International Journal of Pure and Applied Mathematics
Volume 77 No. 2 2012, 219-231
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

APPROXIMATE ADDITIVE MAPPINGS RELATED TO A CAUCHY ADDITIVE FUNCTIONAL INEQUALITY

Mi Hyun Han[1], Won-Gil Park[2 §]
[1]Departament of Mathematics
Chungnam National University
220, Yuseong-Gu, Daejeon, 305-764, REPUBLIC OF KOREA
[2]Department of Mathematics Education
College of Education
Mokwon University
Daejeon, 302-729, REPUBLIC OF KOREA

Abstract: In this paper, we prove the generalized Hyers-Ulam stability of the following additive functional inequality

$$\left\|f(x)+f(y)+f(-z)\right\| \le \left\|f(x+y)-f(z)\right\|$$

in Banach spaces. We investigate the stability of the above functional inequality in non-Archimedean Banach spaces.

AMS Subject Classification: 39B82, 46Bxx
Key Words: additive functional inequality, Banach space, non-Archimedean Banach space

1. Introduction and Preliminaries

In 1940, S. M. Ulam [1] suggested the stability problem of functional equations concerning the stability of group homomorphisms. In the next year, D. H. Hyers [2] gave a first (partial) affirmative answer to the question of Ulam for

Received: January 20, 2012

§Correspondence author

Banach spaces. Thereafter, we call that type the Hyers-Ulam stability.

In 1994, a generalization of the Rassias theorem was obtained by Găvruta as follows [3].

Suppose $(\mathcal{G},+)$ is an abelian group, $\mathcal{E}$ is a Banach space, and that the so-called admissible control function $\varphi : \mathcal{G}^2 \to \mathbb{R}$ satisfies

$$\widetilde{\varphi}(x,y) := \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{2^j}\varphi\big(2^j x, 2^j y\big) < \infty$$

for all $x, y \in \mathcal{G}$. If $f : \mathcal{G} \to \mathcal{E}$ is a mapping with

$$\big\|f(x+y) - f(x) - f(y)\big\| \le \varphi(x,y)$$

for all $x, y \in \mathcal{G}$, then there exists a unique mapping $T : \mathcal{G} \to \mathcal{E}$ such that $T(x+y) = T(x) + T(y)$ and $\big\|f(x) - T(x)\big\| \le \widetilde{\varphi}(x,x)$ for all $x, y \in \mathcal{G}$.

During the last decades, several stability problems of functional equations have been investigated by a number of mathematicians, see ([4]-[6]) and references therein for more detailed information.

A *valuation* is a function $|\cdot|$ from a field $\mathcal{K}$ into $[0,\infty)$ such that, for all $r, s \in \mathcal{K}$, the following conditions hold: (i) $|r| = 0$ if and only if $r = 0$, (ii) $|rs| = |r||s|$, (iii) $|r+s| \le |r| + |s|$.

A field $\mathcal{K}$ is called a *valued field* if $\mathcal{K}$ carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. If the triangle inequality is replaced by $|r+s| \le \max\{|r|, |s|\}$ for all $r, s \in \mathcal{K}$, then the function $|\cdot|$ is called a *non-Archimedean valuation* and the field is called a *non-Archimedean field.* Clearly, $|1| = |-1| = 1$ and $|n| \le 1$ for all $n \in \mathbb{N}$.

Definition 1. Let $\mathcal{X}$ be a vector space over a scalar field $\mathcal{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : \mathcal{X} \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions: (a) $\|x\| = 0$ if and only if $x = 0$, (b) $\|rx\| = |r|\|x\|$, (c) $\|x+y\| \le \max\{\|x\|, \|y\|\}$ for all $x, y \in \mathcal{X}$ and all $r \in \mathcal{K}$. Then $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean space.

Definition 2. Let $\{x_n\}$ be a sequence in a non-Archimedean normed space $\mathcal{X}$.

(1) The sequence $\{x_n\}$ is called a Cauchy sequence if, for any $\varepsilon > 0$, there is a positive integer N such that $\|x_n - x_m\| \le \varepsilon$ for all $n, m \in \mathbb{N}$.

(2) The sequence $\{x_n\}$ is said to be convergent if, for any $\varepsilon > 0$, there is a positive integer N and $x \in \mathcal{X}$ such that $\|x_n - x\| \le \varepsilon$ for all $n \in \mathbb{N}$. Then the point $x \in \mathcal{X}$ is called the limit of the sequence $\{x_n\}$, which is denote by $\lim_{n\to\infty} x_n = x$.

