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ON THE GOLDBACH CONJECTURE

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Abstract: The author had published a 14-page paper on the solutions for the twin primes conjecture in the International Mathematical Journal in 2003. This paper, which comprises of 3 parts that are each self-contained, presents various approaches to the Goldbach conjecture, which is a related problem.

AMS Subject Classification: 11-XX
Key Words: partitions, induction, contradiction, elimination, symmetry, different approaches

1. Part 1

Theorem. *Every even number after 2 is the sum of 2 primes.*
Proof 1.
Lemma. *By Euclid's proof the primes are infinite.*

The prime number theorem, which had been proven, states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log\, n$ as $n$approaches infinity is 1, which is expressed by the formula:

lim $\pi(n)/(n/\log\, n)$ $= 1$, where $\pi(n)$ is approximately equal to $(n/\log\, n)$
$n \to \infty$

The function $\pi(n)$ represents the number of primes less than or equal to the number n. This function measures the distribution of the prime numbers.

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With it, we compute the ratio $n/\pi(n)$ which says what fraction of the numbers up to a given point are primes. (It is actually the reciprocal of this fraction.) The following is the result of a computation:

n $\pi(n)$ $n/\ \pi(n)$

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10 4 (a) 2.5
100 25 (b) 4.0
1,000 168 (c) 6.0
10,000 1,229 (d) 8.1
100,000 9,592 (e) 10.4
1,000,000 78,498 (f) 12.7
10,000,000 664,579 (g) 15.0
100,000,000 5,761,455 (h) 17.4
1,000,000,000 50,847,534 (i) 19.7
10,000,000,000 455,052,512 (j) 22.0

It is noticeable that as one moves from 1 power of 10 to the next, the ratio $n/\ \ \pi(n)$ increases by about 2.3, e.g., 22.0 - 19.7 = 2.3. As $\log_e 10 = 2.30258 \ldots$, we may thus regard $\pi(n)$ as approximately equal to $n/\log n$.

We have the following partitions with the primes described in the "$\pi(n)$" column above:

1) With (a) above, we have the following "prime + prime = even number" combinations:

a) prime a + prime a: 4 x 4 "prime + prime" combinations
b) prime a + prime b: 4 x 25 "prime + prime" combinations
c) prime a + prime c: 4 x 168 "prime + prime" combinations
d) prime a + prime d: 4 x 1,229 "prime + prime" combinations
e) prime a + prime e: 4 x 9,592 "prime + prime" combinations
f) prime a + prime f: 4 x 78,498 "prime + prime" combinations
g) prime a + prime g: 4 x 664,579 "prime + prime" combinations
h) prime a + prime h: 4 x 5,761,455 "prime + prime" combinations
i) prime a + prime i: 4 x 50,847,534 "prime + prime" combinations
j) prime a + prime j: 4 x 455,052,512 "prime + prime" combinations

For example, for (j) above, a prime described in (a) in the "$\pi(n)$" column above plus a

prime described in (j) in the "$\pi(n)$" column above give an even number, and there are

4 x 455,052,512 such "prime + prime = even number" combinations.

2) With (b) above, we have the following "prime + prime = even number" combinations:

a) prime b + prime a: 25 x 4 “prime + prime” combinations
b) prime b + prime b: 25 x 25 “prime + prime” combinations
c) prime b + prime c: 25 x 168 “prime + prime” combinations
d) prime b + prime d: 25 x 1,229 “prime + prime” combinations
e) prime b + prime e: 25 x 9,592 “prime + prime” combinations
f) prime b + prime f: 25 x 78,498 “prime + prime” combinations
g) prime b + prime g: 25 x 664,579 “prime + prime” combinations
h) prime b + prime h: 25 x 5,761,455 “prime + prime” combinations
i) prime b + prime i: 25 x 50,847,534 “prime + prime” combinations
j) prime b + prime j: 25 x 455,052,512 “prime + prime” combinations

3) With (c) above, we have the following “prime + prime = even number” combinations:

a) prime c + prime a: 168 x 4 “prime + prime” combinations
b) prime c + prime b: 168 x 25 “prime + prime” combinations
c) prime c + prime c: 168 x 168 “prime + prime” combinations
d) prime c + prime d: 168 x 1,229 “prime + prime” combinations
e) prime c + prime e: 168 x 9,592 “prime + prime” combinations
f) prime c + prime f: 168 x 78,498 “prime + prime” combinations
g) prime c + prime g: 168 x 664,579 “prime + prime” combinations
h) prime c + prime h: 168 x 5,761,455 “prime + prime” combinations
i) prime c + prime i: 168 x 50,847,534 “prime + prime” combinations
j) prime c + prime j: 168 x 455,052,512 “prime + prime” combinations

4) With (d) above, we have the following “prime + prime = even number” combinations:

a) prime d + prime a: 1,229 x 4 “prime + prime” combinations
b) prime d + prime b: 1,229 x 25 “prime + prime” combinations
c) prime d + prime c: 1,229 x 168 “prime + prime” combinations
d) prime d + prime d: 1,229 x 1,229 “prime + prime” combinations
e) prime d + prime e: 1,229 x 9,592 “prime + prime” combinations
f) prime d + prime f: 1,229 x 78,498 “prime + prime” combinations
g) prime d + prime g: 1,229 x 664,579 “prime + prime” combinations
h) prime d + prime h: 1,229 x 5,761,455 “prime + prime” combinations
i) prime d + prime i: 1,229 x 50,847,534 “prime + prime” combinations
j) prime d + prime j: 1,229 x 455,052,512 “prime + prime” combinations

5) With (e) above, we have the following “prime + prime = even number” combinations:

a) prime e + prime a: 9,592 x 4 “prime + prime” combinations
b) prime e + prime b: 9,592 x 25 “prime + prime” combinations
c) prime e + prime c: 9,592 x 168 “prime + prime” combinations

d) prime e + prime d: 9,592 x 1,229 “prime + prime” combinations
e) prime e + prime e: 9,592 x 9,592 “prime + prime” combinations
f) prime e + prime f: 9,592 x 78,498 “prime + prime” combinations
g) prime e + prime g: 9,592 x 664,579 “prime + prime” combinations
h) prime e + prime h: 9,592 x 5,761,455 “prime + prime” combinations
i) prime e + prime i: 9,592 x 50,847,534 “prime + prime” combinations
j) prime e + prime j: 9,592 x 455,052,512 “prime + prime” combinations

6) With (f) above, we have the following “prime + prime = even number” combinations:

a) prime f + prime a: 78,498 x 4 “prime + prime” combinations
b) prime f + prime b: 78,498 x 25 “prime + prime” combinations
c) prime f + prime c: 78,498 x 168 “prime + prime” combinations
d) prime f + prime d: 78,498 x 1,229 “prime + prime” combinations
e) prime f + prime e: 78,498 x 9,592 “prime + prime” combinations
f) prime f + prime f: 78,498 x 78,498 “prime + prime” combinations
g) prime f + prime g: 78,498 x 664,579 “prime + prime” combinations
h) prime f + prime h: 78,498 x 5,761,455 “prime + prime” combinations
i) prime f + prime i: 78,498 x 50,847,534 “prime + prime” combinations
j) prime f + prime j: 78,498 x 455,052,512 “prime + prime” combinations

