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REMARKS ON THE MAXIMAL RANK CONJECTURE

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Abstract: Here we point out a key lemma in an old paper which should be improved to get a weak form of the Maximal Rank Conjecture.

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1. Introduction

Let $C \subset \mathbb{P}^r$ be any projective curve. The curve C is said to have *maximal rank* if for every integer $x > 0$ the restriction map $\rho_{C,x} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(x)) \to H^0(C, \mathcal{O}_C(x))$ has maximal rank, i.e. either it is injective or it is surjective. For all integers g, r, d set $\rho(g, r, d) := g-(r+1)(g+r-d) = (r+1)d-rg-r(r+1)$ (the Brill-Noether number for g^r_d's on a curve with genus g). Fix integers r, d, g such that $r \geq 3$, $g \geq 0$ and either $d \geq g+r$ or $d-r < g \leq d-r+\lfloor (d-r-2)/(r-2) \rfloor$. There is an irreducible component $W(d, g, r)$ of the Hilbert scheme of $\mathbb{P}^r$ which is generically smooth and of dimension $(r+1)d-(r-3)(g-1)$ such that a general $C \in W(d, r; g)$ has the following properties (see [5] for the case $r = 3$, [7] for the case $r \geq 4$):

(a) C is a smooth and connected non-degenerate curve with degree d, genus g and $h^1(C, N_C) = 0$;

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(b) if $d \geq g + r$, then $h^1(C, \mathcal{O}_C(1)) = 0$;

(c) if $d < g + r$, then C is linearly normal and $h^1(C, \mathcal{O}_C(2)) = 0$;

(d) if $\rho(g, r, d) \geq 0$, then C has general moduli;

(e) if $\rho(g, r, d) < 0$, then the general fiber of the natural rational map $W(d, g, r) \dashrightarrow \mathcal{M}_g$ has dimension $\dim(\mathrm{Aut}(\mathbb{P}^r)) = r^2 + 2r$, i.e. $W(d, g, r)$ has the right number of moduli in the sense of [14].

The Maximal Rank Conjecture for the triple (d, g, r) asks if a general $C \in \mathcal{H}_{d,g,r}$ has maximal rank (see [1] for a stronger form of it). We say that the Maximal Rank Conjecture holds in $\mathbb{P}^r$ is it holds for all triples (d, g, r) such that $\rho(d, g, r) \geq 0$. The Maximal Rank Conjecture is true in $\mathbb{P}^r$ if $r = 3, 4, 5$ (see [5], [2], [3]). It is also known for all triples (d, g, r) with $d \geq g + r$, i.e. in the non-special range (see [4],[6]). It is true if $\rho(d, g, r) \geq 0$ and $2d + 1 - g \leq \binom{r+2}{2}$ (see [8]). For a fixed r there are only finitely many pairs (d, g) for which it is not proved that a general element of $W(d, g, r)$ (see [7], Theorem in the Introduction and Proposition 3.1). See Proposition 2 for a less vague statement. The main aim of this paper is to tell the reader that the tools contained in [7] are very strong even to have non-asymptotic results, but that the key to get better results is to improve [7], Lemma 5.2.

Fix integers r, d, g such that $W(d, g, r)$ is defined. If $(r, d, g) = (r, r, 0)$, then we say that the triple (d, g, r) has *critical value* 1. If $(r, d, g) \neq (r, r, 0)$, then the it critical value of (r, d, g) is the minimal integer k such that $kd + 1 - g \leq \binom{r+k}{r}$. A curve $C \in W(d, g, r)$ has maximal rank if and only if $h^1(\mathbb{P}^r, \mathcal{I}_C(k)) = 0$ (i.e. the restriction map $\rho_{C,k} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k)) \to H^0(C, \mathcal{O}_C(k))$ is surjective) and $h^0(\mathbb{P}^r, \mathcal{I}_C(k-1)) = 0$ (i.e. the restriction map $\rho_{C,k-1} : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(k-1)) \to H^0(C, \mathcal{O}_C(k-1))$ is injective), where k is the critical value of the triple (d, g, r). We prove the following results.