(3) If every Cauchy sequence in $\mathcal{X}$ converges, then the non-Archimedean normed space $\mathcal{X}$ is called a non-Archimedean Banach space.

C. Park, Y. S. Cho and M.-H. Han [7] proved the generalized Hyers-Ulam stability of the additive functional inequality

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|$$

in Banach spaces. In 2011, J. R. Lee, C. Park and D. Y. Shin [8] prove the generalized Hyers-Ulam stability of the additive functional inequality $\|f(2x) + f(2y)+2f(z)\| \leq \|2f(x+y+z)\|$ in Banach spaces. A. Ebadian, N. Ghobadipour, Th. M. Rassias and M. Gordji [9] showed the generalized Hyers-Ulam stability of the functional inequalities

$$\left\|f\left(\frac{x+y+z}{4}\right) + f\left(\frac{3x-y-4z}{4}\right) + f\left(\frac{4x+3z}{4}\right)\right\| \leq \|2f(x)\|,$$
$$\left\|f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right)\right\| \leq \|f(x)\|$$

in non-Archimedean spaces.

Now, we consider the following functional inequality:

$$\|f(x) + f(y) + f(-z)\| \leq \|f(x + y) - f(z)\|. \tag{1}$$

In this paper, we investigate the generalized Hyers-Ulam stability of the functional inequality (1). In Section 2, using direct method we prove the stability of (1) in Banach spaces. In Section 3, we investigate the stability of (1) in non-Archimedean Banach spaces.

2. Stability of Functional Inequality (1)

Throughout this section, let $\mathcal{X}$ be a normed linear space and $\mathcal{Y}$ a Banach space.

Lemma 3. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then it is additive if and only if it satisfies*

$$\|f(x) + f(y) + f(-z)\| \leq \|f(x + y) - f(z)\| \tag{2}$$

for all $x, y, z \in \mathcal{X}$.

Proof. If f is additive, then clearly

$$\|f(x) + f(y) + f(-z)\| = \|f(x + y) - f(z)\|$$

for all $x, y, z \in \mathcal{X}$.

Assume that f satisfies (2). Letting $x = y = z = 0$ in (2), we gain $\|3f(0)\| \leq 0$ and so $f(0) = 0$. Putting $y = 0$ and $z = x$ in (2), we get $\|f(x) + f(-x)\| \leq 0$ and so $f(-x) = -f(x)$ for all $x \in \mathcal{X}$. Replacing x by $2x$ and setting $y = -x$ and $z = x$, we obtain

$$\|f(2x) - 2f(x)\| \leq 0$$

and so $f(2x) = 2f(x)$ for all $x \in \mathcal{X}$. Taking $x = -2y - 2z$ and replacing y and z by $2y$ and $-2z$, respectively, in (2), we see that

$$\| -f(2y + 2z) + 2f(y) + 2f(z)\| \leq 0$$

for all $y, z \in \mathcal{X}$. Thus we see that $f(y + z) = f(y) + f(z)$ for all $y, z \in \mathcal{X}$. □

Theorem 4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying*

$$\|f(x) + f(y) + f(-z)\| \leq \|f(x + y) - f(z)\| + \varphi(x, y, z), \tag{3}$$

$$\widetilde{\varphi}(x, y, z) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi\big((-2)^j x, (-2)^j y, (-2)^j z\big) < \infty \tag{4}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - A(x)\| \leq \frac{1}{2} \widetilde{\varphi}(x, x, 2x) \tag{5}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $(-2)^j x, (-2)^j x, -(-2)^{j+1} x$, respectively, and dividing by 2^{j+1} in (3), we get

$$\left\| \frac{f((-2)^j x)}{(-2)^j} - \frac{f((-2)^{j+1} x)}{(-2)^{j+1}} \right\| \leq \frac{1}{2^{j+1}} \varphi\big((-2)^j x, (-2)^j x, -(-2)^{j+1} x\big)$$

for all $x \in \mathcal{X}$ and all nonnegative integers j. From the above inequality, we have

