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MODELING, STABILITY ANALYSIS AND SCHEDULING DESIGN FOR A CLASS OF QUEUING SYSTEMS BY MEANS OF TIMED PETRI NETS, LYAPUNOV METHODS AND MAX-PLUS ALGEBRA

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Abstract: A queuing system, is a dynamical system whose state evolves in time by the occurrence of events at possibly irregular time intervals. Place-transitions Petri nets (commonly called Petri nets) are a graphical and mathematical modeling tool applicable to queuing systems in order to represent its states evolution. Timed Petri nets are an extension of Petri nets, where now the timing at which the state changes is taken into consideration. One of the most important performance issues to be considered in a queuing system is its stability. Lyapunov stability theory provides the required tools needed to aboard the stability problem for queuing systems modeled with timed Petri nets whose mathematical model is given in terms of difference equations. By proving practical stability one is allowed to preassigned the bound on the queuing systems dynamics performance. Moreover, employing Lyapunov methods, a sufficient condition for the stabilization problem is also obtained. It is shown that it is possible to restrict the queuing systems state space in such a way that boundedness is guaranteed. However, this restriction results to be vague. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra, which is assigned to the timed Petri net graphical model. Moreover, by using max-plus algebra a schedule for the queuing system is set.

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1. Introduction

A queuing system, is a dynamical system whose state evolves in time by the occurrence of events at possibly irregular time intervals. Place-transitions Petri nets (commonly called Petri nets) are a graphical and mathematical modeling tool that can be applied to queuing systems in order to represent its states evolution. Petri nets are known to be useful for analyzing the systems properties in addition of being a paradigm for describing and studying information processing systems. Timed Petri nets are an extension of Petri nets, where now the timing at which the state changes is taken into consideration. This is of critical importance since it allows to consider useful measures of performance as for example: how long does the queuing system spends at a given state etc. For a detailed discussion of Petri net theory see [1] and the references quoted therein. One of the most important performance issues to be considered in a queuing system is its stability. Lyapunov stability theory provides the required tools needed to aboard the stability problem for queuing systems modeled with timed Petri nets whose mathematical model is given in terms of difference equations [2]. By proving practical stability one is allowed to preassigned the bound on the queuing systems dynamics performance. Moreover, employing Lyapunov methods, a sufficient condition for the stabilization problem is also obtained. It is shown that it is possible to restrict the queuing systems state space in such a way that boundedness is guaranteed. However, this restriction results to be vague. This inconvenience is overcome by considering a specific recurrence equation, in the max-plus algebra, which is assigned to the the timed Petri net graphical model. Moreover, by using max-plus algebra a schedule for the queuing system is set. This paper proposes a new methodology consisting in combining Lyapunov theory with max-plus algebra to give a precise solution to the stability and scheduling design problem for queuing systems modeled with timed Petri nets. The presented methodology applied to queuing systems is new and results to be innovative. It is worth mentioning the work done in [3] where the stability for parallel queuing systems is addressed following a stochastic approach. The paper is organized as follows. In Section 2, Lyapunov theory for queuing systems modeled with Petri nets is given. Section 3, presents max-plus algebra. In Section 4, generalized eigenmodes and recurrence equations are discussed. Section 5, introduces an algorithm for computing generalized eigenmodes of reducible matrices. In Section 6, the solution to the stability problem for queuing systems modeled with Petri nets is considered. In Section 7 the modeling, stability analysis and scheduling design for queuing systems is addressed. Finally, the paper ends with some conclusions.

2. Lyapunov Stability and Stabilization of Queuing Systems Modeled with Petri Nets

The solution to the stability problem for queuing systems, whose model is obtained employing timed Petri nets, is achieved thanks to the theory of vector Lyapunov functions and comparison principles. The methodology shows that it is possible to restrict the systems state space in such a way that boundedness is guaranteed.

Notation. $N = \{0, 1, 2, ...\}$, $R_+ = [0, \infty)$, $N^+_{n_0} = \{n_0, n_0 + 1, ..., n_0 + k, ...\}$, $n_0 \geq 0$. Given $x, y \in R^n$, we usually denote the relation "$\leq$" to mean componentwise inequalities with the same relation, i.e., $x \leq y$ is equivalent to $x_i \leq y_i, \forall i$. A function $f(n, x)$, $f : N^+_{n_0} \times R^n \to R^n$ is called nondecreasing in x if given $x, y \in R^n$ such that $x \geq y$ and $n \in N^+_{n_0}$ then, $f(n, x) \geq f(n, y)$.

Consider systems of first ordinary difference equations given by

$$x(n + 1) = f[n, x(n)],\ x(n_o) = x_0,\ n \in N^+_{n_0} \tag{1}$$

where $n \in N^+_{n_0}$, $x(n) \in R^n$ and $f : N^+_{n_0} \times R^n \to R^n$ is continuous in $x(n)$.

Definition 1. The n vector valued function $\Phi(n, n_0, x_0)$ is said to be a solution of (1) if $\Phi(n_0, n_0, x_0) = x_0$ and $\Phi(n + 1, n_0, x_0) = f(n, \Phi(n, n_0, x_0))$ for all $n \in N^+_{n_0}$.

Definition 2. The system (1) is said to be

i). Practically stable, if given (λ, A) with $0 < \lambda < A$, then

$$|x_0| < \lambda \Rightarrow |x(n, n_0, x_0)| < A,\ \forall n \in N^+_{n_0},\ n_0 \geq 0;$$

ii). Uniformly practically stable, if it is practically stable for every $n_0 \geq 0$.

The following class of function is defined.

Definition 3. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing.

Consider a vector Lyapunov function $v(n, x(n))$, $v : N^+_{n_0} \times R^n \to R^p_+$ and define the variation of v relative to (1) by

$$\Delta v = v(n + 1, x(n + 1)) - v(n, x(n)) \tag{2}$$

Then, the following result concerns the practical stability of (1).

Theorem 4. *[4] Let $v : N^+_{n_0} \times R^n \to R^p_+$ be a continuous function in x, define the function $v_0(n, x(n)) = \sum_{i=1}^p v_i(n, x(n))$ such that satisfies the estimates*

$$b(|x|) \leq v_0(n, x(n)) \leq a(|x|) \text{ for } a, b \in \mathcal{K} \text{ and}$$

$$\Delta v(n, x(n)) \leq w(n, v(n, x(n)))$$

for $n \in N^+_{n_0}$, $x(n) \in R^n$, where $w : N^+_{n_0} \times R^p_+ \to R^p$ is a continuous function in the second argument.

Assume that: $g(n, e) \triangleq e + w(n, e)$ is nondecreasing in e, $0 < \lambda < A$ are given and finally that $a(\lambda) < b(A)$ is satisfied.

Then, the practical stability properties of

$$e(n+1) = g(n, e(n)),\ e(n_0) = e_0 \geq 0. \tag{3}$$

imply the practical stability properties of system (1).

Proof. Let us suppose that $e(n+1)$ is practically stable for $(a(\lambda), b(A))$ then, we have that $\sum_{i=1}^p e_{0_i} < a(\lambda) \Rightarrow \sum_{i=1}^p e_i(n, n_0, e_0) < b(A)$ for $n \geq n_0$ where $e_i(n, n_0, e_0)$ is the vector solution of (3). Let $\|x_0\| < \lambda$, we claim that

$$\|x(n, n_0, x_0)\| < A \text{ for } n \geq n_0.$$

If not, there would exist $n_1 \geq n_0$ and a solution $x(n, n_0, x_0)$ such that $\|x(n_1)\| \geq A$ and $\|x(n)\| < A$ for $n_0 \leq n < n_1$. Choose $e_0 = v(n_0, x_0)$ then $v(n, x(n)) \leq e(n, n_0, e_0)$ for all $n \geq n_0$. (If not $v(n, x(n)) \leq e(n, n_0, e_0)$ and $v(n+1, x(n+1)) > e(n+1, n_0, e_0) \Rightarrow g(n, e(n)) = e(n+1, n_0, e_0) < v(n+1, x(n+1)) = \Delta v(n, x_0) + v(n, x(n)) \leq w(n, v(n)) + v(n, x(n)) = g(n, v(n)) - v(n, x(n)) + v(n, x(n)) = g(n, v(n)) \leq g(n, e(n))$ which is a contradiction). Hence we get that $b(A) \leq b(\|x(n_1)\|) \leq v_0(n_1, x(n_1)) \leq \sum_{i=1}^p e_i(n_1, n_0, e_0) < b(A)$, which can not hold therefore, system (1) is practically stable. $\square$

Corollary 5. *In Theorem (4):*

i). If $w(n, e) \equiv 0$ we get uniform practical stability of (1) which implies structural stability.

ii). If $w(n, e) = -c(e)$, for $c \in \mathcal{K}$, we get uniform practical asymptotic stability of (1).