Proposition 1. *Fix an integer $r \geq 4$. For every integer $k \geq 2$ let $\Psi(k)$ be the integer defined in Definition 1. Fix integers d, g such that $W(d, g, r)$ is defined, (d, g, r) has critical value k and $d \geq g + r - \Psi(k)$. Then a general element of $W(d, g, r)$ has maximal rank.*

Proposition 2. *For any integer $r \geq 4$ let Δ_r be the integer introduced in Section 3. Assume that the Maximal Rank Conjecture is true for all (d, g) such that (d, g, r) has critical value at most Δ_r. Then the Maximal Rank Conjecture is true in $\mathbb{P}^r$.*

As in [7] we work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. Preliminaries

We fix a hyperplane H of $\mathbb{P}^r$. We often write $W(d,g,H)$ instead of $W(d,g,r-1)$ when we are working with curves lying in H.

Remark 1. Take a triple (d,g,r) such that $r \geq 3$ and $W(d,g,r)$ is defined. Let C be a general element of $W(d,g,r)$. We have $h^1(C,\mathcal{O}_C(2)) = 0$ (use the definition of $W(d,g,r)$ given in [7], §1.1, and several Mayer-Vietoris exact sequences). Hence $h^0(C,\mathcal{O}_C(t)) = td+1-g$ for all $t \geq 2$. Hence (assuming $g \neq \emptyset$), C has maximal rank if and only if $h^0(\mathcal{I}_C(k-1)) = 0$ and $h^1(\mathcal{I}_C(t)) = 0$ for all $t \geq k$. If $k \geq 3$ Castelnuovo-Mumford's lemma implies that if $h^1(\mathcal{I}_C(k)) = 0$, then $h^1(\mathcal{I}_C(t)) = 0$ for all $t > k$. If $d \geq g+r$, then C is non-special and hence it has maximal rank (see [6]). If $d < g+r$, then C is linearly normal. If $\rho(d,g,r) \geq 0$ and C has critical value 2, then C is projectively normal (see [8], Theorem 1).

The following remark is often called Castelnuovo's lemma or Horace lemma.

Remark 2. Fix a closed subscheme $W \subset \mathbb{P}^r$. Let $\mathrm{Res}_H(W)$ be the residual scheme of W with respect to H, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_H$ as its ideal sheaf. If W is reduced, then $\mathrm{Res}_H(W)$ is the union of the irreducible components of W not contained in H. For any $t \in \mathbb{Z}$ we have the following exact sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_H(W)}(t-1)\mathcal{I}_W(t) \to \mathcal{I}_{W\cap H,H}(t) \to 0 \tag{1}$$

From (1) we get

$$h^i(\mathcal{I}_W(t)) \leq h^i(\mathcal{I}_{\mathrm{Res}_H(W)}(t-1)) + h^i(H,\mathcal{I}_{W\cap H,H}(t))$$

for all $i \geq 0$ and all $t \in \mathbb{Z}$.

See [7], Lemma 2.5, for a lemma related to the following one.

Lemma 1. *Fix non-negative integers a,b,k such that $0 \leq b \leq r$ and write $g := a(r+1)+b$ and $d := r+ra+b+k$. Assume $k \geq 0$ and take a general $S \subset H$ such that $\sharp(S) \leq r+a+k$. Then there is $A \in W(d,g,r)$ such that A is smooth, A intersects transversally H, $S \subset A$ and $h^1(A,N_A) = 0$.*

Proof. Fix a general $S_0 \subset H$ such that $\sharp(S_0) = r$, general $P_i \in H$, $1 \leq i \leq a$, and a general $S' \subset H$ such that $\sharp(S') = k$. Let $D \subset \mathbb{P}^r$ be a rational normal curve such that $D \cap H = S_0$. Fix general $E_i \subset D$, $1 \leq i \leq a$, such that $\sharp(E_i) = r+2$. There is a unique rational normal curve $A_i \subset \mathbb{P}^r$ such that $E_i \cup \{P_i\} \subset A_i$. For general P_i and E_i we may find $E_1,\dots,E_a$ such that