$$\begin{aligned} \left\| \frac{f((-2)^m x)}{(-2)^m} - \frac{f((-2)^n x)}{(-2)^n} \right\| &\leq \sum_{j=m}^{n-1} \left\| \frac{f((-2)^j x)}{(-2)^j} - \frac{f((-2)^{j+1} x)}{(-2)^{j+1}} \right\| \\ &\leq \sum_{j=m}^{n-1} \frac{1}{2^{j+1}} \varphi\big((-2)^j x, (-2)^j x, -(-2)^{j+1} x\big) \end{aligned} \tag{6}$$

for all $x \in \mathcal{X}$ and all nonnegative integers m, n with $m < n$. By the condition (4), the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{f((-2)^n x)}{(-2)^n}\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} \frac{f((-2)^n x)}{(-2)^n}$$

for all $x \in \mathcal{X}$. Taking $m = 0$ and letting n tend to ∞ in (6), we have the inequality (5).

Replacing x, y, z by $(-2)^n x, (-2)^n y, (-2)^n z$, respectively, and dividing by 2^n in (3), we obtain

$$\begin{aligned}\left\|\frac{f((-2)^n x)}{(-2)^n} + \frac{f((-2)^n y)}{(-2)^n} + \frac{f(-(-2)^n z)}{(-2)^n}\right\| \\ \leq \left\|\frac{f((-2)^n (x+y))}{(-2)^n} - \frac{f((-2)^n z)}{(-2)^n}\right\| \\ + \frac{1}{2^n}\varphi\big((-2)^n x, (-2)^n y, (-2)^n z\big)\end{aligned}$$

for all $x, y, z \in \mathcal{X}$ and all nonnegative integers n. Since (4) gives that

$$\lim_{n\to\infty} \frac{1}{2^n}\varphi((-2)^n x, (-2)^n y, (-2)^n z) = 0$$

for all $x, y, z \in \mathcal{X}$, letting n tend to ∞ in the above inequality, we see that A satisfies the inequality (2) and so it is additive by Lemma 3.

Let $A' : \mathcal{X} \to \mathcal{Y}$ be another mapping satisfying (5). Since both A and A' are additive, we have

$$\begin{aligned}&\big\|A(x) - A'(x)\big\| = \frac{1}{2^n}\big\|A((-2)^n x) - A'((-2)^n x)\big\| \\ &\leq \frac{1}{2^n}\big(\big\|A((-2)^n x) - f((-2)^n x)\big\| + \big\|f((-2)^n x) - A'((-2)^n x)\big\|\big) \\ &\leq \frac{1}{2^n}\widetilde{\varphi}\big((-2)^n x, (-2)^n x, -(-2)^{n+1} x\big) \\ &= \sum_{j=n}^{\infty} \frac{1}{2^j}\varphi\big((-2)^j x, (-2)^j x, -(-2)^{j+1} x\big)\end{aligned}$$

which goes to zero as $n \to \infty$ for all $x \in \mathcal{X}$ by (4). Therefore, A is a unique additive mapping satisfying (5), as desired. □

Corollary 5. *Let $p < 1$ and θ be positive real numbers, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + f(-z)\| \leq \|f(x+y) - f(z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $\|f(x) - A(x)\| \leq \frac{2+2^p}{2-2^p}\theta\|x\|^p$ for all $x \in \mathcal{X}$.

Proof. Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function given by

$$\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. We get

$$\widetilde{\varphi}(x, y, z) = \frac{2\theta}{2 - 2^p}(\|x\|^p + \|y\|^p + \|z\|^p) < \infty$$

for all $x, y, z \in \mathcal{X}$. Applying Theorem 4, there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the desired inequality. □

Theorem 6. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. If there is a function $\varphi : X^3 \to [0, \infty)$ satisfying* (3) *and*

$$\widetilde{\varphi}(x, y, z) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x}{(-2)^j}, \frac{y}{(-2)^j}, \frac{z}{(-2)^j}\right) < \infty \tag{7}$$

for all $x, y, z \in \mathcal{X}$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - A(x)\big\| \leq \frac{1}{2}\widetilde{\varphi}(x, x, 2x) \tag{8}$$

for all $x \in \mathcal{X}$.