Definition 6. A Petri net is a 5-tuple, $PN = \{P, T, F, W, M_0\}$ where:

$P = \{p_1, p_2, ..., p_m\}$is a finite set of places,
$T = \{t_1, t_2, ..., t_n\}$ is a finite set of transitions,
$F \subset (P \times T) \cup (T \times P)$ is a set of arcs,
$W : F \to N_1^+$ is a weight function,
M_0: $P \to N$ is the initial marking,
$P \cap T = \varnothing$ and $P \cup T \neq \varnothing$.

Definition 7. The clock structure associated with a place $p_i \in P$ is a set $\mathbf{V} = \{V_i : p_i \in P\}$ of clock sequences $V_i = \{v_{i,1}, v_{i,2}, ...\}$, $v_{i,k} \in R^+$, $k = 1, 2, ...$

The positive number $v_{i,k}$, associated to $p_i \in P$, called holding time, represents the time that a token must spend in this place until its outputs enabled transitions $t_{i,1}, t_{i,2}, ...$, fire. Some places may have a zero holding time while others not. Thus, we partition P into subsets P_0 and P_h, where P_0 is the set of places with zero holding time, and P_h is the set of places that have some holding time.

Definition 8. A timed Petri net is a 6-tuple $TPN = \{P, T, F, W, M_0, \mathbf{V}\}$ where $\{P, T, F, W, M_0\}$ are as before, and $\mathbf{V} = \{V_i : p_i \in P\}$ is a clock structure. A timed Petri net is a timed event petri net when every $p_i \in P$ has one input and one output transition, in which case the associated clock structure set of a place $p_i \in P$ reduces to one element $V_i = \{v_i\}$

A PN structure without any specific initial marking is denoted by N. A Petri net with the given initial marking is denoted by (N, M_0). Notice that if $W(p, t) = \alpha$ (or $W(t, p) = \beta$) then, this is often represented graphically by α, (β) arcs from p to t (t to p) each with no numeric label.

Let $M_k(p_i)$ denote the marking (i.e., the number of tokens) at place $p_i \in P$ at time k and let $M_k = [M_k(p_1), ..., M_k(p_m)]^T$ denote the marking (state) of PN at time k. A transition $t_j \in T$ is said to be enabled at time k if $M_k(p_i) \geq W(p_i, t_j)$ for all $p_i \in P$ such that $(p_i, t_j) \in F$. It is assumed that at each time k there exists at least one transition to fire. If a transition is enabled then, it can fire. If an enabled transition $t_j \in T$ fires at time k then, the next marking for $p_i \in P$ is given by

$$M_{k+1}(p_i) = M_k(p_i) + W(t_j, p_i) - W(p_i, t_j). \tag{4}$$

Let $A = [a_{ij}]$ denote an $n \times m$ matrix of integers (the incidence matrix) where $a_{ij} = a_{ij}^+ - a_{ij}^-$ with $a_{ij}^+ = W(t_i, p_j)$ and $a_{ij}^- = W(p_j, t_i)$. Let $u_k \in \{0, 1\}^n$ denote a firing vector where if $t_j \in T$ is fired then, its corresponding firing vector is $u_k = [0, ..., 0, 1, 0, ..., 0]^T$ with the one in the j^{th} position in the vector and zeros everywhere else. The matrix equation (nonlinear difference equation) describing the dynamical behavior represented by a PN is:

$$M_{k+1} = M_k + A^T u_k \tag{5}$$

where if at step k, $a_{ij}^{-} < M_k(p_j)$ for all $p_i \in P$ then, $t_i \in T$ is enabled and if this $t_i \in T$ fires then, its corresponding firing vector u_k is utilized in the difference equation to generate the next step. Notice that if M' can be reached from some other marking M and, if we fire some sequence of d transitions with corresponding firing vectors $u_0, u_1, ..., u_{d-1}$ we obtain that

$$M' = M + A^T u, \ u = \sum_{k=0}^{d-1} u_k. \tag{6}$$

Let (N_{n0}^m, d) be a metric space where $d : N_{n0}^m \times N_{n0}^m \to R_+$ is defined by

$$d(M_1, M_2) = \sum_{i=1}^{m} \zeta_i \mid M_1(p_i) - M_2(p_i) \mid ; \zeta_i > 0$$

and consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a PN

$$M' = M + A^T u, \ u = \sum_{k=0}^{d-1} u_k \tag{7}$$

where, $M \in N^m$, denotes the marking (state) of the PN, $A \in Z^{n\times m}$, its incidence matrix and $u \in N^n$, is a sequence of firing vectors. Then, the following results concerns in what to the stability problem means.

Proposition 9. *Let PN be a Petri net. PN is uniform practical stable if there exists a Φ strictly positive m vector such that*

$$\Delta v = u^T A \Phi \leq 0 \tag{8}$$

Moreover, PN is uniform practical asymptotic stable if the following equation holds

$$\Delta v = u^T A \Phi \leq -c(e), \ \text{for } c \in \mathcal{K} \tag{9}$$

Proof. Pick as our Lyapunov function candidate $v(M) = M^T \Phi$ with Φ an m vector (to be chosen). One can verify that v satisfies all the conditions of Theorem (4), and that one obtains uniform practical (asymptotic) stability if there exists a strictly positive vector Φ such that equation (8) holds. $\square$

Lemma 10. *Let suppose that Proposition (9) holds then,*

$$\Delta v = u^T A\Phi \leq 0 \Leftrightarrow A\Phi \leq 0 \tag{10}$$

Proof. ($\Leftarrow$) This is immediate from the fact that u is positive. ($\Rightarrow$) Since $u^T A\Phi = 0$ holds for every $u \Rightarrow A\Phi = 0$. If $u^T A\Phi < 0$ again since u is positive $A\Phi < 0$. □

Remark 11. Notice that since the state space of a TPN is contained in the state space of the same now not timed PN, stability of PN implies stability of the TPN.

2.1. Lyapunov Stabilization

Notice, that in the solution of the stability problem, the u vector does not play any role, so why not to take advantage of it in order to get some specific behavior.

Consider the matrix difference equation which describes the dynamical behavior of the discrete event system modeled by a Petri net

$$M' = M + A^T u$$

We are interested in finding a firing sequence vector, control law, such that system (7) remains bounded.

Definition 12. Let PN be a Petri net. PN is said to be stabilizable if there exists a firing transition sequence with transition count vector u such that system (7) remains bounded.

Proposition 13. *Let PN be a Petri net. PN is stabilizable if there exists a firing transition sequence with transition count vector u such that the following equation holds*

$$\Delta v = A^T u \leq 0 \tag{11}$$

Proof. Define as our vector Lyapunov function

$$v(M) = [v_1(M), v_2(M), ..., v_m(M)]^T,$$

where $v_i(M) = M(p_i)$, $1 \leq i \leq m$ we can verify that all the conditions of Theorem (4) are satisfied and, that one obtains uniform practical stability if there exists a fireable transition sequence with transition count vector u such that equation (11) holds. Therefore, we conclude that PN is stabilizable.

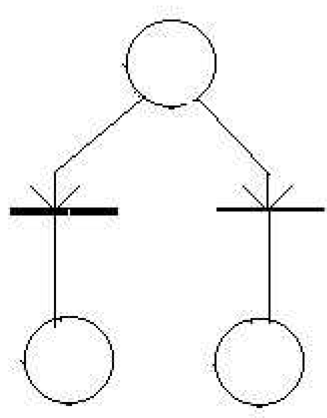

Figure 1

Remark 14. This result was first stated and proved in [5] and it relies in the use of vector Lyapunov functions. It is important to underline that by fixing a particular u, which satisfies (11), we restrict the state space to those markings (states) that are finite. The technique can be utilized to get some type of regulation and/or eliminate some undesirable events (transitions). Notice that in general (8)$\not\Rightarrow$(11) and that the opposite is also true (this is illustrated with the following two examples).

(8)$\not\Rightarrow$(11) Consider the Petri net model shown in Figure 1.

The incidence matrix which represents the model is

$$A = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \tag{12}$$

Then, picking $\Phi = [1, 1, 1]$ uniform practical stability is concluded. However, there is no u such that $A^T u \leq 0$.

(11)$\not\Rightarrow$(8). Consider the Petri net model depicted in Figure 2.

The structure is typical of an unbounded Petri net model in which the marking in p_1 can grow indefinitely due to the repeated firing of t_1. However, by taking $u = [k, k]$, $k > 0$ equation (11) is satisfied therefore, the system becomes bounded i.e., is stabilizable.