$E_i \cap E_j = \emptyset$, $A_i \cap D = E_i$ for all i and A_i intersects quasi-transversally D. Hence $U := D \cup (\bigcup_{i=1}^{a} A_i)$ is a nodal curve, $h^1(U, N_u) = 0$ and $U \in W(r+ra, (r+1)a, r)$ (see [7], Corollary 2.4). For each $P \in S'$ let L_P be the general line through P intersecting D. Take as A a smoothing of the union of U, b general secant lines of D and the k lines L_P, $P \in S'$. □

Lemma 2. *Fix integers $g \geq 0$ and $d \geq g + r - 1$. Let β be the maximal integer such that $(\beta - 1 - r - g)\lfloor (r-1)/2 \rfloor \leq g$. If $d < g + 2(r-1)$ set $\alpha := \beta + d - g - r + 1$. If $d \geq g + 2(r-1)$, then set $\alpha := \beta + \lfloor r(d-g-r+1)/(r-1) \rfloor - 1$. Fix a general $S \subset H$ such that $\sharp(S) \leq \alpha$. There there is a smooth $B \in W(d, g, H)$ such that $S \subset B$.*

Proof. If $d = g+r-1$, then use [6], Lemma 1.5, for $n := r-1$. Now assume $d > g+r-1$. Take $S = S_1 \sqcup S_2$ with $\sharp(S_1) = \beta$. Take $A \in W(g+r-1, g, H)$ such that $S_1 \subset H$ (see [6], Lemma 1.5). Fix a general $P \in A$. If $d \leq g + 2(r-1)$, then $\sharp(S_2) = d - g - r + 1$. Hence we may find $T \in W(d-g-r+1, 0, r)$ such that $T \supset \{P\} \cup S_2$. For general P and S_2 we may also find T with the additional condition that $\{P\} = T \cap A$ and that $A \cup T$ has a node at P. Hence $A \cup T \in W(d, g, H)$. By construction we have $S \subset A \cup T$.

Now assume $d \geq g + 2(r-1)$. The normal bundle of a general smooth rational curve $M \subset H$ such that $\deg(M) = d - g - r + 1$ is balanced, i.e. each of its rank 1 factors has$r+2$ (see [13], [12]). Hence there is one such curve T containing a general subset of H with cardinality $\lfloor r(d-g-r+1)/(r-1) \rfloor$ (see [11], Theorem 1.5). Hence we may find T containing $\{P\} \cup S_2$.

Then we deform $A \cup T$ to a smooth B and follow S in the deformation (we do not claim that we may deform $A \cup T$ to a smooth B among a family of curves containing S). □

3. Propositions 1 and 2

For each integer $k \geq 2$ let $g_{r,k}$ be the maximal non-negative integer such that $k(\lceil r g_{r,k}/(r+1) \rceil) + 1 - g_k \leq \binom{r+k}{r}$. Let $d_{r,k}$ be the maximal integer such that $k d_{r,k} + 1 - g_{r,k} \leq \binom{r+k}{r}$. Since $k(\lceil r g_{r,k}/(r+1) \rceil) + 1 - g_{r,k} \leq \binom{r+k}{r}$, we have $d_{r,k} \geq \lceil r g_{r,k}/(r+1) \rceil$. The maximality of the integer $g_{r,k}$ easily gives

$$\binom{r+k}{r} - k + 1 \leq k d_{r,k} + 1 - g_{r,k} \leq \binom{r+k}{r} \tag{2}$$

$$d_{r,k} \in \{\lceil r g_{r,k}/(r+1) \rceil, \lceil r g_{r,k}/(r+1) \rceil + 1\} \tag{3}$$

Set $g_{1,r} := 0$ and $d_{1,r} := r$. Set $e_{r,k} := \binom{r+k}{k} - (kd_{r,k} + 1 - g_{r,k})$. We have $0 \le e_{r,k} \le k-1$ and

$$kd_{r,k} + 1 - g_{r,k} + e_{r,k} = \binom{r+k}{r} \tag{4}$$

From (4) for the integers k and $k-1$ we get

$$d_{r,k-1} + k(d_{r,k} - d_{r,k-1}) + -g_{r,k-1} + e_{r,k-1} + g_{r,k} - e_{r,k} = \binom{r+k-1}{r-1} \tag{5}$$