Proof. Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{x}{(-2)^n}, \frac{2x}{(-2)^n}$, respectively, and multiplying by 2^{n-1} in (3), we have

$$\begin{aligned}\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^{n-1} f\left(\frac{x}{(-2)^{n-1}}\right)\right\| \\ \leq 2^{n-1}\varphi\left(\frac{x}{(-2)^n}, \frac{x}{(-2)^n}, \frac{2x}{(-2)^n}\right)\end{aligned}$$

for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. From the above inequality, for $m > n$, we get

$$\left\|(-2)^n f\left(\frac{x}{(-2)^n}\right) - (-2)^m f\left(\frac{x}{(-2)^m}\right)\right\| \tag{9}$$

$$\leq \sum_{j=n+1}^{m} \left\| (-2)^j f\left(\frac{x}{(-2)^j}\right) - (-2)^{j-1} f\left(\frac{x}{(-2)^{j-1}}\right) \right\|$$
$$\leq \sum_{j=n+1}^{m} 2^{j-1} \varphi\left(\frac{x}{(-2)^j}, \frac{x}{(-2)^j}, \frac{2x}{(-2)^j}\right)$$

for all $x \in \mathcal{X}$. From (7), the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is a Banach space, the sequence $\left\{(-2)^n f\left(\frac{x}{(-2)^n}\right)\right\}$ converges for all $x \in \mathcal{X}$. So one can define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty} (-2)^n f\left(\frac{x}{(-2)^n}\right)$$

for all $x \in \mathcal{X}$. To prove that A satisfies (8), putting $n = 0$ and letting $m \to \infty$ in (9), we have

$$\left\| f(x) - A(x) \right\| \leq \sum_{j=1}^{\infty} 2^{j-1} \varphi\left(\frac{x}{(-2)^j}, \frac{x}{(-2)^j}, \frac{2x}{(-2)^j}\right) = \frac{1}{2} \widetilde{\varphi}(x, x, 2x)$$

for all $x \in \mathcal{X}$.

Replacing x, y, z by $\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}$, respectively, and multiplying by 2^n in (3), we obtain

$$\left\| (-2)^n f\left(\frac{x}{(-2)^n}\right) + (-2)^n f\left(\frac{y}{(-2)^n}\right) + (-2)^n f\left(\frac{-z}{(-2)^n}\right) \right\|$$
$$\leq \left\| (-2)^n f\left(\frac{x+y}{(-2)^n}\right) - (-2)^n f\left(\frac{z}{(-2)^n}\right) \right\|$$
$$+ 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right)$$

for all $x, y, z \in \mathcal{X}$ and all $n \in \mathbb{N}$. Since (7) gives that

$$\lim_{n\to\infty} 2^n \varphi\left(\frac{x}{(-2)^n}, \frac{y}{(-2)^n}, \frac{z}{(-2)^n}\right) = 0$$

for all $x, y, z \in \mathcal{X}$, if we let $n \to \infty$ in the above inequality, then we have

$$\|A(x) + A(y) + A(-z)\| \leq \|A(x + y) - A(z)\|$$

and so A is additive by Lemma 3. The rest of the proof is similar to the corresponding part of the proof of Theorem 4. $\square$

Corollary 7. *Let $p > 1$ and θ be positive real numbers, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|f(x) + f(y) + f(-z)\| \le \|f(x+y) - f(z)\| + \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. Then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $\|f(x) - A(x)\| \le \frac{2+2^p}{2^p-2}\theta\|x\|^p$ for all $x \in \mathcal{X}$.

Proof. Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function given by

$$\varphi(x, y, z) := \theta(\|x\|^p + \|y\|^p + \|z\|^p)$$

for all $x, y, z \in \mathcal{X}$. We have

$$\widetilde{\varphi}(x, y, z) = \frac{2^p\theta}{2^p - 2}(\|x\|^p + \|y\|^p + \|z\|^p) < \infty$$

for all $x, y, z \in \mathcal{X}$. Applying Theorem 6, there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the desired inequality. □

3. Non-Archimedean Stability of Functional Inequality (1)

We investigate the generalized Hyers-Ulam stability of the functional inequality (1). Throughout this section, let $\mathcal{X}$ be a non-Archimedean normed space and $\mathcal{Y}$ a non-Archimedean Banach space.