7) With (g) above, we have the following “prime + prime = even number” combinations:

a) prime g + prime a: 664,579 x 4 “prime + prime” combinations
b) prime g + prime b: 664,579 x 25 “prime + prime” combinations
c) prime g + prime c: 664,579 x 168 “prime + prime” combinations
d) prime g + prime d: 664,579 x 1,229 “prime + prime” combinations
e) prime g + prime e: 664,579 x 9,592 “prime + prime” combinations
f) prime g + prime f: 664,579 x 78,498 “prime + prime” combinations
g) prime g + prime g: 664,579 x 664,579 “prime + prime” combinations
h) prime g + prime h: 664,579 x 5,761,455 “prime + prime” combinations
i) prime g + prime i: 664,579 x 50,847,534 “prime + prime” combinations
j) prime g + prime j: 664,579 x 455,052,512 “prime + prime” combinations

8) With (h) above, we have the following “prime + prime = even number” combinations:

a) prime h + prime a: 5,761,455 x 4 “prime + prime” combinations
b) prime h + prime b: 5,761,455 x 25 “prime + prime” combinations
c) prime h + prime c: 5,761,455 x 168 “prime + prime” combinations
d) prime h + prime d: 5,761,455 x 1,229 “prime + prime” combinations
e) prime h + prime e: 5,761,455 x 9,592 “prime + prime” combinations
f) prime h + prime f: 5,761,455 x 78,498 “prime + prime” combinations

g) prime h + prime g: 5,761,455 x 664,579 "prime + prime" combinations
h) prime h + prime h: 5,761,455 x 5,761,455 "prime + prime" combinations
i) prime h + prime i: 5,761,455 x 50,847,534 "prime + prime" combinations
j) prime h + prime j: 5,761,455 x 455,052,512 "prime + prime" combinations

9) With (i) above, we have the following "prime + prime = even number" combinations:

a) prime i + prime a: 50,847,534 x 4 "prime + prime" combinations
b) prime i + prime b: 50,847,534 x 25 "prime + prime" combinations
c) prime i + prime c: 50,847,534 x 168 "prime + prime" combinations
d) prime i + prime d: 50,847,534 x 1,229 "prime + prime" combinations
e) prime i + prime e: 50,847,534 x 9,592 "prime + prime" combinations
f) prime i + prime f: 50,847,534 x 78,498 "prime + prime" combinations
g) prime i + prime g: 50,847,534 x 664,579 "prime + prime" combinations
h) prime i + prime h: 50,847,534 x 5,761,455 "prime + prime" combinations
i) prime i + prime i: 50,847,534 x 50,847,534 "prime + prime" combinations
j) prime i + prime j: 50,847,534 x 455,052,512 "prime + prime" combinations

10) With (j) above, we have the following "prime + prime = even number" combinations:

a) prime j + prime a: 455,052,512 x 4 "prime + prime" combinations
b) prime j + prime b: 455,052,512 x 25 "prime + prime" combinations
c) prime j + prime c: 455,052,512 x 168 "prime + prime" combinations
d) prime j + prime d: 455,052,512 x 1,229 "prime + prime" combinations
e) prime j + prime e: 455,052,512 x 9,592 "prime + prime" combinations
f) prime j + prime f: 455,052,512 x 78,498 "prime + prime" combinations
g) prime j + prime g: 455,052,512 x 664,579 "prime + prime" combinations
h) prime j + prime h: 455,052,512 x 5,761,455 "prime + prime" combinations
i) prime j + prime i: 455,052,512 x 50,847,534 "prime + prime" combinations
j) prime j + prime j: 455,052,512 x 455,052,512 "prime + prime" combinations

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The above partitions/"prime + prime = even number" combinations are evidently progressively more "overwhelming" and repetitive. It is not surprising that computer searches completed in 2000 had verified that all even numbers up to 400 trillion (4 x 10^{14}), which is not a small list, are sums of 2 primes, while in 2008, a distributed computer search ran by Tomas Oliveira e Silva, a researcher at the University of Aveiro, Portugal, had further verified the Goldbach conjecture up to 12 x 10^{17}.

The infinitude of the primes, as per the above lemma, together with the infinitude of the even numbers, however imply that the above partitions/"prime + prime = even number" combinations would become increasingly more "overwhelming", dense, and repetitive towards infinity (the Goldbach conjecture becoming evidently stronger and stronger the higher up the infinite list of prime numbers/even numbers we go), hence "ensuring" the continuity (without any breaks or gaps) of the even numbers, and would be so all the way to infinity, thus proving that every even number after 2 is the sum of 2 primes.

Proof 2.

Lemma. *According to the principle of complete induction in set theory, if a set of natural numbers contains 1 and, for each n, it contains $n + 1$ whenever it contains all numbers less than $n + 1$, then it must contain every natural number, e.g., complete induction proves that every natural number is a product of primes.*

By the above lemma, every even number after 2 in the infinite set of the integers is the sum of 2 primes; as per the distributed computer search completed in 2008 at the University of Aveiro, Portugal, stated above, 12 x 10^{17} in the infinite set of the integers is the largest even number found to be the sum of 2 primes while all the consecutive even numbers before it, from 4 to (12 x 10^{17}) - 2, which is not a small list of numbers (it is in fact a long, impressive list, obtainable only with the help of modern computer technology), are also found to be sums of 2 primes - the principle of complete induction implies that all even numbers after 12 x 10^{17} in the infinite set of the integers must also be sums of 2 primes, i.e., it implies that every even number after 2 in the infinite set of the integers must be the sum of 2 primes - in other words, the Goldbach conjecture must be true.

Proof 3.

Lemma. *By Euclid's proof the primes are infinite.*

We make use of the proof by "reductio ad absurdum" here. For this indirect proof, we assume that the Goldbach conjecture is false. (Before we proceed further, we should again note that a long, impressive list of consecutive even numbers, from 4 to 12 x 10^{17}, had already been verified to be sums of 2 primes, and, these partitions/"prime + prime = even number" combinations would become increasingly more "overwhelming", dense, and repetitive towards infinity (the Goldbach conjecture becoming evidently stronger and stronger the higher up the infinite list of prime numbers/even numbers we go), as is described above. The moot question now is, of course, whether after 12 x 10^{17} there would be an even number in the infinite list of even numbers which is the last, or, largest, even number which is the sum of 2 primes - this largest even number, if it exists (thereby proving the falsehood of the Goldbach conjecture), must (of necessity) be the sum of 2 primes which are each the largest existing prime. Before we continue, this point should be clearly held in mind.) This assumption implies that there is a limit to the even numbers which are sums of 2 primes and that there is a largest even number (e) which is, and must necessarily be, the sum of 2 primes that are each the largest existing prime ($e = x + x$, this largest even number, e, representing the ultimate limit of the even numbers which are sums of 2 primes, the 2 primes which add up to give e being of necessity each the largest existing prime (x)). This is of course a contradiction of the above lemma, which would imply that the lemma is false. But the lemma cannot be false - it is in fact a theorem (which had been proven by Euclid); there cannot be a largest existing prime (x) - the primes are infinite. This means that our assumption that the Goldbach conjecture is false is untenable and that the Goldbach conjecture must be true, i.e., every even number after 2 must be the sum of 2 primes. As a matter of fact, the above lemma implies that there would be an infinite number of double primes which sum up to an even number.

