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A VARIATIONAL PRINCIPLE AND ITS APPLICATION

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Abstract: Assume that A is a bounded selfadjoint operator in a Hilbert space H. Then, the variational principle

$$\max_v \frac{|(Au,v)|^2}{(Av,v)} = (Au,u) \tag{*}$$

holds if and only if $A \geq 0$, that is, if $(Av,v) \geq 0$ for all $v \in H$. We define the left-hand side in (*) to be zero if $(Av,v)=0$. As an application of this principle it is proved that

$$C = \max_{\sigma \in L^2(S)} \frac{|\int_S \sigma(t)dt|^2}{\int_S\int_S \frac{\sigma(t)\sigma(s)dsdt}{4\pi|s-t|}}, \tag{**}$$

where $L^2(S)$ is the L^2-space of real-valued functions on the connected surface S of a bounded domain $D \in \mathbb{R}^3$, and C is the electrical capacitance of a perfect conductor D.

The classical Gauss' principle for electrical capacitance is an immediate consequence of (*).

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1. Introduction

In many applications a physical quantity of interest can be expressed as a quadratic form. For example, consider electrical charge distributed on the surface of a perfect conductor with density $\sigma(t)$. If the conductor is charged to a

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potential $u = 1$, then the equation for $\sigma(t)$ is

$$A\sigma := \int_S \frac{\sigma(t)dt}{4\pi r_{st}} = 1, \quad s \in S, \quad r_{st} := |s - t|, \tag{1}$$

where dt is the element of the surface area, S is the surface of the conductor D, and $D \in \mathbb{R}^3$ is a bounded domain with a connected smooth boundary S. The total charge on S is $Q = \int_S \sigma(t)dt$. The physical quantity of interest is electrical capacitance C of the conductor D. Since $Q = Cu$ and $u = 1$ (see equation (1)), it follows that

$$C = \int_S \sigma(t)dt = (A\sigma, \sigma),$$

where $(f, g) := \int_S f\overline{g}dt$ is the inner product in the Hilbert space $H = L^2(S)$, and the overbar stands for complex conjugate.

Let us introduce a general theory. Let $A = A^\star$ be a linear selfadjoint bounded operator in a Hilbert space H. Consider an equation $Au = f$.

We are interested in a quantity (Au, u) and want to find a variational principle that allows one to calculate and estimate this quantity. Let us write $A \geq 0$ if and only if $(Av, v) \geq 0$ for all v, and say in this case that A is non-negative. If $(Av, v) > 0$ for all $v \neq 0$, we write $A > 0$ and say that A is positive.

The following variational principle is our main abstract result.

Theorem 1.1. *Let $A = A^\star$ be a linear bounded selfadjoint operator. Formula*

$$(Au, u) = \max_{v \in H} \frac{|(Av, u)|^2}{(Av, v)} \tag{2}$$

holds if and only if $A \geq 0$.

Remark 1. We define the right-hand side in (2) to be zero if $(Av, v) = 0$.

Theorem 1 can be proved also for unbounded selfadjoint operators A. In this case maximization is taken over $v \in D(A)$, where $D(A)$ is the domain of A, a linear dense subset of H.

In Section 2, Theorem 1.1 is proved. Let us illustrate this theorem by an example.

Example 1. Let A be defined in (1). In Section 2, we prove the following lemma.

Lemma 1.2. *The operator A in equation (1) is positive in $H = L^2(S)$.*

From Theorem 1.1, Lemma 1.2, and equation (1) it follows that the electrical capacitance C can be calculated by the following variational principle:

$$C = \max_{v \in L^2(S)} \frac{|\int_S v(t)dt|^2}{\int_S \int_S \frac{v(t)\overline{v(s)}dsdt}{4\pi r_{\mathrm{st}}}}. \tag{3}$$

This variational principle for electrical capacitance is an application of the abstract variational principle formulated in Theorem 1.

Formula (3) can be rewritten as

$$C^{-1} = \min_{v \in L^2(S)} \frac{\int_S \int_S \frac{v(t)\overline{v(s)}dsdt}{4\pi r_{\mathrm{st}}}}{|\int_S v(t)dt|^2}. \tag{4}$$

In particular, setting $v = 1$ in (3), one gets

$$C \geq \frac{4\pi|S|^2}{J}, \quad J := \int_S \int_S \frac{dsdt}{r_{st}}, \tag{5}$$

where $|S|$ is the surface area of S.

In [3] the following approximate formula for the capacitance is derived:

$$C^{(0)} = \frac{4\pi|S|^2}{J}.$$

This formula is zero-th approximation of an iterative process for finding $\sigma(t)$, the equilibrium charge distribution on the surface S of a perfect conductor charged to the potential $u = 1$.