Theorem 8. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a mapping satisfying*

$$\lim_{n\to\infty} \frac{\varphi(2^n x, 2^n y, 2^n z)}{|2|^n} = 0 \tag{10}$$

for all $x, y, z \in \mathcal{X}$. Suppose that the limit

$$\widetilde{\varphi}(x) := \lim_{n\to\infty} \max_{0\le j<n} \left\{ \frac{\varphi(2^j x, 2^j x, 2^{j+1}x)}{|2|^j}, \frac{\varphi(2^{j+1}x, -2^{j+1}x, 0)}{|2|^j} \right\} \tag{11}$$

exists for all $x \in \mathcal{X}$ and $f : \mathcal{X} \to \mathcal{Y}$ is a mapping with $f(0) = 0$ such that

$$\big\|f(x) + f(y) + f(-z)\big\| \le \big\|f(x+y) - f(z)\big\| + \varphi(x, y, z) \tag{12}$$

for all $x, y, z \in \mathcal{X}$. Then there exists an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\|f(x) - A(x)\big\| \le \frac{1}{|2|}\widetilde{\varphi}(x) \tag{13}$$

for all $x \in \mathcal{X}$. *Moreover, if*

$$\lim_{k\to\infty}\lim_{n\to\infty}\max_{k\le j<n+k}\left\{\frac{\varphi(2^j x, 2^j y, 2^{j+1} z)}{|2|^j}, \frac{\varphi(2^{j+1}x, -2^{j+1}y, 0)}{|2|^j}\right\} = 0,$$

then A *is the unique additive mapping satisfying* (13).

Proof. Putting $y = x$ and $z = 2x$ in (12), we have

$$\left\|2f(x) + f(-2x)\right\| \le \varphi(x, x, 2x)$$

for all $x \in \mathcal{X}$. Replacing $x = 2x$, $y = -2x$ and $z = 0$ in (12), we get

$$\left\|f(2x) + f(-2x)\right\| \le \varphi(2x, -2x, 0)$$

for all $x \in \mathcal{X}$. It follows from the above two inequalities that

$$\left\|2f(x) - f(2x)\right\| \le \max\left\{\varphi(x, x, 2x), \varphi(2x, -2x, 0)\right\} \tag{14}$$

for all $x \in \mathcal{X}$. Replacing x by $2^n x$ and dividing by $|2|^{n+1}$ in the above inequality, we obtain

$$\left\|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right\| \le \frac{1}{|2|^{n+1}}\max\left\{\varphi(2^n x, 2^n x, 2^{n+1}x), \varphi(2^{n+1}x, -2^{n+1}x, 0)\right\}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. It follows from (10) and the above inequality that the sequence $\left\{\frac{f(2^n x)}{2^n}\right\}$ is a Cauchy sequence. Because of the completeness of $\mathcal{Y}$, we define a mapping $A : \mathcal{X} \to \mathcal{Y}$ by

$$A(x) := \lim_{n\to\infty}\frac{f(2^n x)}{2^n}$$

for all $x \in \mathcal{X}$. Using induction one can show that

$$\left\|\frac{f(2^n x)}{2^n} - f(x)\right\| \le \frac{1}{|2|}\max_{0\le k<n}\left\{\frac{\varphi(2^k x, 2^k x, 2^{k+1}x)}{|2|^k}, \frac{\varphi(2^{k+1}x, -2^{k+1}x, 0)}{|2|^k}\right\}$$

for all $x \in \mathcal{X}$ and all nonnegative integers n. Taking $n \to \infty$ in the above inequality and using (11), we can get (13).

It follows from (12) that

$$\begin{aligned}\big\|A(x)+A(y)+A(-z)\big\| &= \lim_{n\to\infty}\frac{1}{|2|^n}\big\|f(2^nx)+f(2^ny)+f(-2^nz)\big\|\\ &\le \lim_{n\to\infty}\frac{1}{|2|^n}\big[\,\big\|f(2^nx+2^ny)-f(2^nz)\big\|+\varphi(2^nx,2^ny,2^nz)\,\big]\\ &= \big\|A(x+y)-A(z)\big\|\end{aligned}$$

for all $x,y,z\in\mathcal{X}$. Therefore, the mapping $A:\mathcal{X}\to\mathcal{Y}$ is additive by Lemma 3.