Let Δ_r be the first integer t such that for all $k \ge t-2$ the following inequalities are satisfied:

$$d_{r,k-1} \ge \binom{r+k-2}{r-2} + \lfloor (d_{r,k,} - d_{r,k-1})/r \rfloor + 2r + 2k \tag{6}$$

$$d_{r,k} - d_{r,k-1} \ge (d_{r,k} - d_{r,k-1})/r + 2k + r \tag{7}$$

Of course, (7) is usually much weaker than (6). The key problem is to improve [7], Lemma 5.2, to be able to avoid the term $\binom{r+k-2}{r-2}$ in the right hand side of (6).

It is easy to check that $g_{r,k} \ge k-1$ for all k. Hence $g_{r,k} - e_{r,k} \ge 0$ and $W(d_{r,k}, g_{r,k} - e_{r,k}, r)$ is defined.

Consider the following assertion $R(k)$, $k \ge 1$:

$R(k)$, $k \ge 1$: A general $C \in W(d_{r,k}, g_{r,k} - e_{r,k}, r)$ has maximal rank.

Fix any $C \in W(d_{r,k}, g_{r,k} - e_{r,k}, r)$. By (4) (d, g, r) has critical value k and C has maximal rank $\iff h^1(\mathcal{I}_C(k)) = 0 \iff h^0(\mathcal{I}_C(k)) = 0$.

Lemma 3. *Fix an integer $k \ge \Delta_r - 1$. If $R(k-1)$ is true, then $R(k)$ is true.*

Proof. Fix a general $A \in W(d_{r,k-1}, g_{r,k-1} - e_{r,k-1}, r)$. Since we assumed that $R(k-1)$ is true, we have $h^i(\mathcal{I}_A(k-1)) = 0$, $i = 0, 1$. It is easy to check that $g_{r,k} - e_{r,k} \ge g_{r,k-1} - e_{r,k-1}$ and that $d_{r,k} - d_{r,k-1} \ge k$, but we do not need these inequalities, because we assumed (7). Fix a general $S \subset H$ such that $\sharp(S) = 1 + g_{r,k} - e_{r,k} - g_{r,k-1} + e_{r,k-1} - (d_{r,k} - d_{r,k-1} + 1 - r)$. The inequality (6) and Lemma 1 show that we may find A containing S. Fix a general $B \in W(d_{r,k} - d_{r,k-1}, d_{r,k} - d_{r,k-1} + 1 - r, H)$. By Lemma 2 and (7) we may find B containing S. Set $X := A \cup B$. We have $X \in W(d_{r,k}, g_{r,k} - e_k, r)$ (see [7], proof of Lemma 2.5). Hence by semicontinuity it is sufficient to prove $h^i(\mathcal{I}_X(k)) = 0$,

$i = 0, 1$. We have $\mathrm{Res}_H(X) = A$. Since $h^i(\mathcal{I}_A(k-1)) = 0$, it is sufficient to prove $h^i(H, \mathcal{I}_{X\cap H}(k)) = 0$ (Remark 2). We have $X \cap H = B \sqcup ((A\cap H) \setminus S)$. Since B may be considered as a general non-special curve of H with prescribed degree and prescribed genus, it has maximal rank (see [6]). Hence the definitions of $e_{r,k}$ and $e_{r,k-1}$ give $h^1(H, \mathcal{I}_B(k)) = 0$ and $h^0(H, \mathcal{I}_B(k)) = \sharp((A\cap H) \setminus S)$. Write $g_{r,k-1} = a(r-1) + b$ with $0 \le b \le r-2$. We may degenerate A to a nodal curve $A_1 \cup A_2$ with $A_1 \in W(r + a(r-2), a(r-1), r)$, A_1 transversal to H, $A_1 \cap B = \emptyset$ and A_2 a smooth rational curve intersecting A_1 at exactly $b+1$ general points of A_1. By (6) we have $\deg(A_2) - \sharp(A_2 \cap B) \ge \binom{r+k-2}{r-2}$. Hence $h^0(B, \mathcal{O}_B(k)) + \deg(A_1 \cap H) \le \binom{r+k-2}{r-1}$. By [7], Lemma 5.2, for general A_1 we have $h^1(H, \mathcal{I}_{B\cup(A_1\cap H)}(k)) = 0$, i.e. $h^0(H, \mathcal{I}_{B\cup(A_1\cap H)}(k)) = \deg(A_2) - \sharp(S)$. For general A_2 we may assume that $A_2 \cap H$ is formed by $\deg(A_2)$ general points of H, exactly $1 + g_{r,k} - e_{r,k} - g_{r,k-1} + e_{r,k-1} - (d_{r,k} - d_{r,k-1} + 1 - r)$, being in B. Hence $h^i(H, \mathcal{I}_{((A_1\cup A_2)\cap H)\cup B}(k)) = 0$, $i = 0, 1$. By semicontinuity we get $R(k)$. □