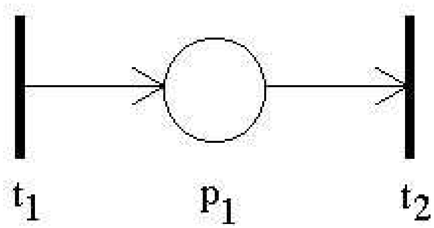

Figure 2

Remark 15. Notice that by firing all the transitions in the same proportion i.e., $u = [k, k]$, $k > 0$ an unbounded PN becomes stable. This guarantees that there is no possibility that the marking will grow without bound at any place between two transitions. This basic idea motivates the definition of stability for TPN which will be given in Section 6.

3. Max-Plus Algebra [6, 7]

In this section the concept of max-plus algebra is defined. Its algebraic structure is described. Matrices and graphs are presented. The spectral theory of matrices is discussed. Finally the problem of solving linear equations is addressed.

3.1. Basic Definitions

Notation. $\mathbb{N}$ is the set of natural numbers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ is the set of positive real numbers, $\epsilon = -\infty$, $e = 0$, $\mathbb{R}_{\max} = \mathbb{R} \cup \{\epsilon\}$, $\underline{n} = 1, 2, ..., n$

Let $a, b \in \mathbb{R}_{\max}$ and define the operations $\oplus$ and $\otimes$ by:

$$a \oplus b = \max(a, b) \text{ and } a \otimes b = a + b. \tag{13}$$

Notice that: $a \oplus \epsilon = \epsilon + a = a$ and $a \otimes e = e \otimes a = a$, $\forall a \in \mathbb{R}_{\max}$.)

Definition 16. The set $\mathbb{R}_{\max}$ with the two operations $\oplus$ and $\otimes$ is called a max-plus algebra and is denoted by $\Re_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \epsilon, e)$.

Definition 17. A semiring is a nonempty set R endowed with two operations $\oplus_R$, $\otimes_R$, and two elements ϵ_R and e_R such that:

- $\oplus_R$ is associative and commutative with zero element ϵ_R;
- $\otimes_R$ is associative, distributes over $\oplus_R$, and has unit element e_R,
- $\in_R$is absorbing for $\otimes_R$ i.e., $a \otimes_R \epsilon = \epsilon_R \otimes a = a$, $\forall a \in R$..

Such a semiring is denoted by $\Re = (R, \oplus_R, \otimes_R, \epsilon, e)$. In addition if $\otimes_R$ is commutative then R is called a commutative semiring, and if $\oplus_R$ is such that $a \oplus_R a = a$, $\forall a \in R$ then it is called idempotent.

Theorem 18. *The max-plus algebra* $\Re_{\max} = (\mathbb{R}_{\max}, \oplus, \otimes, \epsilon, e)$ *has the algebraic structure of a commutative and idempotent semiring.*

Proof. The proof follows immediately using the definitions given by equation (13) (in a similar way to the case for addition and multiplication over the reals) just being careful when one substitutes multiplication for the max operation. As for example in the distributive property for $a, b, c \in \mathbb{R}_{\max}$, it holds that: $a \otimes (b \oplus c) = a + max(b, c) = max(a + b, a + c) = (a \otimes b) \oplus (a \otimes c)$. □

3.2. Matrices and Graphs

Let $\mathbb{R}_{\max}^{n \times n}$ be the set of $n \times n$ matrices with coefficients in $\mathbb{R}_{\max}$ with the following operations:

- The sum of matrices $A, B \in \mathbb{R}_{\max}^{n \times n}$, denoted $A \oplus B$ is defined by:

$$(A \oplus B)_{ij} = a_{ij} \oplus b_{ij} = \max\ (a_{ij}, b_{ij}) \tag{14}$$

  for i and $j \in \underline{n}$.

- The product of matrices $A \in \mathbb{R}_{\max}^{n \times l}, B \in \mathbb{R}_{\max}^{l \times n}$, denoted $A \otimes B$ is defined by:

$$(A \oplus B)_{ik} = \bigoplus_{j=1}^{l} a_{ij} \otimes b_{jk} = \max_{j \in \underline{l}}\ \{a_{ij} + b_{jk}\} \tag{15}$$

  for i and $k \in \underline{n}$. (Notice that the matrix product in general fails to be commutative.)

- The scalar product for $\alpha \in \mathbb{R}_{\max}^{n}$ and $A \in \mathbb{R}_{\max}^{n \times n}$, denoted $\alpha \otimes A$ is defined by:

$$(\alpha \oplus A)_{ij} = \alpha \otimes a_{ij} \tag{16}$$

  for i and $j \in \underline{n}$.

Let $\mathcal{E} \in \mathbb{R}_{\max}^{n\times n}$ denote the matrix with all its elements equal to ϵ and denote by $E \in \mathbb{R}_{\max}^{n\times n}$ the matrix which has its diagonal elements equal to e and all the other elements equal to ϵ. Then, the following result, whose proof is immediate, can be stated.

Theorem 19. *The 5-tuple $\Re_{\max}^{n\times n} = (\mathbb{R}_{\max}^{n\times n}, \oplus, \otimes, \mathcal{E}, E)$ has the algebraic structure of a noncommutative idempotent semiring.*

Definition 20. Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $k \in \mathbb{N}$ then the k-th power of A denoted by $A^{\otimes k}$ is defined by:

$$A^{\otimes k} = \underbrace{A \otimes A \otimes \cdots \otimes A}_{k-\text{times}} \tag{17}$$

where $A^{\otimes 0}$ is set equal to E.

Definition 21. A matrix $A \in \mathbb{R}_{\max}^{n\times n}$ is said to be regular if A contains at least one element distinct from ϵ in each row.

Next, an overview in the theory of graphs will be given, emphasizing the rich relationship that exist between them and matrices.

Definition 22. Let $\mathcal{N}$ be a finite and non-empty set and consider $\mathcal{D} \subseteq \mathcal{N} \times \mathcal{N}$. The pair $G = (\mathcal{N}, \mathcal{D})$ is called a directed graph, where $\mathcal{N}$ is the set of elements called nodes and $\mathcal{D}$ is the set of ordered pairs of nodes called arcs. A directed graph $G = (\mathcal{N}, \mathcal{D})$ is called a weighted graph if a weight $w(i,j) \in \mathbb{R}$ is associated with any arc $(i,j) \in \mathcal{D}$.

Let $A \in \mathbb{R}_{\max}^{n\times n}$ be any matrix, a graph $\mathcal{G}(A)$, called the communication graph of A, can be associated as follows. Define $\mathcal{N}(A) = \underline{n}$ and a pair $(i,j) \in \underline{n} \times \underline{n}$ will be a member of $\mathcal{D}(A) \Leftrightarrow a_{ji} \neq \epsilon$, where $\mathcal{D}(A)$ denotes the set of arcs of $\mathcal{G}(A)$.

Definition 23. A path from node i to node j is a sequence of arcs $p = \{(i_k, j_k) \in \mathcal{D}(A)\}_{k \in \underline{m}}$ such that $i = i_1, j_k = i_{k+1}$, for $k < m$ and $j_m = j$. The path p consists of the nodes $i = i_1, i_2, ..., i_m, j_m = j$ with length m denoted by $\mid p \mid_1 = m$. In the case when $i = j$ the path is said to be a circuit. A circuit is said to be elementary if nodes i_k and i_l are different for $k \neq l$. A circuit consisting of one arc is called a self-loop.

Let us denote by $P(i,j;m)$ the set of all paths from node i to node j of length $m \geq 1$ and for any arc $(i,j) \in \mathcal{D}(A)$ let its weight be given by a_{ij} then the weight of a path $p \in P(i,j;m)$ denoted by $\mid p \mid_w$ is defined to be the sum of the weights of all the arcs that belong to the path. The average weight of a path p is given by $\mid p \mid_w / \mid p \mid_1$. Given two paths, as for example,

$p = ((i_1, i_2), (i_2, i_3))$ and $q = ((i_3, i_4), ((i_4, i_5)$ in $\mathcal{G}(A)$ the concatenation of paths $\circ : \mathcal{G}(A) \times \mathcal{G}(A) \to \mathcal{G}(A)$ is defined as $p \circ q = ((i_1, i_2), (i_2, i_3), (i_3, i_4), (i_4, i_5))$.

The communication graph $\mathcal{G}(A)$ and powers of matrix A are closely related as it is shown in the next theorem, whose proof follows using induction on the length k of the path (see [1]).

Theorem 24. *Let $A \in \mathbb{R}^{n\times n}_{\max}$, then $\forall k \geq 1$:*

$$[A^{\otimes k}]_{ji} = max\{|\, p \,|_w \colon p \in P(i,j;k)\} \tag{18}$$

where $[A^{\otimes k}]_{ji} = \epsilon$ in the case when $P(i,j;k)$ is empty i.e., no path of length k from node i to node j exists in $\mathcal{G}(A)$.