By both induction and contradiction the Goldbach conjecture is hence proved.

2. Part 2

Theorem. *Every even number after 2 is the sum of 2 primes.*

Proof 1. Every even number after 2 is the sum of 2 odd numbers. Every odd number is either a prime which is odd or a composite - product of primes which are odd; notably, every prime with the exception of 2 is an odd number. Every even number after 2 is also a composite, but, a composite with at least 1 even prime factor, namely, 2, while the rest of its prime factors are odd, i.e., it is an even composite.

Therefore, every even number after 2 is the sum of 2 primes which are odd and/or the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd and/or the sum of 2 odd composites whose prime factors are odd, besides being an even composite with at least 1 even prime factor, namely, 2, while the rest of its prime factors are odd.

Lemma:

By Euclid's proof, the primes are infinite; this implies that there would be an infinitude of sums of 2 primes as per the Goldbach conjecture. The even numbers, which are sums of 2 primes as per the conjecture, are also infinite. Thus, there are an infinite number of even numbers which are sums of 2 primes, both the even numbers and sums of 2 primes being infinite.

Corollary:

The odd numbers, which are either prime, every prime with the exception of 2 being an odd number, or composite (have prime factors which are odd), are infinite; this implies that there would be an infinite number of sums of 2 odd numbers, each of which is equal to an even number. Hence, as there is an infinitude of even numbers which are sums of 2 primes, as per the above lemma, and as all primes with the exception of 2 are odd numbers, there are an infinite number of even numbers which are sums of 2 odd numbers that are prime, all the even numbers, sums of 2 odd numbers and primes being infinite; i.e., every even number after 2 is also the sum of 2 odd numbers that are prime.

We thereby see the close interlink or relationship between the primes, even numbers and odd numbers, which are all infinite, which is significant.

The following are thus evident:

a) Every sum of 2 primes which are odd numbers is equal to an even number, as is below in consecutive order:

$2 + 2 = 1 + 3 =$ **4**
$3 + 3 = 1 + 5 =$ **6**
$3 + 5 = 1 + 7 =$ **8**
$5 + 5 = 3 + 7 =$ **10**
$5 + 7 = 1 + 11 =$ **12**
$7 + 7 = 3 + 11 = 1 + 13 =$ **14**
$3 + 13 = 5 + 11 =$ **16**
$7 + 11 = 5 + 13 = 1 + 17 =$ **18**
$7 + 13 = 3 + 17 = 1 + 19 =$ **20**
$11 + 11 = 3 + 19 = 5 + 17 = 11 + 11 =$ **22**
$11 + 13 = 5 + 19 = 7 + 17 = 1 + 23 =$ **24**
$13 + 13 = 3 + 23 = 7 + 19 =$ **26**
$11 + 17 = 5 + 23 =$ **28**
$13 + 17 = 11 + 19 = 7 + 23 = 1 + 29 =$ **30**
$3 + 29 = 13 + 19 = 1 + 31 =$ **32**
$17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 =$ **34**
$17 + 19 = 5 + 31 = 7 + 29 = 13 + 23 =$ **36**
$19 + 19 = 7 + 31 = 1 + 37 =$ **38**
$3 + 37 = 11 + 29 = 17 + 23 =$ **40**

19 + 23 = 5 + 37 = 11 + 31 = 13 + 29 = 1 + 41 =**42**
3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 =**44**
23 + 23 = 3 + 43 = 5 + 41 = 17 + 29 =**46**
5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 =**48**
3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 =**50**
23 + 29 = 5 + 47 = 11 + 41 =**52**
7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 =**54**
3 + 53 = 13 + 43 = 19 + 37 =**56**
29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 =**58**
29 + 31 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 1 + 59 =**60**
31 + 31 = 3 + 59 = 19 + 43 = 1 + 61 =**62**
3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 =**64**
5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 =**66**
7 + 61 = 31 + 37 = 1 + 67 =**68**
3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 =**70**
5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 =**72**
37 + 37 = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 =**74**
3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 =**76**
37 + 41 = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 =**78**
7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 =**80**
41 + 41 = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 =**82**
41 + 43 = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 =1+ 83 =**84**
43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 =**86**
5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 =**88**
7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89 =**90**
3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 =**92**
47 + 47 = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 =**94**
5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**
7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 =**98**
47 + 53 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 =**100**
5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41+ 61 = 43 + 59 = 1 + 101 =**102**

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b) Every sum of 1 prime which is an odd number & 1 odd composite which is the product of primes which are

odd, is equal to the sum of 2 primes which are odd numbers, which are all each equal to an even number, as

is below in consecutive order:

3 + **9** = 5 + 7 = 1 + 11 = **12**

5 + **9** = 3 + 11 = 7 + 7 = 1 + 13 = **14**

7 + **9** = 3 + 13 = 5 + 11 = **16**

3 + **15** =7 + 11 = 5 + 13 = 1 + 17 = **18**

11 + **9** = 3 + 17 = 7 + 13 = 1 + 19 = **20**

13 + **9** = 3 + 19 = 5 + 17 = 11 + 11 = **22**

3 + **21** = 11 + 13 = 5 + 19 = 7 + 17 = 1 + 23 = **24**

17 + **9** = 3 + 23 = 7 + 19 = 13 + 13 = **26**

19 + **9** = 5 + 23 = 11 + 17 = **28**

5 + **25** = 13 + 17 = 11 + 19 = 7 + 23 = 1 + 29 = **30**

23 + **9** = 3 + 29 = 13 + 19 = 1 + 31 = **32**

7 + **27** = 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **34**

3 + **33** = 17 + 19 = 5 + 31 = 7 + 29 = 13 + 23 = **36**

29 + **9** = 7 + 31 = 19 + 19 = 1 + 37 = **38**

31 + **9** = 3 + 37 = 11 + 29 = 17 + 23 = **40**

3 + **39** = 19 + 23 = 5 + 37 = 11 + 31 = 13 + 29 = 1 + 41 = **42**

5 + **39** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **44**

37 + **9** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **46**

3 + **45** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **48**

41 + **9** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **50**

43 + **9** = 5 + 47 = 11 + 41 =23 + 29 = **52**

5 + **49** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **54**

47 + **9** = 3 + 53 = 13 + 43 = 19 + 37 = **56**

3 + **55** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **58**

5 + **55** = 29 + 31 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 1 + 59 = **60**

53 + **9** = 3 + 59 = 19 + 43 = 31 + 31 = 1 + 61 = **62**

7 + **57** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **64**

11 + **55** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **66**

59 + **9** = 7 + 61 = 31 + 37 = 1 + 67 = **68**

61 + **9** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **70**

3 + **69** =5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **72**

5 + **69** = 37 + 37 = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **74**