Now, we consider another additive mapping $A':\mathcal{X}\to\mathcal{Y}$ satisfying (13) to show the uniqueness of A. Then we have

$$\begin{aligned}\big\|A(x)-A'(x)\big\| &= \frac{1}{|2|^k}\big\|A(2^kx)-A'(2^kx)\big\|\\ &\le \frac{1}{|2|^k}\max\big\{\big\|A(2^kx)-f(2^kx)\big\|,\big\|f(2^kx)-A'(2^kx)\big\|\big\}\\ &\le \frac{1}{|2|}\lim_{n\to\infty}\max_{k\le i<n+k}\left\{\frac{\varphi(2^ix,2^ix,2^{i+1}x)}{|2|^i},\frac{\varphi(2^{i+1}x,-2^{i+1}x,0)}{|2|^i}\right\}\end{aligned}$$

for all $x\in\mathcal{X}$. Since the right-hand side of the above inequality tends to zero by taking $k\to\infty$, $\big\|A(x)-A'(x)\big\|=0$ for all $x\in\mathcal{X}$. Therefore, $A=A'$. This completes the proof. □

Corollary 9. *Let p and θ be positive real numbers with $p>1$, and let $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying*

$$\big\|f(x)+f(y)+f(-z)\big\|\le\big\|f(x+y)-f(z)\big\|+\theta\big(\|x\|^p+\|y\|^p+\|z\|^p\big)$$

for all $x,y,z\in\mathcal{X}$. If $|2|<1$, then there exists a unique additive mapping $A:\mathcal{X}\to\mathcal{Y}$ such that $\big\|f(x)-A(x)\big\|\le\frac{2+|2|^p}{|2|}\theta\|x\|^p$ for all $x\in\mathcal{X}$.

Proof. Taking $\varphi(x,y,z):=\theta\big(\|x\|^p+\|y\|^p+\|z\|^p\big)$, we have

$$\lim_{n\to\infty}\frac{\varphi(2^nx,2^ny,2^nz)}{|2|^n}=\lim_{n\to\infty}\frac{|2|^{np}}{|2|^n}\theta(\|x\|^p+\|y\|^p+\|z\|^p)=0$$

for all $x,y,z\in\mathcal{X}$ and all nonnegative integers n. In addition, the limit

$$\begin{aligned}\widetilde{\varphi}(x) &:= \lim_{n\to\infty}\max_{0\le j<n}\left\{\frac{\varphi(2^jx,2^jx,2^{j+1}x)}{|2|^j},\frac{\varphi(2^{j+1}x,-2^{j+1}x,0)}{|2|^j}\right\}\\ &= (2+|2|^p)\theta\|x\|^p\end{aligned}$$

exists for all $x \in \mathcal{X}$. Also,

$$\lim_{k\to\infty} \lim_{n\to\infty} \max_{k\le j<n+k} \left\{ \frac{\varphi(2^j x, 2^j y, 2^{j+1} z)}{|2|^j}, \frac{\varphi(2^{j+1}x, -2^{j+1}y, 0)}{|2|^j} \right\} \\ = \lim_{k\to\infty} |2|^{k(p-1)} \big(\|x\|^p + \|y\|^p + |2|^p \|z\|^p \big) = 0$$

for all $x, y, z \in \mathcal{X}$.

Applying Theorem 8, there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the desired inequality. □

Theorem 10. *Let $\varphi : \mathcal{X}^3 \to [0, \infty)$ be a function satisfying*

$$\lim_{n\to\infty} |2|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0 \tag{15}$$

for all $x, y, z \in \mathcal{X}$ and let the limit

$$\widetilde{\varphi}(x) := \lim_{n\to\infty} \max_{1\le i\le n} \left\{ |2|^i \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}, \frac{2x}{2^i} \right), |2|^i \varphi\left(\frac{2x}{2^i}, \frac{-2x}{2^i}, 0 \right) \right\}$$

exist for all $x \in \mathcal{X}$. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying

$$\big\| f(x) + f(y) + f(-z) \big\| \le \big\| f(x+y) - f(z) \big\| + \varphi(x, y, z)$$

for all $x, y, z \in \mathcal{X}$. Then there is an additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that

$$\big\| f(x) - A(x) \big\| \le \frac{1}{|2|} \widetilde{\varphi}(x) \tag{16}$$

for all $x \in \mathcal{X}$. Moreover, if

$$\lim_{k\to\infty} \lim_{n\to\infty} \max_{k+1\le i\le k+n} \left\{ |2|^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{2z}{2^i} \right), |2|^i \varphi\left(\frac{2x}{2^i}, \frac{-2y}{2^i}, 0 \right) \right\} = 0,$$

then A is the unique additive mapping satisfying (16).