Definition 25. Let $A \in \mathbb{R}^{n\times n}_{\max}$ then define the matrix $A^+ \in \mathbb{R}^{n\times n}_{\max}$ as:

$$A^+ = \bigoplus_{k=1}^{\infty} A^{\otimes k} \tag{19}$$

sometimes known as the shortest path matrix. Where the element $[A^+]_{ji}$ gives the maximal weight of any path from j to i. If in addition one wants to add the possibility of staying at a node then one must include matrix E in the definition of matrix A^+ giving rise to its Kleene star representation defined by:

$$A^* = \bigoplus_{k=0}^{\infty} A^{\otimes k}. \tag{20}$$

Lemma 26. *Let $A \in \mathbb{R}^{n\times n}_{\max}$ be such that any circuit in $\mathcal{G}(A)$ has average circuit weight less than or equal to ϵ. Then it holds that:*

$$A^* = \bigoplus_{k=0}^{n-1} A^{\otimes k}. \tag{21}$$

Proof. Since $A^* = \bigoplus_{k=0}^{\infty} A^{\otimes k} = (\bigoplus_{k=0}^{n-1} A^{\otimes k}) \oplus (\bigoplus_{k\geq n}^{\infty} A^{\otimes k})$ and all paths of length greater than or equal to n are made up of a circuit and a path of length strictly less than n, we have that $A^k \leq A \oplus A^{*2} \oplus \cdots \oplus A^{*(n-1)}$ $\forall k \geq n$, which implies that $A^* = \bigoplus_{k=0}^{n-1} A^{\otimes k}$. $\square$

Definition 27. Let $G = (\mathcal{N}, \mathcal{D})$ be a graph and $i, j \in \mathcal{N}$, node j is reachable from node i, denoted as $i\mathcal{R}j$, if there exists a path from i to j. A graph G is said to be strongly connected if $\forall i, j \in \mathcal{N}, j\mathcal{R}i$. A matrix $A \in \mathbb{R}^{n\times n}_{\max}$ is called irreducible if its communication graph is strongly connected, when this is not the case matrix A is called reducible.

Definition 28. Let $G = (\mathcal{N}, \mathcal{D})$ be a not strongly connected graph and $i, j \in \mathcal{N}$, node j communicates with node i, denoted as $i\mathcal{C}j$, if either $i = j$ or $i\mathcal{R}j$ and $j\mathcal{R}i$.

The relation $i\mathcal{C}j$ defines an equivalence relation in the set of nodes, and therefore a partition of $\mathcal{N}$ into a disjoint union of subsets, the equivalence classes, $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_q$ such that $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup ... \cup \mathcal{N}_q$ or $\mathcal{N} = \bigcup\limits_{i \in \mathcal{N}} [i]$; $[i] = \{j \in \mathcal{N} : iCj\}$.

Given the above partition, it is possible to focus on subgraphs of G denoted by $G_r = (\mathcal{N}_r, \mathcal{D}_r); r \in \underline{q}$ where $\mathcal{D}_r$ denotes the subset of arcs, which belong to $\mathcal{D}$, that have both the begin node and end node in $\mathcal{N}_r$. If $\mathcal{D}_r \neq \varnothing$, the subgraph $G_r = (\mathcal{N}_r, \mathcal{D}_r)$ is known as a maximal strongly connected subgraph of G.

Remark 29. In case of having an isolated node i (i.e., a node that does not communicate with any other node) and which does not even have an arc from it to itself, the associated subgraph is given by $([i], \varnothing)$ which is not strongly connected however, for convenience it will be considered as if it were.

Definition 30. The reduced graph $\widetilde{G} = (\widetilde{\mathcal{N}}, \widetilde{\mathcal{D}})$ of G is defined by setting $\widetilde{\mathcal{N}} = \{[i_1], [i_2], ... [i_q]\}$ and $([i_r], [i_s]) \in \widetilde{\mathcal{D}}$ if $r \neq s$ and there exists an arc $(k, l) \in \mathcal{D}$ for some $k \in [i_r]$ and $l \in [i_s]$.

Let A_{rr} denote the matrix by restricting A to the nodes in $[i_r]$ $\forall r \in \underline{q}$ i.e., $[A_{rr}]_{kl} = a_{kl}$ $\forall k, l \in [i_r]$. Then $\forall r \in \underline{q}$ either A_{rr} is irreducible or is equal to ϵ. Therefore since by construction the reduced graph does not contain any circuits, the original reducible matrix A after a possible relabeling of the nodes in $G(A)$, can be written as:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix} \tag{22}$$

with matrices A_{sr} $1 \leq s < r \leq q$ of suitable size, where each finite entry in A_{sr} corresponds to an arc from a node in $[i_r]$ to a node in $[i_s]$.

Definition 31. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a reducible matrix then, the block upper triangular given by (22) is said to be a normal form of matrix A.

3.2.1. Spectral Theory

Definition 32. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a matrix. If $\mu \in R_{\max}$ is a scalar and $v \in R_{\max}^{n}$ is a vector that contains at least one finite element such that:

$$A \otimes v = \mu \otimes v \tag{23}$$

then, μ is called an eigenvalue and v an eigenvector.

Remark 33. Notice that the eigenvalue can be equal to ϵ and is not necessarily unique. Eigenvectors are certainly not unique indeed, if v is an eigenvector then $\alpha \otimes v$ is also an eigenvector for all $\alpha \in \mathbb{R}$.

Let $\mathcal{C}(A)$ denote the set of all elementary circuits in $\mathcal{G}(A)$ and write:

$$\lambda = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1} \tag{24}$$

for the maximal average circuit weight. Notice that since $\mathcal{C}(A)$ is a finite set, the maximum of (24) is attained (which is always the case when matrix A is irreducible). In case $\mathcal{C}(A) = \emptyset$ define $\lambda = \epsilon$.

Definition 34. A circuit $p \in G(A)$ is said to be critical if its average weight is maximal. The critical graph of A, denoted by $G^c(A) = (\mathcal{N}^c(A), \mathcal{D}^c(A))$, is the graph consisting of those nodes and arcs that belong to critical circuits in $G(A)$.

Lemma 35. *Let assume that $G(A)$ contains at least one circuit then, any circuit in $G^c(A)$ is critical.*

Proof. If this were not the case, we could find a circuit $p \in G^c(A)$, composed of sub-paths, lets say p_i of critical circuits p^c, with weight different from λ (which without loss of generality will be assumed to be equal to e). If this circuit had a weight greater than e then, since p is also a circuit in $G(A)$, it would contradict the assumption that the maximal average circuit weight λ is equal to e. On the other hand, if the weight of it were less than e, since the maximal average circuit weight is $\lambda = e$, the circuit composed of the union of the complements of the paths $p_i \in G^c(A)$, with respect to $G(A)$, must have positive weight, in order to assure that the critical circuits $p^c \in G(A)$, to which the sub-paths p_i belong is critical i.e., has average wight $\lambda = \epsilon$, which is also a contradiction. Therefore, any circuit in $G^c(A)$ is critical. $\square$

Definition 36. Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a matrix and μ an eigenvalue of A with associated eigenvector v then, the support of v consists of the set of nodes of $G(A)$ which correspond to finite entries of v.

Lemma 37. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ be an irreducible matrix then any $v \in R_{\max}^{n}$ which satisfies (23) has all components different from ϵ.*

Proof. Let us assume that the support of v does not cover the whole node set of $G(A)$ then since A is irreducible, there are arcs going from nodes in the support of v going to nodes not belonging to the support of v i.e., there exists a node j in the support of v and a node i not in the support of v with $a_{ij} \neq \epsilon$. But this implies that $[A \otimes v]_i \geq a_{ij} \otimes v_j > \epsilon$ therefore, the support of $A \otimes v$ is larger than the support of v which contradicts (23). □

Next, the most important result of this sub-section is given.

Theorem 38. *If $A \in \mathbb{R}_{\max}^{n\times n}$ is irreducible, then there exists one and only one finite eigenvalue (with possible several eigenvectors). This eigenvalue is equal to the maximal average weight of circuits in $G(A)$:*

$$\lambda(A) = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1} \tag{25}$$

Proof. **Existence of the eigenvalue λ and the eigenvector v.** Consider matrix A_λ with elements $[A_\lambda]_{ij} = a_{ij} - \lambda$, λ finite. The maximum average circuit of A_λ is e. Hence, lemma (26) implies that A_λ^* and A_λ^+ exist. Moreover, from lemma (35), matrix A_λ^+ is such that that $\forall \eta \in \mathcal{N}^c(A) : [A_\lambda^+]_{\eta\eta} = e$. Let $[A]_{.k}$ denote the kth column of matrix A then, since $\forall \eta \in \mathcal{N}^c(A) : [A_\lambda^+]_{\eta\eta} = e \Rightarrow [A_\lambda^*]_{\eta\eta} = e + [A_\lambda^+]_{\eta\eta} = e$, it follows that $[A_\lambda^+]_{.\eta} = [A_\lambda^*]_{.\eta}$. But $A_\lambda^+ = A_\lambda \oplus A_\lambda^*$ which implies that:

$$[A_\lambda \oplus A_\lambda^*]_{.\eta} = [A_\lambda^*]_{.\eta} \Rightarrow A_\lambda \oplus [A_\lambda^*]_{.\eta} = [A_\lambda^*]_{.\eta} \Longleftrightarrow A \oplus [A_\lambda^*]_{.\eta} = \lambda \oplus [A_\lambda^*]_{.\eta}.$$

Hence, it follows that λ is an eigenvalue of matrix A with associated eigenvector v the ηth column of A_λ^* for all $\eta \in \mathcal{N}^c(A)$.