67 + **9** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **76**

3 + **75** = 37 + 41 = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **78**

71 + **9** = 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **80**

73 + **9** = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = 41 + 41 = **82**

3 + **81** = 41 + 43 = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37

+ 47 = 1+ 83 = 84

5 + **81** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **86**

79 + **9** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **88**

3 + **87** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89

= **90**

83 + **9** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **92**

7 + **87** = 47 + 47 = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **94**

3 + **93** = 5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**

89 + **9** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **98**

91 + **9** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **100**

3 + **99** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41+ 61 = 43 + 59 = 1 + 101

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c) Every sum of 2 odd composites which are products of primes which are odd, is equal to the sum of 2 primes

which are odd numbers, which are all each equal to an even number, as is below in consecutive order:

9 + **9** = 5 + 13 = 7 + 11 = 1 + 17 = **18**

9 + **15** = 5 + 19 = 7 + 17 = 11 + 13 = 1 + 23 = **24**

15 + **15** = 7 + 23 = 11 + 19 = 13 + 17 = 1 + 29 = **30**

9 + **25** = **7** + **27** = 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **34**

15 + **21** = 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 = **36**

15 + **25** = 3 + 37 = 11 + 29 = 17 + 23 = **40**

21 + **21** = 5 + 37 = 11 + 31 = 13 + 29 = 19 + 23 = 1 + 41 = **42**

9 + **35** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **44**

21 + **25** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **46**

9 + **39** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **48**

25 + **25** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **50**

25 + **27** = 5 + 47 = 11 + 41 =23 + 29 = **52**

27 + **27** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **54**

21 + **35** = 3 + 53 = 13 + 43 = 19 + 37 = **56**

9 + **49** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **58**

27 + **33** = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 = 1 + 59 = **60**

27 + **35** = 31 + 31 = 3 + 59 = 19 + 43 = 1 + 61 = **62**

9 + **55** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **64**

33 + **33** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **66**

33 + **35** = 7 + 61 = 31 + 37 = 1 + 67 = **68**

35 + **35** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **70**

9 + **63** = 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **72**

35 + **39** = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **74**

21 + **55** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **76**

39 + **39** = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **78**

15 + **65** =7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **80**

25 + **57** = 41 + 41 = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = **82**

39 + **45** = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 41 + 43 = 1 + 83 = **84**

9 + **77** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **86**

25 + **63** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **88**

45 + **45** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89

= 90

15 + **77** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **92**

45 + **49** = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **94**

9 + **87** = 5 + 91 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **96**

49 + **49** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **98**

49 + **51** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **100**

51 + **51** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **102**

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d) From (a), (b) & (c) above, we have the even numbers from 4 to 102 . . . composed as follows:

1) **4** = 2 + 2 = 1 + 3 (sum of 2 primes only)

2) **6** = 3 + 3 = 1 + 5 (sum of 2 primes only)

3) **8** = 3 + 5 = 1 + 7 (sum of 2 primes only)

4) **10** = 5 + 5 = 3 + 7 (sum of 2 primes only)

5) **12** = 5 + 7 = 1 + 11 = **3** + **9** (sum of 1 prime & 1 odd composite)

6) **14** = 3 + 11 = 7 + 7 = 1 + 13 = **5** + **9** (sum of 1 prime & 1 odd composite)

7) **16** = 3 + 13 = 5 + 11 = **7** + **9** (sum of 1 prime & 1 odd composite)

8) **18** = 5 + 13 = 7 + 11 = 1 + 17 = **3** + **15** (sum of 1 prime & 1 odd composite) = **9** + **9** (sum of 2 odd composites)

9) **20** = 3 + 17 = 7 + 13 = 1 + 19 = **11** + **9** (sum of 1 prime & 1 odd composite)

10) **22** = 3 + 19 = 5 + 17 = 11 + 11 = **13** + **9** (sum of 1 prime & 1 odd composite)

11) **24** = 5 + 19 = 7 + 17 = 11 + 13 = 1 + 23 = **3** + **21** (sum of 1 prime & 1 odd composite) = **9** + **15** (sum of 2 odd composites)

12) **26** = 3 + 23 = 7 + 19 = 13 + 13 = **17** + **9** (sum of 1 prime & 1 odd composite)

13) **28** = 5 + 23 = 11 + 17 = **19** + **9** (sum of 1 prime & 1 odd composite)

14) **30** = 7 + 23 = 11 + 19 = 13 + 17 = 1 + 29 = **5** + **25** (sum of 1 prime & 1 odd composite) = **15** + **15** (sum of 2 odd composites)

15) **32** = 3 + 29 = 13 + 19 = 1 + 31 = **23** + **9** (sum of 1 prime & 1 odd composite)

16) **34** = 17 + 17 = 3 + 31 = 5 + 29 = 11 + 23 = 17 + 17 = **7** + **27** (sum of 1 prime & 1 odd composite) = **9** + **25** (sum of 2 odd composites)

17) **36** = 5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 = **3** + **33** (sum of 1 prime & 1 odd composite) = **15** + **21** (sum of 2 odd composites)

18) **38** = 7 + 31 = 19 + 19 = 1 + 37 = **29** + **9** (sum of 1 prime & 1 odd composite)

19) **40** = 3 + 37 = 11 + 29 = 17 + 23 = **31** + **9** (sum of 1 prime & 1 odd composite) = **15** + **25** (sum of 2 odd composites)

20) **42** = 5 + 37 = 11 + 31 = 13 + 29 = 19 + 23 = 1 + 41 = **3** + **39** (sum of 1 prime & 1 odd composite) = **21** + **21** (sum of 2 odd composites)

21) **44** = 3 + 41 = 7 + 37 = 13 + 31 = 1 + 43 = **5** + **39** (sum of 1 prime & 1 odd composite) = **9** + **35** (sum of 2 odd composites)

22) **46** = 3 + 43 = 5 + 41 = 17 + 29 = 23 + 23 = **37** + **9** (sum of 1 prime & 1 odd composite) = **21** + **25** (sum of 2 odd composites)

23) **48** = 5 + 43 = 7 + 41 = 11 + 37 = 17 + 31 = 19 + 29 = 1 + 47 = **3** + **45** (sum of 1 prime & 1 odd composite) = **9** + **39** (sum of 2 odd composites)

24) **50** = 3 + 47 = 7 + 43 = 13 + 37 = 19 + 31 = **41** + **9** (sum of 1 prime & 1 odd composite) = **25** + **25** (sum of 2 odd composites)

25) **52** = 5 + 47 = 11 + 41 =23 + 29 = = **43** + **9** (sum of 1 prime & 1 odd composite) = **25** + **27** (sum of 2 odd composites)

26) **54** = 7 + 47 = 11 + 43 = 13 + 41 = 17 + 37 = 23 + 31 = 1 + 53 = **5** + **49** (sum of 1 prime & 1 odd composite) = **27** + **27** (sum of 2 odd composites)

27) **56** = 3 + 53 = 13 + 43 = 19 + 37 = **47** + **9** (sum of 1 prime & 1 odd composite) = **21** + **35** (sum of 2 odd composites)