Formula (4) yields a well-known Gauss' principle (see [2]), which says that if the total charge $Q = \displaystyle\int_S v(t)dt$ is distributed on the surface S of a perfect conductor with a density $v(t)$ and $u(s)$ is the corresponding distribution of the potential on S, then the minimal value of the functional

$$Q^{-2} \int_S \int_S \frac{v(t)\overline{v(s)}dsdt}{4\pi r_{st}} = \min \tag{6}$$

is equal to C^{-1}, where C is the electrical capacitance of the conductor, and this minimal value is attained at $v(t) = \sigma(t)$, where $\sigma(t)$ solves equation (1).

2. Proofs

Proof of Theorem 1.1. The *sufficiency* of the condition $A \geq 0$ for the validity of (2) is clear: if $A = A^* \geq 0$, then the quadratic form $[u, u] := (Au, u)$ is non-negative and the standard argument yields the Cauchy inequality

$$|(Au, v)|^2 \leq (Au, u)(Av, v). \tag{7}$$

The equality sign in (7) is attained if and only if u and v are linearly dependent. Dividing (7) by (Av, v), one obtains (2), and the maximum in (2) is attained if $v = \lambda u$, λ =const.

Let us prove the *necessity* of the condition $A \geq 0$ for (2) to hold. Let us assume that there exist z and w such that $(Az, z) > 0$ and $(Aw, w) < 0$, and prove that then (2) cannot hold.

Note that if $(Av, v) \leq 0$ for all v, then (2) cannot hold. Indeed, if $(Av, v) \leq 0$ for all v, then (2) implies $|(Bu, v)|^2 \geq (Bu, u)(Bv, v)$, where $B = -A \geq 0$. This is a contradiction to the Cauchy inequality. This contradiction proves that $(Av, v) \leq 0$ for all v cannot hold if (2) holds.

Let us continue the proof of necessity. Take $v = \lambda z + w$, where λ is an arbitrary real number. Then, (2) yields

$$\frac{|(Au, \lambda z + w)|^2}{q(\lambda)} \leq (Au, u), \tag{8}$$

where

$$q(\lambda) := a\lambda^2 + 2b\lambda + c, \quad a := (Az, z) > 0, \quad c = (Aw, w) < 0, \tag{9}$$

and $b := Re(Az, w)$. The polynomial $q(\lambda)$ has two real roots $\lambda_1 < 0$ and $\lambda_2 > 0$, $q^{-1}(\lambda) \to +\infty$ if $\lambda \to \lambda_1 - 0$ or if $\lambda \to \lambda_2 + 0$. The quadratic polynomial $p(\lambda) := |(Au, \lambda z + w)|^2$ has also two roots, and by (2), the ratio $\frac{p(\lambda)}{q(\lambda)}$ is bounded when $\lambda \to \lambda_1 - 0$ and $\lambda \to \lambda_2 + 0$. Therefore, one concludes that $p(\lambda)$ has the same roots as $q(\lambda)$, that is, λ_1 and λ_2 are roots of $p(\lambda)$.

Since $\lambda_1\lambda_2 < 0$ and

$$p(\lambda) = |(Au, z)|^2\lambda^2 + 2\lambda\text{Re}(Au, z)\overline{(Au, w)} + |(Au, w)|^2,$$

it follows that

$$\frac{|(Au, w)|^2}{|(Au, z)|^2} < 0. \tag{10}$$

This is a contradiction which proves that there are no elements z and w such that $(Az, z) > 0$ and $(Aw, w) < 0$.

Theorem 1.1 is proved. □

Proof of Lemma 1.2. It is known that

$$F(\frac{1}{|x|}) := \int_{\mathbb{R}^3} \frac{e^{-i\zeta\cdot x}}{|x|}dx = \frac{4\pi}{|\zeta|^2} > 0, \tag{11}$$

where the Fourier transform F is understood in the sense of distributions (see, e.g., [1]). Therefore,

$$(A\sigma, \sigma) = \int_S \int_S \frac{\sigma(t)\overline{\sigma(s)}}{4\pi|s-t|}dsdt = \int_{\mathbb{R}^3} \frac{|F\sigma(\zeta)|^2}{|\zeta|^2}d\zeta \geq 0, \tag{12}$$

which proves Lemma 1.2. □

In (12), $F\sigma(\zeta)$ is the Fourier transform of the distribution $\sigma(t)$ with support on the surface S. There are many results about the rate of decay of the Fourier transform of a function (measure) supported on a surface. For example, if the Gaussian curvature of the surface S is strictly positive, then (see [4])

$$F\sigma(\zeta) := \int_S \sigma(t)e^{-i\zeta\cdot t}dt = O\left(\frac{1}{|\zeta|}\right), \quad |\zeta| \to \infty, \quad \zeta \in \mathbb{R}^3, \tag{13}$$

provided that $\sigma(t)$ is sufficiently smooth.

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