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THE MAXIMAL RANK CONJECTURE FOR LINEARLY NORMAL CURVES $C \subset \mathbb{P}^r$ with $h^1(C, \mathcal{O}_C(1)) = 1$

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Abstract: Let $C \subset \mathbb{P}^r$ be a general linearly normal curve with prescribed genus and $h^1(C, \mathcal{O}_C(1)) = 1$. Here we prove that C has maximal rank, i.e. that for all integers t the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(C, \mathcal{O}_C(t))$ is either injective or surjective.

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1. Introduction

Let $C \subset \mathbb{P}^r$ be any projective curve. The curve C is said to have *maximal rank* if for every integer $x > 0$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective.

For any curve X and any spanned $L \in \mathrm{Pic}(X)$ let $h_L : X \to \mathbb{P}^r$, $r := h^0(X, L) - 1$, denote the morphism induced by the complete linear system $|L|$. Here we prove the following result, which improves one of the results in [9].

Theorem 1. *Fix integers $g > r \geq 3$ and set $d := g + r - 1$. Fix a general $X \in \mathcal{M}_g$ and a general $L \in W^r_d(X)$. Then L is very ample, $h^0(X, L) = r + 1$ and the curve $h_L(X) \subset \mathbb{P}^r$ has maximal rank.*

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Take g, r, d, X as in Theorem 1. We have $\rho(g,r,d) := (r+1)g - r(g+r-1) + r(r+1) > 0$. Hence Brill-Noether theory gives $W^r_d(X) \neq \emptyset$, that $W^r_d(X)$ is irreducible of dimension $\rho(g,r,d)$ and that $W^r_d(X) \neq W^{r+1}_d(X)$, i.e. $h^0(X,L) = r+1$ for a general $L \in W^r_d(X)$. ([1], Ch. V). Hence $h^1(X,L) = 1$ for a general $L \in W^r_d(X)$. For this range of triples (g,r,d) it is very easy to prove that a general $L \in W^r_d(X)$ is very ample (e.g., see the proof of [8], Theorem at pages 26-27). Hence to prove Theorem 1 it is sufficient to prove that $h_L(X)$ has maximal rank.

2. Proof of Theorem 1

We fix a hyperplane H of $\mathbb{P}^r$. For any closed subscheme $A \subset \mathbb{P}^x$ and any integer $t \geq 0$ let $r_{A,t,x} : H^0(\mathbb{P}^x, \mathcal{O}_{\mathbb{P}^x}(t)) \to H^0(A, \mathcal{O}_A(t))$ denote the restriction map. If $x = r$ we often write $r_{A,t}$ instead of $r_{A,t,r}$. If $x = r-1$ and $\mathbb{P}^{r-1}$ is the hyperplane H of $\mathbb{P}^{r-1}$, then we often write $r_{A,t,H}$ instead of $r_{A,t,r-1}$.

The following remark is often called Castelnuovo's lemma or Horace lemma.

Remark 1. Fix a closed subscheme $W \subset \mathbb{P}^r$. Let $\mathrm{Res}_H(W)$ be the residual scheme of W with respect to H, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_H$ as its ideal sheaf. If W is reduced, then $\mathrm{Res}_H(W)$ is the union of the irreducible components of W not contained in H. For any $t \in \mathbb{Z}$ we have the following exact sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_H(W)}(t-1) \to \mathcal{I}_W(t) \to \mathcal{I}_{W\cap H,H}(t) \to 0 \qquad (1)$$

From (1) we get

$$h^i(\mathcal{I}_W(t)) \leq h^i(\mathcal{I}_{\mathrm{Res}_H(W)}(t-1)) + h^i(H, \mathcal{I}_{W\cap H,H}(t))$$

for all $i \geq 0$ and all $t \in \mathbb{Z}$.

For all integers $r \geq 3$, $g \geq 0$ and $d \geq g + r$ let $Z(d,g,r)$ denote the closure in the Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^r)$ of $\mathbb{P}^r$ of the set of all smooth, connected and non-degenerate curves $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C) = d$ and $h^1(C, \mathcal{O}_C(1)) = 0$. For all integers $g > r \geq 3$ let $Z'(g,r)$ denote the closure in $\mathrm{Hilb}(\mathbb{P}^r)$ of the set of all smooth, connected and non-degenerate curves $C \subset \mathbb{P}^r$ such that $p_a(C) = g$, $\deg(C) = g + r - 1$ and $h^1(C, \mathcal{O}_C(1)) = 1$ (since C is assumed to be non-degenerate the latter condition is equivalent to the linear normality of C). Obviously $Z(d,g,r)$ and $Z'(g,r)$ are irreducible. If $r \geq 4$ we write $Z(d,g,H)$ instead of $Z(d,g,r-1)$ to stress that any $U \in Z(d,g,H)$ is contained in H. If $1 \leq d \leq r-2$ let $Z(d,0,H)$ denote the closure in $\mathrm{Hilb}(H)$ of the set of all degree d smooth rational curves $T \subset H$ spanning a linear subspace of dimension d.