28) **58** = 29 + 29 = 5 + 53 = 11 + 47 = 17 + 41 = 29 + 29 = **3** + **55** (sum of 1 prime & 1 odd composite) = **9** + **49** (sum of 2 odd composites)

29) **60** = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31 = 1 + 59 = **5** + **55** (sum of 1 prime & 1 odd composite) = **27** + **33** (sum of 2 odd composites)

30) **62** = 3 + 59 = 19 + 43 = 31 + 31 = 1 + 61 = **53** + **9** (sum of 1 prime & 1 odd composite) = **27** + **35** (sum of 2 odd composites)

31) **64** = 3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = **7** + **57** (sum of 1 prime & 1 odd composite) = **9** + **55** (sum of 2 odd composites)

32) **66** = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37 = **11** + **55** (sum of 1 prime & 1 odd composite) = **33** + **33** (sum of 2 odd composites)

33) **68** = 7 + 61 = 31 + 37 = 1 + 67 = **59** + **9** (sum of 1 prime & 1 odd composite) = **33** + **35** (sum of 2 odd composites)

34) **70** = 3 + 67 = 11 + 59 = 17 + 53 = 23 + 47 = 29 + 41 = **61** + **9** (sum of 1 prime & 1 odd composite) = **35** + **35** (sum of 2 odd composites)

35) **72** = 5 + 67 = 11 + 61 = 13 + 59 = 19 + 53 = 29 + 43 = 31 + 41 = 1 + 71 = **3** + **69** (sum of 1 prime & 1 odd composite) = **9** + **63** (sum of 2 odd composites)

36) **74** = 3 + 71 = 7 + 67 = 13 + 61 = 31 + 43 = 37 + 37 = 1 + 73 = **5** + **69** (sum of 1 prime & 1 odd composite) = **35** + **39** (sum of 2 odd composites)

37) **76** = 3 + 73 = 5 + 71 = 17 + 59 = 23 + 53 = 29 + 47 = **67** + **9** (sum of 1 prime & 1 odd composite) = **21** + **55** (sum of 2 odd composites)

38) **78** = 5 + 73 = 7 + 71 = 11 + 67 = 31 + 47 = 37 + 41 = **3** + **75** (sum of 1 prime & 1 odd composite) = **39** + **39** (sum of 2 odd composites)

39) **80** = 7 + 73 = 13 + 67 = 19 + 61 = 37 + 43 = 1 + 79 = **71** + **9** (sum of 1 prime & 1 odd composite) = **15** + **65** (sum of 2 odd composites)

40) **82** = 3 + 79 = 11 + 71 = 23 + 59 = 29 + 53 = 41 + 41 = **73** + **9** (sum of 1 prime & 1 odd composite) = **25**
+ **57** (sum of 2 odd composites)

41) **84** = 5 + 79 = 11 + 73 = 13 + 71 = 17 + 67 = 23 + 61 = 31 + 53 = 37 + 47 = 41 + 43 = 1 + 83 = **3** + **81**
(sum of 1 prime & 1 odd composite) = **39** + **45** (sum of 2 odd composites)

42) **86** = 43 + 43 = 3 + 83 = 7 + 79 = 13 + 73 = 19 + 67 = 43 + 43 = **5** + **81** (sum of 1 prime & 1 odd
composite) = **9** + **77** (sum of 2 odd composites)

43) **88** = 5 + 83 = 17 + 71 = 29 + 59 = 41 + 47 = **79** + **9** (sum of 1 prime & 1 odd composite) = **25** + **63** (sum
of 2 odd composites)

44) **90** = 7 + 83 = 11 + 79 = 17 + 73 = 19 + 71 = 23 + 67 = 29 + 61 = 31 + 59 = 37 + 53 = 43 + 47 = 1 + 89
= 3 + 87 (sum of 1 prime & 1 odd composite) = **45** + **45** (sum of 2 odd composites)

45) **92** = 3 + 89 = 13 + 79 = 19 + 73 = 31 + 61 = 1 + 91 = **83** + **9** (sum of 1 prime & 1 odd composite) = **15**
+ **77** (sum of 2 odd composites)

46) **94** = 5 + 89 = 11 + 83 = 23 + 71 = 41 + 53 = 47 + 47 = **7** + **87** (sum of 1 prime & 1 odd composite) = **45**
+ **49** (sum of 2 odd composites)

47) **96** = 5 + 91 =7 + 89 =13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53 = **3** + **93** (sum of 1
prime & 1 odd composite) = **9** + **87** (sum of 2 odd composites)

48) **98** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **89** + **9** (sum of 1 prime & 1 odd composite) = **49**
+ **49** (sum of 2 odd composites)

49) **100** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **91** + **9** (sum of 1 prime & 1 odd
composite) = **49** + **51** (sum of 2 odd composites)

50) **102** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 +
101 = **3** + **99** (sum of 1 prime & 1 odd composite) = **51** + **51** (sum of 2 odd composites)

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(The above is only a partial or incomplete listing of sums of 1 prime & 1 odd composite, and, sums of 2 odd composites, each of which is equal to the sum of 2 primes as well as an even number. For example, in the list of compositions for the even numbers 4 to 102 . . . above, in Item (48), we could also have other "combinations" such as: **98** = 7 + 91 = 19 + 79 = 31 + 67 = 37 + 61 = 1 + 97 = **25** + **73** (sum of 1 prime & 1 odd composite) = **21** + **77** (sum of 2 odd composites), etc., in Item (49), we could also have other "combinations" such as: **100** = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = **31** + **69** (sum of 1 prime & 1 odd composite) = **45** + **55** (sum of 2 odd composites), etc., and, in Item (50), we could also have other "combinations" such as: **102** = 5 + 97 = 11 + 91 = 13 + 89 = 19 + 83 = 23 + 79 = 29 + 73 = 31 + 71 = 41 + 61 = 43 + 59 = 1 + 101 = **17** + **85** (sum of 1 prime & 1 odd composite) = **21** + **81** (sum of 2 odd composites), etc.. That is, there are more "combinations" than those shown in the above listing.)

In (d) above, in the list of compositions for the 50 consecutive even numbers 4 to 102 . . . , the even numbers 4, 6, 8 and 10 are only formed through the summing of 2 primes and not at all through the summing of 1 prime and 1 odd composite, or, the summing of 2 odd composites, which are impossibilities here. These sums of 2 primes are present (always present) throughout the whole list of compositions, from 4 right through to 102, while this is not the case for the sums of 1 prime and 1 odd composite, and, the sums of 2 odd composites.

We reason here by the process of elimination, through analysing the information in (d) above which pertains to the compositions of the 50 consecutive even numbers 4 to 102 . . . taken from the infinite list of even numbers. We stated at the beginning the following about the even numbers after 2:

Firstly, every even number after 2 is:

A) The sum of 2 odd numbers.

(Every odd number is either a prime which is odd or a composite - product of primes which are odd.

Notably, every prime with the exception of 2 is an odd number.)

Secondly, every even number after 2 is also (the below-mentioned is the logical consequence of (A) above):

1) The sum of 2 primes which are odd.

2) And/or the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd.

