0 \tag{10}$$

Thus, in the large, oil displacement by water can be described by the quasi-linear first-order hyperbolic equation(10) in the unknown $\sigma(\xi, \tau)$.

Then by equation (6) and equation (7)

$$\frac{\partial \sigma}{\partial \tau} = \frac{\sigma_{i,j+1} - \sigma_{i,j}}{\Delta \tau}$$

And

$$\frac{\partial f(\sigma)}{\partial \xi} = \frac{f(\sigma_{i,j}) - f(\sigma_{i-1,j})}{\Delta \xi} \tag{11}$$

Substituting equs(6) and (7) in equation (10) yields

$$\frac{\sigma_{i,j-1} - \sigma_{i,j}}{\Delta \tau} = \frac{f(\sigma_{i-1,j}) - f(\sigma_{i,j})}{\Delta \xi}$$

$$\sigma_{i,j+1} = \frac{\Delta \tau}{\Delta \xi}[f(\sigma_{i-1,j}) - f(\sigma_{i,j})] + \sigma_{i,j} \tag{12}$$

Since

$$f(\sigma) = \frac{\mu_0 k_H^*(\sigma)}{\mu_0 k_H^*(\sigma) + k_p^*(\sigma)}$$

Then equation (12) can be expressed as (Maksimov,Rybitskaya, 1976)

$$\sigma_{i,j+1} = \frac{\Delta \tau}{\Delta \xi}[\frac{\mu_0 k_H^*(\sigma_{i-1,j})}{\mu_0 k_H^*(\sigma_{i-1,j}) + k_p^*(\sigma_{i-1,j})} - \frac{\mu_0 k_H^*(\sigma_{i,j})}{\mu_0 k_H^*(\sigma_{i,j}) + k_p^*(\sigma_{i,j})} + \sigma_{i,j} \tag{13}$$

Equation (13) is a formula for the unknown saturation $\sigma_(i, j+1)$ at the $(i, j+1)th$ mesh point in terms of known saturations along the jth time-row.

As a numerical example let us solve equation (13) which satisfies the following conditions:

$$\begin{aligned} W = 1 \quad &, \qquad m = 1; \\ \sigma(\xi, 0) = 0.2 \quad &, \qquad \xi > 0 \\ \sigma(0, \tau) = 0.85 \quad &, \qquad \tau > 0, \end{aligned}$$

Phase permeabilties were choosen according to (Charny, 1963)

$$k_H^*(\sigma) = \begin{cases} 0 & , 0 \le \sigma \le 0.2 \\ (\frac{\sigma - 0.2}{0.8})^{3.5} & , 0.2 \le \sigma \le 1 \end{cases}$$

$$k_p^*(\sigma) = \begin{cases} 0 & , 0.85 \le \sigma \le 1 \\ (\frac{0.85 - \sigma}{0.85})^{2.8}(1 + 2.4\sigma) & , 0 \le \sigma \le 0.85 \end{cases}$$

Let $\Delta \xi = 0.01, (N = 100), \Delta \tau = 0.001$

For the case $\mu_0 = 1$, the solution of equation (13) obtained by applying this finite-difference equation to the boundary and initial values is recorded.

By similar way a finite-difference solution may be obtained for the case when $\mu_0 = 2$ and 4.

The value of the water saturation at $\tau = 0.05; 0.1; 0.3; 0.4$ for the cases of $\mu_0 = 1; 2; 4$ were calculated.

Figure 1 shows the saturation distribution at different moments of time τ for $\mu_0 = 1$.

It indicates that the flood front, will advance a distance during the time, and that the saturation is changing sharply in a very small range.

Moreover, it shows that the results for the Buckley-Leevertt problem solved by using the finite-difference method confirm the discontinuity.

In order to study the effect of μ_0 on the water saturation distribution, fig 2 have been drawn, it represents the distribution of water saturation at the same moments of time for $\mu_0 = 2$.

Figure 3 represents the water saturation distribution for different moments of time for $\mu_0 = 4$.

Figure 4 represents the water saturation distribution at the same moment of time ($\tau = 0.1$) for different values of μ_0 ($\mu_0 = 1; 2; 4$).

Figure 4 indicates that μ_0 has a great influence on the water-flooding process, farther more, it shows that the small values of μ_0 gives a sharper displacement front is close to the piston model solution.

Figure 5 represents water saturation distribution for different values of μ_0 for $\tau = 0.3$.

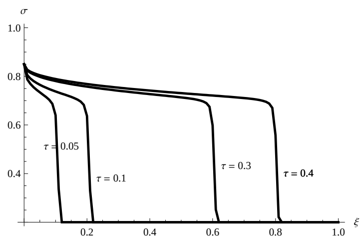

Figure 1: Water saturation for different moments of time with $\mu_0 = 1$

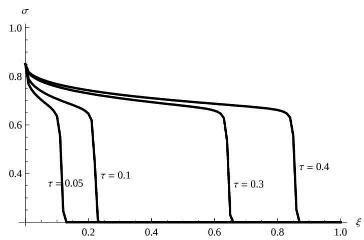

Figure 2: Water saturation for different moments of time with $\mu_0 = 2$

4. Conclusion

The classical mathematical theory of two-phase displacement of incompressible, immiscible liquids in a porous medium has been modeled, and yields the Buckley-Leverett quasi-linear first-order hyperbolic equation.

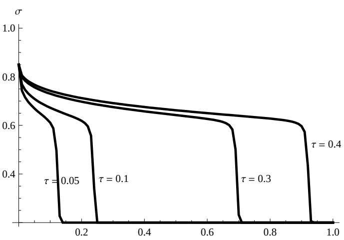

Figure 3: Water saturation for different moments of time with $\mu_0 = 4$

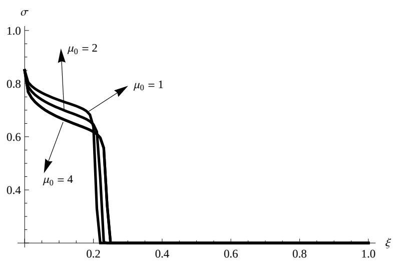

Figure 4: Water saturation for different values of μ_0 for $\tau = 0.1$

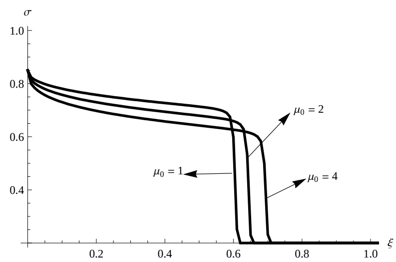

Figure 5: Water saturation for different values of μ_0 for $\tau = 0.3$

The solution of the one-dimensional problem of displacement of oil by water from a homogeneous reservoir using the finite-difference method, consists of a discontinuity or a saturation jump at the displacement front.

As a result of solving the Buckley-Leverett problem using the numerical method, some conclusions have been reached stating that, the flood front is moving and it will advance a distance during the time, and that the saturation will be changing sharply in a very small range.

In addition, the influence of changing the values of μ_0 on the water-flooding process has been noticed, and a general statement could be formed as, the smaller values of μ_0 will lead to a sharper displacement front.

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