Uniqueness. Suppose $\mu \neq \lambda$ satisfies (23) and pick any circuit $\gamma = ((\eta_1, \eta_2), (\eta_2, \eta_3), ..., (\eta_l, \eta_{l+1})) \in G(A)$ of length $l = |\gamma|_1$ with $\eta_{l+1} = \eta_1$. Then, since $a_{\eta_{k+1}\eta_k} \neq \epsilon$ with $k \in \underline{l}$, it follows that $a_{\eta_{k+1}\eta_k} \oplus v_{\eta_k} \leq \mu \oplus v_{\eta_{k+1}}$, $k \in \underline{l}$, where lemma (37) assures that all components of $v \neq \epsilon$, but this implies that $\bigotimes_{k=1}^{l} a_{\eta_{k+1}\eta_k} \oplus v_{\eta_k} \leq \mu^{\otimes l} \oplus \bigotimes_{k=1}^{l} v_{\eta_{k+1}}$ which in conventional algebra can be written as: $\sum_{k=1}^{l} a_{\eta_{k+1}\eta_k} + v_{\eta_k} \leq \mu \times l + \sum_{k=1}^{l} v_{\eta_{k+1}}$ which is reduced to $\sum_{k=1}^{l} a_{\eta_{k+1}\eta_k} \leq \mu \times l$ or $|\gamma|_W \leq \mu \times l \Rightarrow \frac{|\gamma|_W}{|\gamma|_l} \leq \mu$. But since this holds for every circuit in $G(A)$ μ has to be equal to λ. □

3.2.2. Linear Equations

Theorem 39. *Let $A \in \mathbb{R}_{\max}^{n\times n}$ and $b \in \mathbb{R}_{\max}^{n}$. If the communication graph $G(A)$ has maximal average circuit weight less than or equal to e, then $x = A^* \otimes b$ solves the equation $x = (A \otimes x) \oplus b$. Moreover, if the circuit weights in $G(a)$ are negative then, the solution is unique.*

Proof. **Existence.** By lemma (26) A^* exists. Substituting the proposed solution into the equation one gets:

$$x = (A\otimes[A^*\otimes b])\oplus b = (A\otimes A^*\otimes b)\oplus(e\oplus b) = [(A\otimes A^*)\oplus e]\otimes b = [A\otimes A^*]\oplus b = A^*\oplus b.$$

Uniqueness. Let y be another solution of $x = (A \otimes x) \oplus b$ then substituting $y = b \oplus (A \otimes y)$ it follows that: $y = b \oplus (A \otimes b) \oplus (A^{\otimes 2} \otimes y)$, iterating once and once again, one gets: $y = b\oplus(A\otimes b)\oplus(A^{\otimes 2}\otimes b)\oplus ... \oplus(A^{\otimes(k-1)}\otimes b)\oplus(A^{\otimes k}\otimes y) = [\bigoplus_{l=0}^{k-1}(A^{\otimes l} \otimes b)] \otimes (A^{\oplus k} \oplus y)$. Now, since by assumption circuits have negative weight the right side of the above equation, as k goes to ∞ tend to $\mathcal{E}$ while the left side, using lemma (26), tends to $A^* \otimes b$ therefore, $y = x$. □

4. Generalized Eigenmodes and Recurrence Equations

This section starts by introducing the concept of generalized eigenmode. Once this has been done, the section continues by discussing, how to compute the generalized eigenmode for recurrence equations for the cases of irreducible and reducible matrices. Finally, higher order recurrence relations are considered.

Definition 40. Let $A \in \mathbb{R}_{\max}^{n\times n}$ be a regular matrix, a pair of vectors $(\eta, v) \in \mathbb{R}^n \times \mathbb{R}^n$ is called a generalized eigenmode of A if for all $k \geq 0$:

$$A \oplus (k \times \eta + v) = (k + 1) \times \eta + v \tag{26}$$

Remark 41. It is important to underline that the second vector v in a generalized eigenmode is not unique. Indeed, if (η, v) is a generalized eigenmode then the pair $(\eta, \nu \oplus v)$ $\forall\nu \in \mathbb{R}$, also works.

Theorem 42. *Consider the inhomogeneous recurrence equation*

$$x(k + 1) = A \otimes x(k) \oplus \bigoplus_{j=1}^{m} B_j \otimes u_j(k),\ k \geq 0 \tag{27}$$

with $A \in \mathbb{R}_{\max}^{n\times n}$ *irreducible with eigenvalue* $\lambda = \lambda(A)$, *or* $A \in \mathbb{R}_{\max}$ $A = \epsilon$ *with* $\lambda = \epsilon$, $\{B_j\}_{j=1}^{m} \in \mathbb{R}_{\max}^{n\times m_j}$ *for some appropriate* $m_j \geq 1$ *matrices different from* $\mathcal{E}$, $u_j(k) \in \mathbb{R}^{m_j}$ *such that* $u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$, $k \geq 0$, *with* $\tau_j \in \mathbb{R}$ *and* $w_j \in \mathbb{R}^{m_j}$. *Denote* $\tau = \bigoplus_{j\in \underline{m}} \tau_j$. *Then, there exists an integer* $K \geq 0$ *and a vector* $v \in \mathbb{R}^n$ *such that the sequence* $x(k) = v \otimes \mu^{\otimes k}$ *with* $\mu = \lambda \otimes \tau$ *satisfies equation (27) for all* $k \geq K$.

Proof. The proof is given by considering two possible cases.

Case $\lambda > \tau$. Since A is irreducible, theorem (38) and lemma (37), guarantee the existence of the eigenvalue λ with associated finite eigenvector $v \in \mathbb{R}^n$. Choose v such that $v \oplus \lambda > \bigoplus_{j=1}^{m} B_j \otimes w_j$, this can always be done since if not, it is possible to replace v by $v \otimes \rho$, ρ an arbitrary but fixed real number which can be picked as big as desired (see remark (33)). Set $\mu = \lambda > \tau_j$ $\forall j \in \underline{m}$ then, $\forall k \geq 0$ it follows that: $v \otimes \mu^{\otimes(k+1)} = A \otimes v \otimes \mu^{\otimes k}$ and since $\mu^{\otimes(k+1)} \geq \bigoplus_{j=1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}$ it implies that $v \otimes \mu^{\otimes(k+1)} = A \otimes v \otimes \mu^{\otimes k} \oplus \bigoplus_{j=1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}$. Therefore, equation (27) is satisfied $\forall k \geq 0$.

Case $\lambda \leq \tau$. Sub-case (1): A is a matrix. Recall that $\tau = \bigoplus_{j\in\underline{m}} \tau_j$ and assume that the maximum is attained by the first r τ‘s, which can always be accomplished by a proper renumbering of the sequences $u_j(k)$, $j \in \underline{m}$. Now, look at the equation:

$$s = A_\tau \otimes s \oplus \bigoplus_{j=1}^{r} (B_j)_\tau \otimes w_j, \tag{28}$$

where A_τ and $(B_j)_\tau, j \in \underline{m}$ are obtained from their original matrices A and (B_j) by subtracting τ from all of its finite elements. Because $\lambda \leq \tau$, the communication graph of A_τ only contains circuits with a non-positive weight therefore, from theorem (39) a solution v exists, further since $(A_\tau)^*$ is completely finite (A_τ is strongly connected) and $\bigoplus_{j=1}^{r} (B_j)_\tau \otimes w_j$ contains at least one finite element it implies that v is finite i.e., $v \in \mathbb{R}^n$. But this implies that;

$$v \otimes \tau = A \otimes v \oplus \bigoplus_{j=1}^{r} B_j \otimes w_j.$$

Then, setting $\mu = \tau = \tau_j$, $j = 1, 2, ..., r$ it follows that:

$$v \otimes \mu^{\otimes(k+1)} = A \otimes v \otimes \mu^{\otimes k} \oplus \bigoplus_{j=1}^{r} B_j \otimes w_j \otimes \tau_j^{\otimes k}, \forall k \geq 0$$

which leads to:

$$v \otimes \mu^{\otimes(k+1)} \leq A \otimes v \otimes \mu^{\otimes k} \oplus \bigoplus_{j=1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}.$$

However since $\mu > \tau_j$ for $j = r+1, r+2, ..., m$, there exists an integer $K \geq 0$, as large as needed such that $\forall k \geq K$ $v \otimes \mu^{\otimes(k+1)} \geq \bigoplus_{j=r+1}^{m} B_j \otimes w_j \otimes \tau_j^{\otimes k}$. Therefore, equation (27) is satisfied $\forall k \geq K$.