3) And/or the sum of 2 odd composites whose prime factors are odd.

Evidently, at least 1 of (1), (2) & (3) above has to be the "atom" or building-block of the even numbers. In (d) above, we observe the following:

1. All the 50 consecutive even numbers 4 to 102 ... in (d) above taken from the infinite list of even numbers are sums of 2 primes.

2. It is impossible for each of the even numbers 4, 6, 8 & 10 in (d) above to be the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd.

iii) It is impossible for each of the even numbers 4, 6, 8, 10, 12, 14, 16, 20, 22, 26, 28, 32 & 38 in

(d) above to be the sum of 2 odd composites whose prime factors are odd.

It is evident from (i), (ii) & (iii) above that neither (2) nor (3) can be the "atom" or building-block of the even numbers since they are "incomplete". As (1) - the sum of 2 primes which are odd - is "complete", i.e., always present in the 50 consecutive even numbers 4 to 102 ... in (d) above, unlike (2) & (3), it evidently is the "atom" or building-block of the even numbers. That is, every even number after 2 is evidently the sum of 2 primes which are odd. In fact, a distributed computer search completed in 2008 at the University of Aveiro, Portugal, had verified this for all even numbers up to 12 x 10^{17}, which is not a small list. Definitely, due respectively to (ii) & (iii) above, we cannot say that every even number after 2 is the sum of 1 prime which is odd and 1 odd composite whose prime factors are odd, or, every even number after 2 is the sum of 2 odd composites whose prime factors are odd.

By the above lemma and corollary, the infinitudes of the primes, even numbers and odd numbers indeed imply that there are an infinite number of sums of 2 primes which are odd numbers, which are each equal to an even number. As the sums of 2 primes which are odd numbers are evidently the "atoms" or building-blocks of the even numbers, it also implies that they are infinite, since the even numbers are infinite.

Hypothetically, if on the other hand just 1 of the 3 items stated above, primes, even numbers and odd numbers, were finite, the above-said sums of 2 primes which are odd numbers, each of which is equal to an even number, would be finite. The primes, even numbers and odd numbers are evidently intricately linked, with the primes playing the part of building-blocks of both the even and odd numbers through various "combinations" as is described below. However, as the primes, even numbers and odd numbers are intricately linked, the finiteness (or, infinity) of any 1 of them implies the finiteness (or, infinity) of the other 2, and vice versa. These 3 items are evidently "close comrades-in-arm" working together to give special meaning to the integers. As these 3 are all infinite, it indeed implies that there is an infinitude of even numbers which are infinitely the sums of 2 primes that are odd and infinite.

Proof 2.

Lemma. *According to the precepts of fractal geometry and group theory, symmetry is a very important, intrinsic part of nature. There is symmetry all around us and within us. There is evident symmetry in human bodies, the structures of viruses and bacteria, polymers and ceramic materials, the permutations of numbers, the universe and many others, even the movements of prices in financial markets, the growths of populations, the sound of music, the flow of blood through our circulatory system, the behaviour of people en masse, etc.. In other words, regularity, pattern, order, uniformity or symmetry is evident everywhere.*

The above-mentioned most basic, always present sums of 2 primes, each of which is equal to an even number, which are evidently the "atoms" or building-blocks of the even numbers, are characterised by the feature of symmetry (in 2008, a distributed computer search ran by Tomas Oliveira e Silva, a researcher at the University of Aveiro, Portugal, had verified that all even numbers up to 12 x 10^{17}, which is no small list of numbers, are sums of 2 primes, a regularity, uniformity, order, pattern, symmetry). Thus, by the above lemma, every even number after 2 is naturally or inherently the sum of 2 primes, i.e., there is an infinitude of sums of 2 primes which are each equal to an even number.

Hence, the confirmation of the following generalisation pertaining to the integers, whereby it is indeed evident that the primes play a very important role:

Let a prime = p, &, a composite = c = p x p . . . . .

a) Every even number after 2 = p + p = *c = *p x p . . . . &/V = c + p = (p x p . . . .) + p

&/V = c + c = (p x p . . . .) + (p x p . . . .) (in *c = *p x p . . . . here, which is an even

composite, 1 or more of the p's are 2, the only even prime, e.g., 6 = 2 x 3, 8 = 2 x

2 x 2, 10 = 2 x 5, 18 = 2 x 3 x 3, 20 = 2 x 2 x 5, 24 = 2 x 2 x 2 x 3, etc.)

b) Every odd number = p V c = p x p . . . . (in c = p x p . . . . here, which is an odd

composite, like the c = p x p . . . .'s in (a) above, all the p's are odd, e.g., 9 = 3 x 3, 15

= 3 x 5, 21 = 3 x 7, 25 = 5 x 5, 63 = 3 x 3 x 7, 99 = 3 x 3 x 11, etc.)

It is easy to see that the Goldbach conjecture is valid, i.e., every even number after 2 is the sum of 2 primes.

3. Part 3

Theorem. *Every even number after 2 is the sum of 2 primes.*

Solution. The prime numbers are evidently the atoms or building-blocks of the integers. The integers are either primes (not divisible by other integers except 1) or composites (divisible by other integers, e.g., the prime numbers), and, even (the sums of 2 primes as conjectured by Goldbach) or odd (primes, or, composites whereby they are divisible by prime factors). Therefore, to determine whether the conjecture that every even number (except the number 2) is the sum of 2 primes is true, it would be appropriate to analyse the evident atoms or building-blocks of the even numbers, viz., the prime numbers. For the solution to this conjecture we note that the primes (vide Euclid's proof) and the even numbers are infinite, which implies that this conjecture should be true.

We here analyse the "behaviour" of the first 2,400 consecutive prime numbers (divided into 12 batches of consecutive primes, each subsequent batch with an increment of 200 primes), leaving out 2 (because it is an even prime) and commencing with 3, which is the $2^{\text{nd.}}$ consecutive prime, the latter to be the first prime in our list of 2,400 consecutive primes (3 to 21,391), as follows:

(1) 200 Consecutive Primes From 3 To 1,229

(a) Even numbers (obtained by summing of 2 primes) = 6 to 2,458

(b) No. of even numbers = 1,227

(c) No. of primes = 200

(d) Average no. of even numbers "generated" by each of these 200 consecutive primes =

1,227 ÷200 = **6.14**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 200 primes = 200 x 200 = 40,000

(f) Average no. of summings of 2 primes/permutations for each of the 1,227 even

numbers = 40,000 ÷1,227 = **32.60**

(2) 400 Consecutive Primes From 3 To 2,749

(a) Even numbers (obtained by summing of 2 primes) = 6 to 5,498

(b) No. of even numbers = 2,747

(c) No. of primes = 400

(d) Average no. of even numbers "generated" by each of these 400 consecutive primes =

2,747 ÷400 = **6.87**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 400 primes = 400 x 400 = 160,000

(f) Average no. of summings of 2 primes/permutations for each of the 2,747 even
numbers = 160,000 ÷2,747 = **58.25**