Proof. It follows from (14) that

$$\left\| 2^n f\left(\frac{x}{2^n} \right) - 2^{n-1} f\left(\frac{x}{2^{n-1}} \right) \right\| \\ \le |2|^{n-1} \max \left\{ \varphi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{2x}{2^n} \right), \varphi\left(\frac{2x}{2^n}, \frac{-2x}{2^n}, 0 \right) \right\}$$

for all $x \in \mathcal{X}$ and all $n \in \mathbb{N}$. It follows from (15) and the above inequality that the sequence $\left\{ 2^n f\left(\frac{x}{2^n} \right) \right\}$ is a Cauchy sequence for all $x \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\left\{ 2^n f\left(\frac{x}{2^n} \right) \right\}$ converges. Thus one can define the mapping $A : \mathcal{X} \to \mathcal{Y}$ by $A(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n} \right)$ for all $x \in \mathcal{X}$. The rest of the proof is similar to the corresponding part of the proof of Theorem 8. □

Corollary 11. *Let p and θ be positive real numbers with $p < 1$, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\big\|f(x) + f(y) + f(-z)\big\| \le \big\|f(x+y) - f(z)\big\| + \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in \mathcal{X}$. If $|2| < 1$, then there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ such that $\big\|f(x) - A(x)\big\| \le \frac{2+|2|^p}{|2|^p}\theta\|x\|^p$ for all $x \in \mathcal{X}$.

Proof. Let $\varphi : \mathcal{X}^3 \to [0,\infty)$ be a function given by

$$\varphi(x, y, z) := \theta\big(\|x\|^p + \|y\|^p + \|z\|^p\big)$$

for all $x, y, z \in \mathcal{X}$. We get $\lim_{n\to\infty} |2|^n \varphi\big(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\big) = 0$ and

$$\begin{aligned}\widetilde{\varphi}(x) &= \lim_{n\to\infty} \max_{1\le i\le n} \left\{ |2|^i \varphi\left(\frac{x}{2^i}, \frac{x}{2^i}, \frac{2x}{2^i}\right), |2|^i \varphi\left(\frac{2x}{2^i}, \frac{-2x}{2^i}, 0\right)\right\} \\ &= \lim_{n\to\infty} \max_{1\le i\le n} \left\{ |2|^{i(1-p)}(2 + |2|^p)\theta\|x\|^p, |2|^{i(1-p)}(2\cdot|2|^p)\theta\|x\|^p \right\} \\ &= |2|^{1-p}(2 + |2|^p)\theta\|x\|^p\end{aligned}$$

for all $x, y, z \in \mathcal{X}$. Moreover,

$$\begin{aligned}&\lim_{k\to\infty} \lim_{n\to\infty} \max_{k+1\le i\le k+n} \left\{ |2|^i \varphi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{2z}{2^i}\right), |2|^i \varphi\left(\frac{2x}{2^i}, \frac{-2y}{2^i}, 0\right)\right\} \\ &\qquad = \lim_{k\to\infty} |2|^{(k+1)(1-p)}\big(\|x\|^p + \|y\|^p + |2|^p\|z\|^p\big) = 0\end{aligned}$$

for all $x, y, z \in \mathcal{X}$.

Applying Theorem 10, there exists a unique additive mapping $A : \mathcal{X} \to \mathcal{Y}$ satisfying the desired inequality. □

References

[1] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York (1960).

[2] D.H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, **27** (1941), 222-224.

[3] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *Journal of Mathemetical Analysis and Applications*, **184** (1994), 431-436.

[4] S. Czerwik (Ed.), *Stability of Functional Equations of Ulam-Hyers-Rassias Type*, Hadronic Press, Palm Harbor, Florida (2003).

[5] S.-M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, Florida (2001).

[6] Th.M. Rassias (Ed.), *Functional Equations, Inequalities and Appications*, Kluwer Academic Publishers, Dordrecht, Boston, London (2003).

[7] C. Park, Y.S. Cho, M.-H. Han, Functional inequalities associated with Jordan-von Neumann-type additive functional equations, *J. Inequal. Appl.*, **2007** (2007), Article ID 41820, 13 pages.

[8] J.R. Lee, C. Park, D.Y. Shin, Stability of an additive functional inequality in proper CQ^*-algebras, *Bull. Korean Math. Soc.*, **48** (2011), 853-871.

[9] A. Ebadian, N. Ghobadipour, Th.M. Rassias, M. Eshaghi Gordji, Functional inequalities associatied with Cauchy addtive functional equation in non-Archimedean spaces, *Discreate Dynamics in Nature and Society*, **2011** (2011), Article ID 929824, 14 pages.