Lemma 4. *Fix an integer $k \ge \Delta_r$. Assume that the Maximal Rank Conjecture is true in $\mathbb{P}^r$ for all (d, g) such that (d, g, r) has critical value $\le \Delta_r$. Then $R(k)$ is true for all $k \ge 1$.*

Proof. If $k \le \Delta_r$, then $R(k)$ is true, because $g_{r,k} - e_{r,k} \ge 0$ and $\rho(d_{r,k}, g_{r,k} - e_{r,k}) \ge 0$. Hence Lemma 3 proves $R(k)$ for all $k \ge \Delta_r$. □

Proof of Proposition 2. Fix an integer $k \ge \Delta_r$ and (d, g) such that $\rho(d, g, r) \ge 0$ and (d, g, r) has critical value k. Since $W(d, g, r)$ is irreducible, the semicontinuity theorem for cohomology shows that it is sufficient to find $X_1, X_2 \in W(d, g, r)$ such that $h^1(\mathcal{I}_{X_1}(k)) = 0$ and $h^0(\mathcal{I}_{X_2}(k-1)) = 0$. We only prove the existence of X_1, since the proof of the existence of X_2 is similar. Take a general $Y \in W(d_{r,k-1}, g_{r,k-1} - e_{k-1}, r)$. We have $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$.

(a) Here we assume $d \ge d_{r,k-1} + r - 1$ and $g \ge g_{r,k-1} - e_{r,k-1} + d - d_{r,k-1} - r + 1$. Fix a general $S \subset H$ such that $\sharp(S) = 1 + g - (g_{r,k-1} - e_{r,k-1} + d - d_{r,k-1} - r + 1)$. By (6) and Lemma 1 we may assume $S \subset Y \cap H$. Fix a general $B \in W(d - d_{r,k-1}, d - d_{r,k-1}, H)$. Since B has maximal rank, (5) gives $h^1(H, \mathcal{I}_B(k)) = 0$. By (7) and Lemma 2 we may assume $S \subset B$. For general B we may also assume $B \cap Y = S$. Hence $Y \cup B \in W(d, g, r)$ (see [7], Lemma 2.7). To pass from $h^1(H, \mathcal{I}_B(k)) = 0$ to $h^1(H, \mathcal{I}_{B\cup(Y\cap H)}(k)) = 0$ we need degenerate Y to $A_1 \cup A_2$ as in the proof of Lemma 3.

(b) Here we assume $d \ge d_{r,k-1} + r - 1$ and $g_{r,k-1} - e_{r,k-1} \le g < g_{r,k-1} - e_{r,k-1} + d - d_{r,k-1} - r + 1$. We copy the proof of step (a) with the only difference that we take B with genus $g - g_{r,k-1} + e_{r,k-1}$ and $\sharp(S) = 1$.

(c) Here we assume $g < g_{r,k-1} - e_{r,k-1}$. Since the Maximal Rank Conjecture is true for general non-special embeddings (see [6]), we may assume $d < g + r$. Let $m \ge 1$ be the maximal integer such that $g \ge g_{m,r} - e_{m,r}$ (m exists, because $g_{1,r} = e_{1,r} = 0$.