(3) 600 Consecutive Primes From 3 To 4,421

(a) Even numbers (obtained by summing of 2 primes) = 6 to 8,842

(b) No. of even numbers = 4,419

(c) No. of primes = 600

(d) Average no. of even numbers "generated" by each of these 600 consecutive primes =
4,419 ÷600 = **7.37**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 600 primes = 600 x 600 = 360,000

(f) Average no. of summings of 2 primes/permutations for each of the 4,419 even
numbers = 360,000 ÷4,419 = **81.47**

(4) 800 Consecutive Primes From 3 To 6,143

(a) Even numbers (obtained by summing of 2 primes) = 6 to 12,286

(b) No. of even numbers = 6,141

(c) No. of primes = 800

(d) Average no. of even numbers "generated" by each of these 800 consecutive primes =
6,141 ÷800 = **7.68**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 800 primes = 800 x 800 = 640,000

(f) Average no. of summings of 2 primes/permutations for each of the 6,141 even
numbers = 640,000 ÷6,141 = **104.22**

(5) 1,000 Consecutive Primes From 3 To 7,927

(a) Even numbers (obtained by summing of 2 primes) = 6 to 15,854

(b) No. of even numbers = 7,925

(c) No. of primes = 1,000

(d) Average no. of even numbers "generated" by each of these 1,000 consecutive primes

$$= 7,925 \div 1,000 = 7.93$$

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 1,000 primes = 1,000 x 1,000 = 1,000,000

(f) Average no. of summings of 2 primes/permutations for each of the 7,925 even

numbers = 1,000,000 ÷7,925 = **126.18**

(6) 1,200 Consecutive Primes From 3 To 9,739

(a) Even numbers (obtained by summing of 2 primes) = 6 to 19,478

(b) No. of even numbers = 9,737

(c) No. of primes = 1,200

(d) Average no. of even numbers "generated" by each of these 1,200 consecutive primes

$$= \quad 9,737 \quad \div \quad 1,200 \quad = \quad 8.11$$

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 1,200 primes = 1,200 x 1,200 = 1,440,000

(f) Average no. of summings of 2 primes/permutations for each of the 9,737 even

numbers = 1,440,000 ÷9,737 = **147.89**

(7) 1,400 Consecutive Primes From 3 To 11,677

(a) Even numbers (obtained by summing of 2 primes) = 6 to 23,354

(b) No. of even numbers = 11,675

(c) No. of primes = 1,400

(d) Average no. of even numbers "generated" by each of these 1,400 consecutive primes

$$= \quad 11,675 \quad \div \quad 1,400 \quad = \quad 8.34$$

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 1,400 primes = 1,400 x 1,400 = 1,960,000

(f) Average no. of summings of 2 primes/permutations for each of the 11,675

even numbers = 1,960,000 ÷11,675 = **167.88**

(8) 1,600 Consecutive Primes From 3 To 13,513

(a) Even numbers (obtained by summing of 2 primes) = 6 to 27,026

(b) No. of even numbers = 13,511

(c) No. of primes = 1,600

(d) Average no. of even numbers "generated" by each of these 1,600 consecutive primes

$$= \quad 13,511 \quad \div \quad 1,600 \quad = \quad 8.44$$

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 1,600 primes = 1,600 x 1,600 = 2,560,000
(f) Average no. of summings of 2 primes/permutations for each of the 13,511
even numbers = 2,560,000 ÷13,511 = **189.48**
(9) 1,800 Consecutive Primes From 3 To 15,413
(a) Even numbers (obtained by summing of 2 primes) = 6 to 30,826
(b) No. of even numbers = 15,411
(c) No. of primes = 1,800
(d) Average no. of even numbers "generated" by each of these 1,800 consecutive primes

$$= \quad 15,411 \quad \div \quad 1,800 \quad = \quad 8.56$$

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 1,800 primes = 1,800 x 1,800 = 3,240,000
(f) Average no. of summings of 2 primes/permutations for each of the 15,411
even numbers = 3,240,000 ÷15,411 = **210.24**
(10) 2,000 Consecutive Primes From 3 To 17,393
(a) Even numbers (obtained by summing of 2 primes) = 6 to 34,786
(b) No. of even numbers = 17,391
(c) No. of primes = 2,000
(d) Average no. of even numbers "generated" by each of these 2,000 consecutive
primes = 17,391 ÷2,000 = **8.70**
(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...
etc.) for these 2,000 primes = 2,000 x 2,000 = 4,000,000
(f) Average no. of summings of 2 primes/permutations for each of the 17,391
even numbers = 4,000,000 ÷17,391 = **230.00**
(11) 2,200 Consecutive Primes From 3 To 19,427
(a) Even numbers (obtained by summing of 2 primes) = 6 to 38,854
(b) No. of even numbers = 19,425

(c) No. of primes = 2,200

(d) Average no. of even numbers "generated" by each of these 2,200 consecutive

primes = 19,425 ÷2,200 = **8.83**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 2,200 primes = 2,200 x 2,200 = 4,840,000

(f) Average no. of summings of 2 primes/permutations for each of the 19,425

even numbers = 4,840,000 ÷19,425 = **249.16**

(12) 2,400 Consecutive Primes From 3 To 21,391

(a) Even numbers (obtained by summing of 2 primes) = 6 to 42,782

(b) No. of even numbers = 21,389

(c) No. of primes = 2,400

(d) Average no. of even numbers "generated" by each of these 2,400 consecutive

primes = 21,389 ÷2,400 = **8.91**

(e) No. of summings of 2 primes/permutations (3 + 3, 3 + 5, 3 + 7, 3 + 11, ...

etc.) for these 2,400 primes = 2,400 x 2,400 = 5,760,000

(f) Average no. of summings of 2 primes/permutations for each of the 21,389

even numbers = 5,760,000 ÷21,389 = **269.30**

There would evidently be more and more profuse repetitions and overlaps of the even numbers "generated" by the primes the higher up the infinite list of prime numbers we go, which is significant.

We compare all the (d)s and (f)s in (1) to (12) above, which is as follows:

(d) Average no. of even numbers "generated" by each of the consecutive primes in (1) to

(12) above, as follows according to the listings (1) to (12):

(1) **6.14**, (2) **6.87**, (3) **7.37**, (4) **7.68**, (5) **7.93**, (6) **8.11**, (7) **8.34**, (8) **8.44**, (9) **8.56**, (10) **8.70**,

(11) **8.83**, (12) **8.91**

(f) Average no. of summings of 2 primes/permutations for each of the even numbers

in (1) to (12) above, as follows according to the listings (1) to (12):

(1) **32.60**, (2) **58.25**, (3) **81.47**, (4) **104.22**, (5) **126.18**, (6) **147.89**, (7) **167.88**, (8) **189.48**, (9) **210.24**,

(10) **230.00**, (11) **249.16**, (12) **269.30**

The following is evident from the above information:

(A): (d) Average no. of even numbers "generated" by each of the consecutive primes in the above 12 listings increases continually all the way from the list: (1) 200

Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes

From 3 To 21,391, from **6.14** even numbers per prime number in List (1) to **8.91** even numbers per prime number in List (12).