Sub-case (2): ***A*** **is the scalar** ϵ **with** $\lambda = \epsilon$**.** Take v, solution of (28), as $v = \bigoplus_{j=1}^{r} (B_j)_\tau \otimes w_j$ and proceed exactly as it was done in sub-case (1). □

Remark 43. Notice that in theorem (42) equation (27) is satisfied for all $k \geq K$. However, in the case where it is possible to reinitialize the sequences $u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$, $k \geq 0$, by redefining the vectors w_j for $j \in \underline{m}$ then, it is possible to satisfy equation (27) $\forall k \geq 0$. Indeed, just set $v = v \otimes \mu^{\otimes K}, w_j(k) = w_j(k) \otimes \tau_j^{\otimes K}$, $j \in \underline{m}$. Then, the new sequences $x(k) = v \otimes \mu^{\otimes k}, u_j(k) = w_j(k) \otimes \tau_j^{\otimes k}$ $j \in \underline{m}$ solve our problem $\forall k \geq 0$.

Now, let us consider the recurrence equation:

$$x(k+1) = A \otimes x(k), \ k \geq 0 \tag{29}$$

with A reducible and regular. Recalling what was presented in sub-section (3.2) (see also definition (31)), and using that matrix A is regular, it follows that matrix A can always be rewritten in its normal form i.e.,

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & \cdots & A_{1q} \\ \mathcal{E} & A_{22} & \cdots & \cdots & A_{2q} \\ \mathcal{E} & \mathcal{E} & A_{33} & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & \mathcal{E} & A_{qq} \end{pmatrix} \tag{30}$$

with the conditions that A_{qq} is irreducible, that for $i \in \underline{q-1}$ either A_{ii} is an irreducible matrix or is equal to ϵ, and that the A_{ij} matrices are different from

$\mathcal{E}$ for $i, j = i+1; i \in \underline{q}$. Let the vector $x(k)$ be partitioned according to the normal form given by equation (30) as:

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_q(k) \end{pmatrix}$$

where $x_i(k), i \in \underline{q}$ are vectors of suitable size. Therefore the recurrence equation given by equation (29) can be written as:

$$x(k+1) = A_{ii} \otimes x_i(k) \oplus \bigoplus_{j=1+1}^{q} A_{ij} \otimes x_j(k), i \in \underline{q},\ k \geq 0 \tag{31}$$

Then, the next result follows.

Theorem 44. *Consider the recurrence equation given by equation (31). Assume that A_{qq} is irreducible and that for $i \in \underline{q-1}$ either A_{ii} is an irreducible matrix or is equal to ϵ. Assume also, that the A_{ij} matrices are different from $\mathcal{E}$ for $i, j = i+1; i \in \underline{q}$. Then, there exist finite vectors $v_1, v_2, ..., v_q$ of suitable size and scalars $\xi_1, \xi_2, ..., \xi_q \in \mathbb{R}$ such that the sequences:*

$$x_i(k) = v_i \otimes \xi_i^{\otimes k}, i \in \underline{q}$$

satisfy equation (31) for all $k \geq 0$. The scalars $\xi_1, \xi_2, ..., \xi_q \in \mathbb{R}$ are determined by:

$$\xi_i = \bigoplus_{j \in \mathcal{H}_i} \xi_j \oplus \lambda_i,$$

where $\mathcal{H}_i = \{j \in \underline{q} : j > i, A_{ij} \neq \mathcal{E}\}$.

Proof. The proof follows straightforward by first considering the case $i = q$, for which the result is immediate, and then proceeding backwards step by step. Using, at each step, the result given by theorem (42), whose hypothesis are automatically satisfied. The fact that the theorem holds $\forall k \geq 0$ follows since all the sequences $x_i(k) \in \underline{q}$ can be reinitialized, see remark (43). $\square$

Corollary 45. *Let $A \in \mathbb{R}_{\max}^{n \times n}$ be a reducible and regular matrix, then there exist a pair of vectors $(\eta, v) \in \mathbb{R}^n \times \mathbb{R}^n$, a generalized eigenmode, such that for all $k \geq 0$:*

$$A \oplus (k \times \eta + v) = (k+1) \times \eta + v \tag{32}$$

Proof. From what was discussed above theorem (44) about reducible and regular matrices, and applying it. The pair $\eta = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $v = (v_1, v_2, ..., v_q) \in \mathbb{R}^n$ result to be a generalized eigenmode which satisfies (32) for all $k \geq 0$. □

The result provided by corollary (45) plays a fundamental role in the proposed algorithm for reducible matrices, as will be seen in the next section.

Definition 46. Let $A_m \in \mathbb{R}_{\max}^{n \times n}$ for $0 \leq m \leq M$ and $x(m) \in \mathbb{R}_{\max}^{n}$ for $-M \leq m \leq -1$; $M \geq 0$. Then, the recurrence equation:

$$x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m);\ k \geq 0 \tag{33}$$

is called an Mth order recurrence equation.

Theorem 47. *The Mth order recurrence equation, given by equation (33), can be transformed into a first order recurrence equation $x(k+1) = A \otimes x(k)$; $k \geq 0$ provided that A_0 has circuit weights less than or equal to zero.*

Proof. Since by hypothesis, A_0 has circuit weights less than or equal to zero, lemma (26) allows A_0 to be written as $A_0^* = \bigoplus_{i=0}^{n-1} A_0^{\otimes i}$. Setting $b(k) = \bigoplus_{m=1}^{M} A_m \otimes x(k-m)$ equation (33) reduces to $x(k) = A_0 \otimes x(k) \oplus b(k)$ which by theorem (39) can be rewritten as $x(k) = A_0^* \otimes b(k)$. Finally, defining $\hat{x}(k) = (x^T(k-1), x^T(k-2), ..., x^T(k-M))^T$ and,

$$\hat{A} = \begin{pmatrix} A_0^* \otimes A_1 & A_0^* \otimes A_2 & \cdots & \cdots & A_0^* \otimes A_M \\ E & \mathcal{E} & \cdots & \cdots & \mathcal{E} \\ \mathcal{E} & E & \ddots & & \mathcal{E} \\ \vdots & & \ddots & & \vdots \\ \mathcal{E} & \mathcal{E} & \cdots & E & \mathcal{E} \end{pmatrix}$$

we get that $\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k)$; $k \geq 0$ as desired. □

4.1. Max-Plus Recurrence Equations For Timed Event Petri Nets

With any timed event Petri net, matrices $A_0, A_1, ..., A_M \in \mathbb{N}^n \times \mathbb{N}^n$ can be defined by setting $[A_m]_{jl} = a_{jl}$, where a_{jl} is the largest of the holding times with respect to all places between transitions t_l and t_j with m tokens, for

$m = 0, 1, ..., M$, with M equal to the maximum number of tokens with respect to all places. Let $x_i(k)$ denote the kth time that transition t_i fires, then the vector $x(k) = (x_1(k), x_2(k), ...x_m(k))^T$, called the state of the system, satisfies the Mth order recurrence equation:

$$x(k) = \bigoplus_{m=0}^{M} A_m \otimes x(k-m);\ k \geq 0 \tag{34}$$

Now, assuming that all the hypothesis of theorem (47) are satisfied, and setting $\hat{x}(k) = (x^T(k), x^T(k-1), ..., x^T(k-M+1))^T$, equation (34) can be expressed as:

$$\hat{x}(k+1) = \hat{A} \otimes \hat{x}(k);\ k \geq 0 \tag{35}$$

which is known as the standard autonomous equation.

5. An Algorithm for Computing Generalized Eigenmodes of Reducible Matrices

This section illustrates how by means of theorems (42, 44) and corollary (45), an algorithm for computing a generalized eigenmode for reducible matrices can be proposed. Two numerical examples are included, (see [6]).

Algorithm

1. Take $A \in \mathbb{R}_{\max}^{n \times n}$ a reducible and regular matrix.

2. Using the material presented in (3.2) bring it to the normal form and write it in the form of system (31).

3. Consider the last equation of system (31) i.e., the nth equation, and compute its eigenvalue λ_n with associated eigenvector v_n, set $\xi_n = \lambda_n$ and $j = n$.