(c1) First assume $m \ge \Delta_r$. For all integers t such that $m < t \le k-1$ define the integers $u_{g,t}$ and $v_{g,t}$ by the relations:

$$tu_{g,t} + 1 - g + v_{g,t} = \binom{r+t}{r}, 0 \le v_{g,t} \le t-1 \tag{8}$$

For all integers t such that $m + 1 \le t \le k - 1$ let H'_t denote the following assertion:

H'_t, $m + 1 \le t \le k - 1$: There is $Y_t = E_t \sqcup F_t$ with $E_t \in W(u_{g,t} - v_{g,t}, g, r)$, F_t a disjoint union of $v_{g,t}$ lines and $h^i(\mathcal{I}_{Y_t}(t)) = 0$, $i = 0, 1$.

(c1.1) Here we prove H'_{m+1}. Instead of the curve B used in step (a) we need to add a curve $B' = B'' \sqcup F_{m+1} \subset H$ with F_{m+1} union of $v_{g,m+1}$ lines and $B' \in W(u_{g,m+1} - v_{g,m+1}, g, r)$. The existence of B' as above comes as a small part of the proofs of [6], but we prefer to do it in the following way (at least if $r \ge 5$). Since $m \ge \Delta_r$, the curve B' has critical value $\le k - 1$. We fix a hyperplane H' of H and add in H' $v_{g,t}$ general lines (see [9], we need [6], Lemma 1.5, to handle the postulation of $B' \cap H'$. For $r = 4$, we would need that B' has critical value $\le k - 2$ and then add $v_{g,t}$ disjoint lines in a smooth quadric surface Q of $H = \mathbb{P}^3$; we need [4] to handle the postulation of $B' \cap Q$. We used the assumption $m \ge \Delta_r$ also to use the inequality (6) needed to apply [7], Lemma 5.2.

(c1.2) Here we prove H'_t for all $t \in \{m+1, \dots, k-1\}$. If $t = m+1$, then this is the content of step (c1.1). Assume $t \ge m+2$. We prove that H'_{t-1} implies H'_t taking in H the union of a smooth rational curve and, perhaps disjoint lines, as in the case $g = 0$ of [6]. Since the curve we started with in degree $t - 1$ (say with degree δ and genus γ) satisfies $\delta - r - r\gamma/(r+2) > \binom{t+r-2}{r-2}$, we still may apply [7], Lemma 5.2.

(c2) Now assume $m < \Delta_r$. Since $d < g+r$, (d, g, r) has critical value $k \ge \Delta_r$ and $v_{g,\Delta_r} \le t-1$, we have $g \ge v_{g,\Delta_r}$. Take a general $E \in W(u_{g,\Delta_r}, g - v_{g,\Delta_r}, r)$. Since $(u_{g,\Delta_r}, g - v_{g,\Delta_r}, r)$ has critical value Δ_r (use (8) and we assumed the

Maximal Rank Conjecture for curves in $\mathbb{P}^r$ with critical value $\le \Delta_r$, the curve E has maximal rank. Hence $h^i(\mathcal{I}_E(\Delta_r)) = 0$, $i = 0, 1$, by (8). We start with E and then work as in step (c1.2). $\square$

As in [6] for all integers $t \ge 2$ define the integers $x_{t,r}$ and $y_{t,r}$ by the relations

$$(t-1)x_{t,r} + r + 1 + y_{t,r} = \binom{r+t}{r},\ 0 \le y_{t,r} \le t-2 \tag{9}$$

Set $x_{1,r} = r$ and $y_{1,r} = 0$. We have $x_{t,r} \ge r + y_{t,r}$ for all t (see [6], Lemma 4.3).

Definition 1. Set $\Psi(k) := 0$ if $k \le 3$. Now assume $k \ge 4$. Set $\beta_k := \lfloor (\binom{r+k-3}{r}/r - 2r - rk - \binom{r+k-2}{r-2})/r \rfloor$, $\alpha_k := 2(x_{k-2,r} - x_{k-3,r})/r - 2r - rk$ and $\Psi(k) := \min\{\beta_k, \alpha_k\}$.