(B): (f) Average no. of summings of 2 primes/permutations for each of the even numbers in the above 12 listings increases continually all the way from the list: (1) 200

Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes

From 3 To 21,391, from **32.60** number of summings of 2 primes/permutations per even number in List (1) to **269.30** number of summings of 2 primes/permutations per even number in List (12).

Proof 1.

Lemma. *According to the principle of complete induction in set theory, if a set of natural numbers contains 1 and, for each n, it contains $n + 1$ whenever it contains all numbers less than $n + 1$, then it must contain every natural number, e.g., complete induction proves that every natural number is a product of primes.*

By induction, we now deduce the following:

The larger the list of consecutive primes becomes, the greater would be the average number of even numbers "generated" by each of the primes in the list of consecutive primes (inferred from (A) above).

The larger the list of consecutive primes becomes, the greater would be the average number of summings of 2 primes/permutations for each of the even numbers in the infinite list of even numbers (inferred from (B) above).

Furthermore, the Goldbach conjecture had been tested and found to be correct for every even number up to 12 x 10^{17}, which is not a small list, by a distributed computer search carried out at the University of Aveiro, Portugal, in 2008.

As the primes and the even numbers are infinite, by the above lemma and all the above deductions and information, it could be inferred that the increases stated in (A) and (B) above, with the even numbers each being the sum of 2 primes, continue to infinity.

The validity of the Goldbach conjecture is thereby confirmed - every even number after 2 is the sum of 2 primes.

Proof 2:

Next, we resort to the proof by contradiction. The above deduction would be reversed if, e.g., the following takes place (which is the reversal of the above-mentioned information):

(A): (d) Average no. of even numbers "generated" by each of the consecutive primes in

the above 12 listings decreases continually all the way from the list: (1) 200

Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes

From 3 To 21,391, from **8.91** even numbers per prime number in List (1) to **6.14**

even numbers per prime number in List (12).

(B): (f) Average no. of summings of 2 primes/permutations for each of the even numbers

in the above 12 listings decreases continually all the way from the list: (1) 200

Consecutive Primes From 3 To 1,229 to the list: (12) 2,400 Consecutive Primes

From 3 To 21,391, from **269.30** number of summings of 2 primes/permutations per even number in List (1) to **32.60** number of summings of 2 primes/permutations per even number in List (12).

If this reversed state happens, the implication is that there would reach a point when there are no more batches of 2 prime numbers summing together to form even numbers, in which case the Goldbach conjecture would be false. Evidently this would happen when the prime numbers are finite. As the prime numbers are infinite (as Euclid had proved long ago) this would never happen.

Since the above information indicate otherwise, and, the prime numbers are infinite, we accept the above induction/deduction and infer that the Goldbach conjecture could not be false, i.e., the Goldbach conjecture is true, and, every even number (except 2) is indeed the sum of 2 prime numbers. This concludes the proof by contradiction.

Thus, by both induction and contradiction or *reductio ad absurdum* the validity of the Goldbach conjecture is proved.

4. Conclusion

A number of methods have been adopted in this paper in proving the Goldbach conjecture.

The inductive method, which is a well-established proof, is one of the methods utilised. The following lends support to this inductive proof of the Goldbach conjecture: (a) The characteristic of a mountain or infinite volume of sand is reflected in the characteristic of some grains of sand found there so that studying the characteristic of some grains of sand found there is enough for deducing the characteristic of the mountain or infinite volume of sand, to ascertain the quality of a batch of products it is only necessary to inspect some carefully selected samples from that batch of products and not everyone of the products and to carry out a population census, i.e., find out the characteristics of a population, it is only necessary to carry out a survey on some carefully selected respondents and not the whole population; in like manner, by the same principle, we just need to study a carefully selected list of even numbers, find out whether they are all sums of 2 primes and deduce by induction whether all even numbers after this list would also be sums of 2 primes - this act is rather like extrapolation. (For example, a distributed computer search completed in 2008 at the University of Aveiro, Portugal, had confirmed that every even number up to $12 \text{ x } 10^{17}$, which is no small list of numbers, is the sum of 2 primes. By the principle of induction in this case we could deduce that all the even numbers after $12 \text{ x } 10^{17}$ would also be sums of 2 primes.) (b) Thus, in this way every even number after 2 could be reasonably proved to be the sum of 2 primes. In fact, induction plays an important part in a number of the proofs.

The other argument used to prove the conjecture is the indirect (reductio ad absurdum) method, which had been used by Euclid and other mathematicians after him. Logically, 1 or 2 examples of “contradiction” should be sufficient proof of infinity, for it does not make sense to have a need for an infinite number of cases of “contradiction”, as our proof would then have to be infinitely and impossibly long, an absurdity. This method of proof is “proof by implication” as a result of “contradiction” - which is a “short-cut” and smart way in proving infinity, instead of “proving infinity by counting to infinity”, which is ludicrous, and, impossible. Hence, 1 or 2 cases of “contradiction” should be sufficient for implying that there would be an infinitude of even numbers which are sums of 2 primes, which of course also tacitly implies that there would be an infinitude of the number of cases of such “contradiction”. (Euclid evidently had this logical point in mind when he formulated the indirect (reductio ad absurdum) proof of the infinity of the primes.) This method of proof had been cleverly used by a number of mathematicians, not the least by the great German mathematician, David Hilbert. For example, Hilbert had used an indirect method (the “reductio ad absurdum” proof) to prove Gordan’s Theorem without having to show an actual “construction”, a proof which had been accepted by his peers.

There is also the involvement of concepts from set theory, group theory, geometry, etc..

One important query here, which many might not have considered: What if the list of prime numbers is not infinite? Of course, if that is the case, the Goldbach conjecture would be false. It would then have been absurd for the Goldbach conjecture to have been conceived at all. However, the list of primes is infinite (vide Euclid's proof). This gives credence to the Goldbach conjecture.

A very important related point, in fact a most important point, must be highlighted here. If the Goldbach conjecture were indeed false, there must be an ultimate (largest) even number which is (and must necessarily be) the result of the summation of 2 primes that are each the largest existing prime. It must be noted that this is actually an impossibility, as there can never be a largest existing prime - by Euclid's proof, the primes are infinite (refer to Part 1, Proof 3 above). Hence, the Goldbach conjecture cannot be false, and, by both *reduction ad absurdum* (contradiction), and, induction (wherein all even numbers up to 12 x 10^{17}, not a small list, had been confirmed to be sums of 2 primes), has to be true.

So far, there has been no indication or confirmation at all that the number of even numbers after the number 2 which are each the sum of 2 primes is finite and the largest existing even number which is the sum of 2 primes has not been found and confirmed. (This would of course be the case if the Goldbach conjecture is true.) On the other hand, practically everyone could intuit that the list of even numbers after the number 2 which are each the sum of 2 primes is infinite. Besides, the evidence, as shown in this paper, is overwhelmingly in support of the infinity of this list.

We have no other more logical choice but to take the stand that every even number after the number 2 is the sum of 2 prime numbers.

In conclusion, we state that the Goldbach conjecture is true - every even number after the number 2 is indeed the sum of 2 primes.

It is evident here that the Goldbach conjecture could be approached in a number of different ways. All this could be food for further thought about the conjecture.

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