Fix integers $r \geq 3$ and $k \geq 2$. Define the integers $a_{r,k}$ and $b_{r,k}$ by the relations

$$k(a_{r,k} + r - 1) + 1 - a_{r,k} + b_{r,k} = \binom{r+k}{r}, \ 0 \leq b_{r,k} \leq k-2 \qquad (2)$$

Set $a_{r,1} := r+1$ and $b_{r,1} = 0$. As in [5], Definition 2.2, we define the integers $g(r,k)$ and $f(r,k)$, $k \geq 2$ by the relations

$$k(g(k,r) + r) + 1 - g(k,r) + f(k,r) = \binom{r+k}{r}, \ 0 \leq f(k,r) \leq k-2 \qquad (3)$$

Set $g(1,r) := 0$ and $f(1,r) := 0$. For any integer $r \geq 3$ and $k \geq 2$ we have if $b_{r,k} = g(k,r)+1$ and $b_{r,k} = f(k,r)+1$ if $b_{r,k} \neq 0$ (or, equivalently, if $f(k,r) \neq k-2$) and $a_{r,k} = g(k,r)+2$ and $b_{r,k} = 0$ if $b_{r,k} = 0$ (or, equivalently, if $f(k,r) = k-2$). These relations allows us to apply the numerical lemmas in [5], §4, loosing at most 2. Using (2) instead of (3) we may extends the proofs in [5], §4, without loosing nothing. Fix an integer $g > r$. The critical value k of the pair (g,r) is the minimal integer k such that $g \leq a_{r,k}$. Fix any smooth and linearly normal $C \in Z'(g,r)$ and let k be the critical value of the pair (g,r). Fix any integer $t \geq 2$. Since $h^1(C, \mathcal{O}_C(2)) = 0$, Riemann-Roch shows that $h^0(C, \mathcal{O}_C(t)) \geq \binom{r+t}{t}$ if and only if $t \geq k$. The curve C has maximal rank if and only if $h^1(\mathcal{I}_C(k)) = 0$ and $h^0(C, \mathcal{I}_C(k-1)) = 0$ (the " only if " part is obvious, the " if " part follows from Castelnuovo-Mumford's lemma if $k \geq 3$, while if $k \leq 2$ one need more, e.g. that a canonically embedded curve is projectively normal and then to add $g-(2r-2)$ general secant lines). Anyway, the case $k=2$ is obvious by [7].

We recall the following Assertion $H(k)$, $k \geq 1$, proved in [5], Lemma 3.1:

$H(k)$, $k \geq 1$: The map $r_{A,k}$ is bijective for a general $A \in Z(g(k,r) + r, g(k,r) - f(k,r))$.

Of course, before proving $H(k)$ we proved that $Z(g(k,r)+r, g(k,r)-f(k,r))$ is well-defined, i.e. $g(k,r) \geq f(k,r)$.

Proof of Theorem 1. Since the case $r=3$ is true by [4], Theorem 1, we may assume $r \geq 4$. To apply the numerical lemmas in [5] we also assume $r \geq 5$ (anyway, the case $r=4$ is known by [2]). Fix an integer $g > r$ and a general $C \in Z'(g,r)$. Let k be the critical value of the pair (g,r). If $k=1$, then C is a canonically embedded smooth curve; hence it is projectively normal; hence it has maximal rank. If $k = 2$, then C is projectively normal ([7]). From now on we assume $k \geq 3$. The assumption that k is the critical value of the pair (g, r) is equivalent to the following two inequalities:

$$k(g + r - 1) + 1 - g \leq \binom{r + k}{r} \tag{4}$$

$$(k - 1)(g + r - 1) + 1 - g > \binom{r + k - 1}{r} \tag{5}$$

From (4) and (5) we get

$$g + r - 2 \leq \binom{r + k - 1}{r - 1} \tag{6}$$

Since $Z'(g, r)$ is irreducible, it is sufficient to find $A, B \in Z'(g, r)$ such that $h^1(\mathcal{I}_A(k)) = 0$ and $h^1(\mathcal{I}_B(k - 1)) = 0$. In steps (a) and (b) we prove the existence of A, while in steps (c) and (d) we prove the existence of B. Take a general $E \in Z(g(k - 1, r) + r, g(k - 1, r) - f(k - 1, r))$. Since $H(k - 1)$ is true ([5], Lemma 3.1) the linear map $r_{E,k-1}$ is bijective. For a general E we may assume that E is transversal to H and that any subset of $E \cap H$ is in linearly general position in H, i.e. H is spanned by any $S' \subset E \cap H$ such that $\sharp(S') = r$. To prove the existence of A and of B we may use induction on the critical value k.

(a) Here we prove the existence of the curve A under the assumption $g \geq g(k-1, r) + r$. Take a general $A_2 \in Z(g - g(k-1, r) + r - 1, g - g(k-1, r), H)$. By [3] (case $r - 1 = 3$) or [5] (case $r - 1 \geq 4$), A_2 has maximal rank. Taking the difference between the equation in (3) and the same equation for the integer $k' := k - 1$ we get the following equation ([5], eq. (4)):

$$(k-2)(g(k, r) - g(k-1, r)) = \binom{r - 1 + k}{r - 1} - g(k, r) + f(k-1, r) - f(k, r) \tag{7}$$

Since $a_{r,k-1} < g \leq a_{r,k}$, $a_{r,k} \in \{g(k, r), g(k, r) + 1\}$, $a_{r,k-1} \in \{g(k - 1, r), g(k - 1, r) + 1\}$, $f(k, r) \leq k - 2$ and $a_{r,k-1} \geq 2r + k$, we have $k(g - g(k - 1, r) + r - 1) + 1 - (g - g(k - 1, r)) \leq \binom{r+k-1}{r-1}$. Since A_2 has maximal rank, the map $r_{A_2,k,H}$ is surjective. Fix $S \subset H$ such that $\sharp(S) = r + 2$ and S is in linearly general position in H. Since any two such sets S are projectively equivalent, we may assume $S \subset A_2$ and $S \subset E \cap H$. For a general A_2 we may also assume $S = (E \cap H) \cap A_2$. Hence $S = E \cap A_2$ and $E \cup A_2$ is a nodal curve of degree $g + r - 1$ with arithmetic genus g. By [6], Lemma 2.7, we have $E \cup A_2 \in Z'(g,r)$. Hence by semicontinuity to prove the existence of A it is sufficient to prove $h^1(\mathcal{I}_{A_2\cup E}(k)) = 0$. Since $r_{A_2,k,H}$ is surjective we have $h^0(H, \mathcal{I}_{A_2}(k)) = \binom{r+k-1}{r-1} - k(g-g(k-1,r)+r-1)+(g-g(k-1,r))-1$. Taking the difference between (4) and the case $k' = k-1$ we get $\sharp(E \cap H) - \sharp(S) \leq h^0(H, \mathcal{I}_{A_2}(k)) = 0$. Since $h^1(\mathcal{I}_E(k-1)) = 0$, Remark 1 shows that to prove $h^1(\mathcal{I}_{A_2\cup E}(k)) = 0$ it is sufficient to prove $h^1(H, \mathcal{I}_{A_2\cup(E\cap H)}(k)) = 0$, i.e. it is sufficient to prove $h^0(H, \mathcal{I}_{(E\cap H)\cup A_2}(k)) = h^0(H, \mathcal{I}_{A_2}(k)) - \sharp(A_2 \cap H) - \sharp(A_2 \cap E)$. Apply the proof of part (b) of [5], Lemma 1.6.