4. Consider the above next $(j-1)$th equation, and compute the eigenvalue of matrix $A_{(j-1)(j-1)}$, called it λ_{j-1}.

5. Is $\lambda_{j-1} > \xi_j$, if this is the case go to 6 if not, go to 7.

6. Set $\xi_{j-1} = \lambda_{j-1}$ and compute v_{j-1} according to the first case of the proof of theorem (42). Go to 8.

7. Set $\xi_{j-1} = \xi_j$ and compute v_{j-1} according to the second case of the proof of theorem (42).

8. Decrease j by one. Is $j \neq 1$ go back to 4 if not finish.

At the end the algorithm provides one pair of vectors $\eta = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $v = (v_1, v_2, ..., v_q) \in \mathbb{R}^n$ which result to be a generalized eigenmode of matrix $A \in \mathbb{R}^{n\times n}_{\max}$.

Remark 48. Theorem (38) can be used for computing the eigenvalues of the irreducible matrices $\{A_{ii}\,;\ i \in \underline{n}\}$. In addition, the power algorithm (see [1]) results of great help for computing the eigenvector in case it comes from the solution of equation (23).

Example 49. Consider the following regular reducible matrix already in its normal form:

$$A = \begin{pmatrix} 1 & 2 & \varepsilon & 7 \\ \varepsilon & 3 & 5 & \varepsilon \\ \varepsilon & 4 & \varepsilon & 3 \\ \varepsilon & 2 & 8 & \varepsilon \end{pmatrix}$$

with $A_{11} = 1$, and $A_{22} = \begin{pmatrix} 3 & 5 & \varepsilon \\ 4 & \varepsilon & 3 \\ 2 & 8 & \varepsilon \end{pmatrix}$.

From A_{22} we get that $\lambda_2 = max\{10/3, 11/2, 9/2, 3\} = 11/2 = \xi_2$ and using the power algorithm or doing algebra that $v_2 = (20, 20.5, 23)$. Now, since $A_{11} = 1$ this implies that $\lambda_1 = 1 \leq \xi_2$ therefore $\xi_1 = \xi_2 = 11/2$ and $v_1 = 24.5$ is obtained as the solution of $(1 \otimes v_1) \oplus 22 \oplus 30 = 11/2 \otimes v_1$. Therefore, the pair $\eta = (11/2, 11/2, 11/2, 11/2)$, $v = (24.5, 20, 41/2, 23)$ results to be a generalized eigenmode. Notice that subtracting 21 to each member of v we get that $\eta = (11/2, 11/2, 11/2, 11/2)$, $v = (6/2, -1, -1/2, 2)$ is also a generalized eigenmode

Example 50. Consider the following regular reducible matrix:

$$\begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 4 & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 16 & \varepsilon & \varepsilon & \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 1/2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

The communication graph $\mathcal{G}(\mathcal{A})$ has five maximal strongly connected subgraphs which implies that its reduced graph, $\widetilde{G} = (\widetilde{\mathcal{N}}, \widetilde{\mathcal{D}})$ turns out to be defined by: $\widetilde{\mathcal{N}} = \{[1], [5], [8], [9], [10]\}$, $\widetilde{\mathcal{D}} = \{([1], [10]), ([1], [5]), ([5], [8]), ([5], [9])\}$, where $[1] = \{1, 2, 3, 4\}, [5] = \{5, 6, 7\}, [8] = \{8\}, [9] = \{9\}$, and $[10] = \{10\}$. Based on the reduced graph, after placing the rows and columns of matrix A in the order $8, 9, 5, 6, 7, 10, 1, 2, 3, 4$ the following normal form of matrix A is obtained:

$$\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 1/2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & -5 & \varepsilon & \varepsilon & \varepsilon & 16 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 9 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & -3 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon & 4 & \varepsilon & 0 \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}$$

with $A_{11} = A_{22} = A_{44} = \epsilon$,

$$A_{33} = \begin{pmatrix} \varepsilon & -5 & \varepsilon \\ \varepsilon & \varepsilon & 0 \\ 9 & \varepsilon & \varepsilon \end{pmatrix}$$

and

$$A_{55} = \begin{pmatrix} \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & -3 & \varepsilon \\ \varepsilon & 4 & \varepsilon & 0 \\ 0 & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}.$$

From A_{55} we get that $\lambda_5 = max\{1/2, -3/4\} = 1/2 = \xi_5$ and doing algebra that $v_5 = (17/2, 9, 25/2, 8)$. Now, since $A_{11} = \epsilon$ this implies that $\lambda_1 = \epsilon \leq \xi_2$ therefore $\xi_4 = \xi_5 = 1/2$ and that $v_1 = 17$. Proceeding with A_{33} we get that $\lambda_3 = 4/3 > \xi_4$ therefore, we obtain that $\xi_3 = 4/3$ and that $v_3 = (24, 91/3, 95/3)$, which is obtained from the solution of $A_{33} \otimes v_3 = \lambda_3 \otimes v_3$ and $\lambda_3 \otimes v_{3_2} > a_{38} \otimes v_{5_2} = 25$. Iterating one more time, we get for A_{22} that $\xi_2 = 4/3$ and $v_2 = 35$. An finally, for A_{11}, $\xi_1 = 4/3$ and $v_1 = 185/6$ Therefore, the pair $\eta = (4/3, 4/3, 4/3, 4/3, 4/3, 1/2, 1/2, 1/2, 1/2, 1/2)$, $v = (185/6, 35, 24, 91/3, 95/3, 17, 17/2, 9, 25/2, 8)$ results to be a generalized eigenmode.

6. The Solution to the Stability Problem for Queuing Systems Modeled with Timed Petri Nets

This section defines what it means for a TPN to be stable, then gathering the results previously presented in the past sections the solution to the problem is obtained.

Definition 51. A TPN is said to be stable if all the transitions fire with the same proportion i.e., if there exists $q \in \mathbb{N}$ such that

$$\lim_{k\to\infty} \frac{x_i(k)}{k} = q, \forall i = 1, ..., n \tag{36}$$

This last definition tell us that in order to obtain a stable TPN all the transitions have to be fired q times. However, it will be desirable to be more precise and know exactly how many times. The answer to this question is given next.

Lemma 52. *Consider the recurrence relation* $x(k+1) = A \otimes x(k), k \geq 0$, $x(0) = x_0 \in \mathbb{R}^n$ *arbitrary.* A *an irreducible matrix and* $\lambda \in \mathbb{R}$ *its eigenvalue then,*

$$\lim_{k\to\infty} \frac{x_i(k)}{k} = \lambda, \forall i = 1, ..., n \tag{37}$$

Proof. Let v be an eigenvector of A such that $x_0 = v$ then,

$$x(k) = \lambda^{\otimes k} \otimes v \Rightarrow x(k) = k\lambda + v \Rightarrow \frac{x(k)}{k} = \lambda + \frac{v}{k} \Rightarrow \lim_{k\to\infty} \frac{x_i(k)}{k} = \lambda$$

□

Now starting with an unstable TPN, collecting the results given by: proposition (13), what has just been discussed about recurrence equations for TPN at the end of subsection (4.1) and the previous lemma (52) plus theorem (38), the solution to the problem is obtained.

7. Modeling, Stability Analysis and Scheduling Design for a Class of Queuing Systems

In this section the modeling, stability analysis and scheduling design for queuing systems is addressed. Two types of queues are considered. The first one formed by one server, while the second one has two servers, (this last one, can be straightforwardly extended to the case with n servers).

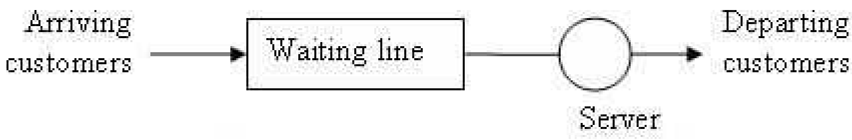

Figure 3: One server queuing system

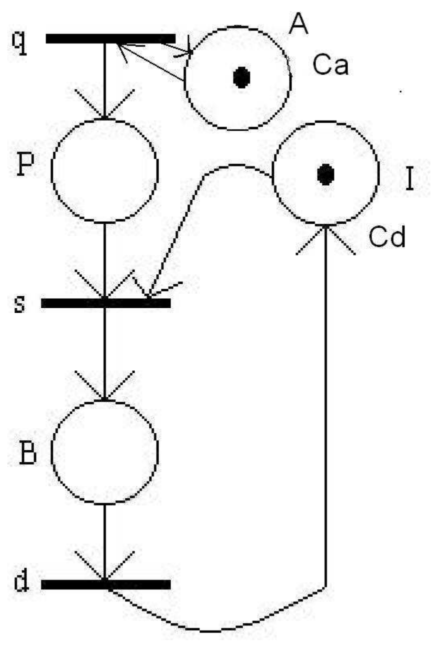

Figure 4: Timed Petri net model

Case I: Consider a simple one server queuing system (Fig 3.) whose TPN model is depicted in Fig 4. Where the events (transitions) that drive the system are: q: customers arrive to the queue, s: service starts, d: the customer departs. The places (that represent the states of the queue) are: A: customers arriving, P: the customers are waiting for service in the queue, B: the customer is being served, I: the server is idle. The holding times associated to the places A and I are Ca and Cd respectively, (with $Ca > Cd$). The incidence matrix that represents the PN model is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Therefore since there does not exists a Φ strictly positive m vector such that $A\Phi \leq 0$ the sufficient condition for stability is not satisfied. Moreover, the PN (TPN) is unbounded since by the repeated firing of q, the marking in P grows indefinitely. However, by taking $u = [k, k, k]; k > 0$ (but unknown), we get that $A^T u \leq 0$. Therefore, the PN is stabilizable which implies that the TPN is stable. Now, let us proceed to determine the exact value of k. From the TPN model we obtain that:

$$A_0 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & Cd \\ \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

and making the required computations that:

$$A_0^* = \begin{pmatrix} 0 & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon \\ 0 & 0 & 0 \end{pmatrix},$$

leading to:

$$\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon \\ Ca & \varepsilon & Cd \\ Ca & \varepsilon & Cd \end{pmatrix}$$

Therefore, $\lambda(A) = \max_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1} = \max\{Ca, Cd\} = Ca$. This means that in order for the TPN to be stable and work properly the speed at which the one server queuing system works has to be equal to Ca or being more precise, that all the transitions must fire at the same speed as the customers arrive i.e., they have to be served as soon as they arrive to the queue which is attained by setting $k = Ca$.

Now, bringing it into its normal form, $\hat{A}$ is expressed as:

$$\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} \varepsilon & Ca & \varepsilon \\ \varepsilon & Ca & Cd \\ \varepsilon & Ca & Cd \end{pmatrix}$$

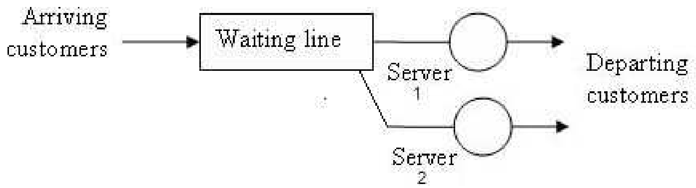

Figure 5: Two server queuing system

where $A_{11} = \epsilon$, and $A_{22} = \begin{pmatrix} Ca & Cd \\ Ca & Cd \end{pmatrix}$.

From A_{22} we get that $\lambda_2 = Ca = \xi_2$ and doing algebra that $v_2 = (v, v), v > 0$. Now, since $A_{11} = \epsilon$ this implies that $\lambda_1 = Ca \leq \xi_2$ therefore $\xi_1 = \xi_2 = Ca$ and $v_1 = v$ is obtained as the solution of $Ca \otimes v = Ca \otimes v_1$. Therefore, the pair $\eta = (Ca, Ca, Ca)$, $v = (v, v, v), v > 0$ results to be a generalized eigenmode and since it satisfies equation (35) it provides a possible schedule for the queue given by:

$$x(k) = k \times [Ca, Ca, Ca]^T + [v, v, v]^T, k \geq 0.$$

Notice that the maximum numerical value attained by the elements that form vector η, which in this case is Ca, determines the highest frequency at which the queue operates (or in other words the slowest one).

Case II: Consider a two server queuing system (Fig 5.) whose TPN model is depicted in Fig 6. Where the events (transitions) that drive the system are: q: customers arrive to the queue, s1, s2: service starts, d1,d2: the customer departs. The places (that represent the states of the queue) are: A: customers arriving, P: the customers are waiting for service in the queue, B1, B2: the customer is being served, I1, I2: the servers are idle. The holding times associated to the places A and I1, I2 are Ca and Cd respectively, (with $Ca > Cd$). The incidence matrix that represents the PN model is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

Therefore since there does not exists a Φ strictly positive m vector such that $A\Phi \leq 0$ the sufficient condition for stability is not satisfied. Moreover, the PN (TPN) is unbounded since by the repeated firing of q, the marking in P grows indefinitely. However, by taking $u = [k, k/2, k/2, k/2, k/2]; k > 0$

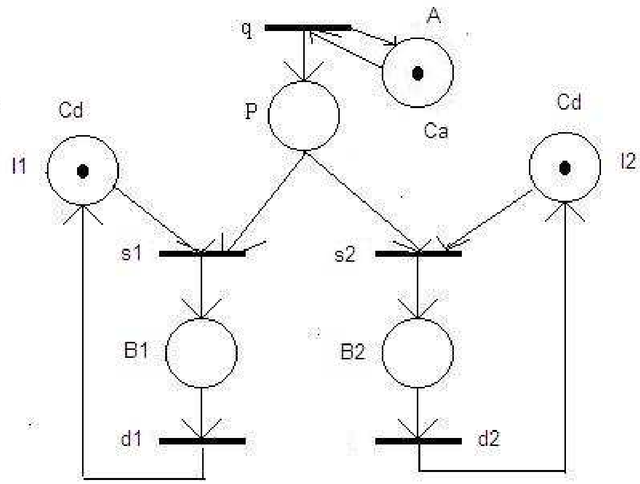

Figure 6: Timed Petri net model

(but unknown) we get that $A^T u \leq 0$. Therefore, the PN is stabilizable which implies that the TPN is stable. Now, let us proceed to determine the exact value of k. From the TPN model we obtain that:

$$A_0 = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & Cd & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & Cd \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}$$

and making the required computations that:

$$A_0^* = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & 0 & \varepsilon & \varepsilon \\ 0 & 0 & \varepsilon & 0 & \varepsilon \\ 0 & \varepsilon & 0 & \varepsilon & 0 \end{pmatrix},$$

leading to:

$$\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} Ca & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ Ca & \varepsilon & \varepsilon & Cd & \varepsilon \\ Ca & \varepsilon & \varepsilon & \varepsilon & Cd \\ Ca & 0 & \varepsilon & Cd & \varepsilon \\ Ca & \varepsilon & \varepsilon & \varepsilon & Cd \end{pmatrix}$$

Therefore, $\lambda(A) = \max\limits_{p \in \mathcal{C}(A)} \frac{|p|_w}{|p|_1} = \max\{Ca, Cd\} = Ca$. This means that in order for the TPN to be stable and work properly the speed at which the two sever queuing system works has to be equal to Ca which is attained by taking $k = Ca$, i.e., the load has to be equally divided between the two servers.

Now, bringing it into its normal form, $\hat{A}$ is expressed as:

$$\hat{A} = A_0^* \otimes A_1 = \begin{pmatrix} \varepsilon & Ca & \varepsilon & Cd & \varepsilon \\ \varepsilon & Ca & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & Ca & \varepsilon & \varepsilon & Cd \\ \varepsilon & Ca & \varepsilon & Cd & \varepsilon \\ \varepsilon & Ca & \varepsilon & \varepsilon & Cd \end{pmatrix}$$

where $A_{11} = \epsilon$, and $A_{22} = \begin{pmatrix} Ca & \varepsilon & \varepsilon & \varepsilon \\ Ca & \varepsilon & \varepsilon & Cd \\ Ca & \varepsilon & Cd & \varepsilon \\ Ca & \varepsilon & \varepsilon & Cd \end{pmatrix}$.

From A_{22} we get that $\lambda_2 = Ca = \xi_2$ and doing algebra that $v_2 = (v, v, v, v), v > 0$. Now, since $A_{11} = \epsilon$ this implies that $\lambda_1 = Ca \leq \xi_2$ therefore $\xi_1 = \xi_2 = Ca$ and $v_1 = v$ is obtained as the solution of $(Ca \otimes v) \oplus (Cd \otimes v) = Ca \otimes v_1$. Therefore, the pair $\eta = (Ca, Ca, Ca, Ca, Ca)$, $v = (v, v, v, v, v), v > 0$ results to be a generalized eigenmode and since it satisfies equation (35) it provides a possible schedule for the queue given by:

$$x(k) = k \times [Ca, Ca, Ca, Ca, Ca]^T + [v, v, v, v, v]^T, k \geq 0.$$

Remark 53. This case is easily extended to the case with n servers, obtaining that $u = [Ca, Ca/n, Ca/n, ..., Ca/n]$, $\eta = (Ca, Ca, ..., Ca)$ and $v = (v, .., v), v > 0$.

8. Conclusions

The main contribution of this paper consists in combining Lyapunov theory with max-plus algebra to give a complete and precise solution to the stability and scheduling design problem for queuing systems modeled with timed Petri nets. The presented methodology applied to queuing systems is new and results to be innovative.

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