Proof of Proposition 1. Since $\Psi(k) = 0$ for $k \le 3$, we may assume $k \ge 4$. By [6] we may assume $d < g + r$. Since $W(d, g, r)$ is irreducible, it is sufficient to prove the existence of $X_i \in W(d, g, r)$, $i = 1, 2$, such that $h^1(\mathcal{I}_{X_1}(k)) = 0$ and $h^0(\mathcal{I}_{X_2}(k-1)) = 0$. In steps (a) and (b) we will check the existence of X_1, while in step (c) we consider the existence of X_2. Set $\epsilon := g + r - d$. By assumption we have $1 \le \epsilon \le \Psi(k)$. Define the integers z, w by the relations

$$(k-1)z + 1 - (z - r - \epsilon) + w = \binom{r+k-1}{r},\ 0 \le w \le k-3 \tag{10}$$

Notice that $W(z, z-r+\epsilon-w, r)$ is defined. We want to prove $h^i(\mathcal{I}_W(k-1)) = 0$, $i = 0, 1$, for a general $W \in W(z, z-r+\epsilon-w, r)$. Fix a general $S \subset H$ such that $\sharp(S) = r + \epsilon + w$. Fix $A \in W(x_{k-2,r}, x_{k-2,r} - r - e_{k-2,r}, r)$ with maximal rank, i.e. such that $h^i(\mathcal{I}_A(k-2)) = 0$, $i = 0, 1$. Since $\epsilon \le \Psi(k)$ and $\Psi(k) \ge \beta_k$, we may assume $S \subset A \cap H$. Notice that $z \ge x_{r,k-1}$. Since $\Psi(k) \ge \alpha_k$, $e_{k-2,r} \le k-3$ and $z \ge x_{k-1}$, we may find $B \in W(z - x_{k-2,r}, z_{k-2,r} - r, r)$ with maximal rank and containing S (Lemma 2). We may also assume $A \cap B = S$. By [6], Lemma 1.5, we get $h^i(H, \mathcal{I}_{(B \cup (A \cap H}(k-1)) = 0$, $i = 0, 1$. Hence a $h^i(\mathcal{I}_W(k-1)) = 0$, $i = 0, 1$, for a general $W \in W(z, z-r+\epsilon-w, r)$.

(a) Here we assume $d \ge z + rw$. Take a general $W \in W(z, z-r+\epsilon-w, r)$. In particular we assume that W is transversal to H. Take a general $E \subset H$ such that $E \in W(d-z, d-z-r+1, H)$ and $\sharp(E \cap (W \cap H)) = r + w$. To get $h^1(H, \mathcal{I}_{E \cup (W \cap H)}(k)) = 0$ and hence to obtain the existence of X_1 in this case we need to check that we may apply [7], Lemma 5.2; write $z-r+\epsilon-w = a(r+1)+b$ with $0 \le b \le r-2$. We need $z \ge ar + \binom{r+k-2}{r-2} + r(\epsilon + w)$. This is true, because $\epsilon \le \Psi(k)$, $x_{r,k-2} \ge \binom{r+k-2}{r}/r$ and $\Psi(k) \le \beta_k$.

(b) Here we assume $d < z + rw$. Since $g - d = z - \epsilon$ and $w \le k - 3$, we get $kd+1-g \le z+(k-1)z+krw-g \le z+(k-1)z+krw-r-\epsilon \le \binom{r+k-1}{r}+z+r(k-3)$. Since $z \le \binom{r+k-1}{/}(k-2)$, we get $kd+1-g \le \binom{r+k}{r} - w$. As in step (a) we find the existence of $W' \in W(z, z-r+\epsilon, r)$ such that $h^1(\mathcal{I}_{W'}(k-1)) = 0$. Then we add in H a general $B \in W(d-z, d-z-r)$ such that $\sharp(W' \cap B) = r$.

(c) Here we look at the existence of X_2. Essentially, we make the same proof starting with $k-2$ instead of k. The definition of the integer $\Psi(k)$ is done so that that lemmas 1 and 2 may be applied even with $k-2$ instead of k. □

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