(b) Here we prove the existence of the curve A under the assumption $g \leq g(k-1,r)+r-1$. By (3) for $k' := k-1$ we get $k(g+r-1)+1-g \leq \binom{r+k}{r} - k+1$. Let E' be a general element of $Z'(a_{k-1},r)$. Since the pair $(a_{k-1,r}, r)$ has critical value $k-1$, the inductive assumption gives $h^1(\mathcal{I}_{E'}(k)) = 0$. We may also assume that E' is transversal to H and that each subsets of $E' \cap H$ are in linearly general position in H. Take $S' \subset E' \cap H$ such that $\sharp(S') = \min\{r, g-a_{k-1,r}+1\}$. Set $\epsilon := \max\{g-g(k-1,r)-r+1, 0\}$. Take a general $A_3 \in Z(g-a_{k-1,r}, \epsilon, r)$ containing S' (it exists, because either $\epsilon = 0$ and $\sharp(S') = g-a_{k-1,r}-1$ or $\epsilon = r$ and $g - a_{k-1,r} \geq r-1$) . Since any two subsets of H with cardinality $\sharp(S')$ and in linearly general position are projectively equivalent, we may assume $S' = E' \cap A_3$. Repeat the proof of step (a) taking E', A_3 and S' instead of E, A_2 and S, respectively.

(c) Here we prove the existence of the curve B under the assumption $g \geq g(k-1,r)+r+1$. There is $A_1 \in Z(g-1-g(k-1,r), g-1-g(k-1,r)-r)$ such that $\sharp(A_1 \cap E) = r+2$ and $E \cup A_1$ is nodal. By [6], Lemma 2.7, we have $E \cup A_1 \in Z'(g,r)$. Hence by the semicontinuity theorem for cohomology to prove the existence of B it is sufficient to prove the injectivity of $r_{E\cup A_1,k-1}$. Since $r_{E,k-1}$ is injective, $r_{E\cup A_1,k-1}$ is injective.

(d) Here we prove the existence of the curve B under the assumption $g \leq g(k-1,r)+r$. Fix a general $F \in Z(g(k-2,r)+r, g(k-2,r)-f(k-2,r))$. Since $H(k-2)$ is true ([5], Lemma 3.1) the linear map $r_{F,k-2}$ is bijective (here we use that $k \geq 3$). For a general F we may assume that F is transversal to H and that any subset of $F \cap H$ is in linearly general position in H. Fix $S_1 \subset F \cap H$ such that $\sharp(S_1) = r+1$ (it exists, because $k \geq 3$ and $\deg(F) = a_{r,k-2}+r-1 \geq a_{r,1}-r-1 = 2r$). Since $g > a_{r,k-1}$ and $g(k-1,r)-g(k-2,r)$, we have $g-a_{r,k-2}-1 \geq r-1$. Take a general $A_3 \in Z(g-a_{r,k-2}-1, g-a_{r,k-2}-r, H)$ containing S_1 (it exists, because $g-a_{r,k-2}-1 \geq r-1$). We may also assume $S_1 = F \cap A_3$. Hence $F \cup A_3$ is a nodal curve of degree $g+r-1$ and arithmetic genus g spanning $\mathbb{P}^r$, By [6], Lemma 2.7, we have $F \cup A_3 \in Z'(g,r)$. By semicontinuity to prove the existence of the curve B it is sufficient to prove $h^0(\mathcal{I}_{A_3\cup F}(k-1)) = 0$.

Since A_3 is a general non-special curve of its degree and genus, it has maximal rank ([3],[5]). Hence either $h^1(H, \mathcal{I}_{A_3}(k-1)) = 0$ or $h^0(H, \mathcal{I}_{A_3}(k-1)) = 0$. First assume $h^0(H, \mathcal{I}_{A_3}(k-1)) = 0$. Since $h^0(\mathcal{I}_F(k-1)) = 0$, Remark 1 implies $h^0(\mathcal{I}_{A_3 \cup F}(k-1)) = 0$. Now assume $h^1(H, \mathcal{I}_{A_3}(k-1)) = 0$, i.e. $h^0(H, \mathcal{I}_{A_3}(k-1)) = \binom{k+r-2}{r-1} - (k-1)(g - a_{r,k-2} - 1) - 1 + (g - a_{r,k-2} - r)$. Taking the difference of the equation in (3) for the integer $k' := k-1$ with the same equation for the integer $k' := k-2$ we get

$$a_{r,k-2} + (k-1)(a_{r,k-1} - a_{r,k-2}) + b_{r,k-1} - b_{r,k-1} = \binom{r+k-2}{r-1} \qquad (8)$$

Since $g > a_{r,k-1}$ and $b_{r,k-1} \leq k-3$, we get $\sharp(F \cap H) - \sharp(F \cap A_3) \geq h^0(H, \mathcal{I}_{A_3}(k-1))$. Hence we may apply [5], Lemma 1.4, and get $h^0(H, \mathcal{I}_{A_3 \cup ((F \cap H)}(k-1)) = 0$. Apply Remark 1. $\square$

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