International Journal of Pure and Applied Mathematics
Volume 77 No. 3 2012, 337-348
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

SHORT-STEP PRIMAL-DUAL TARGET-FOLLOWING ALGORITHMS FOR THE CONVEXE QUADRATIQUE PROBLEMS

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Abstract: In this paper we propose a method for convex quadratic programming with the property that starting from an initial feasible point, it generates iterates that simultaneously get closer to optimality and closer to centrality. The iterates follow so-called targets, that are updated with Short-steps. Newton's method is used to find an iterate close to a target. We propose an algorithm with the best theoretical polynomial complexity namely $O(\sqrt{n}\log(\frac{n}{\varepsilon}))$, iteration bound. For its numerical performances some strategies are used. Finally, we have tested this algorithm on some convexe quadratique problems.

AMS Subject Classification: 65K05, 90C20, 90C51
Key Words: interior points methods, convex quadratic programming, short-step method, primal-dual target following algorithm, polynomial complexity

1. Introduction

Interior point methods ($IPMs$) are among the most effective methods for solving wide classes of optimization problems because of their polynomial complexity and their numerical efficiency. Since the seminal work of Karmarkar [5], many researchers have proposed and analyzed various $IPMs$ for linear optimisation (LO) and a large amount of results have deen reported. In this paper we

Received: November 25, 2011

deal with the complexity analysis and the numerical implementation of a short-step primal-dual interior point algorithm. These algorithms are based on the strategy of the central path and on a method for finding a new search directions. This technique was used by Darvay [1, 2] for linear optimization (LO) and for lineary constrained convex optimisation (LCO). Here, we reconsider this technique to the convex quadratic programming (CQP) case where we show also that this short-step algorithm deserves the best current polynomial complexity namely $\mathrm{O}(\sqrt{n}\log(\frac{n}{\varepsilon}))$ iteration bound which is analogous to (LO) and (LCO) cases. Finally, we concentrate on its numerical implementation where a strategy on the update barrier parameter μ is made for its numerical performances. The algorithm is applied on some convex quadratic problems

The paper is organized as follows: in Section 2 the statement of the problem is presented. In Section 3, we deal with the new search directions and the description of the algorithm. In Section 4 we state its complexity analysis. In Section 5 its numerical implementation is stated. Finally, in Section 6, a conclusion is given.

2. Statement of the Problem

To be more specific we introduce the convex quadratic programming problem in standard form:

$$(PCQP)\begin{cases} \min\ c^t x + \frac{1}{2}x^t Qx, \\ Ax = b, \\ x \succeq 0. \end{cases} \tag{1}$$

Here Q is a $n \times n$ matrix assumed to be positive semidefinite, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$ and A is a $m \times n$ matrix.

The dual of ($PCQP$) is:

$$(DCQD)\begin{cases} \max\ b^t y - \frac{1}{2}x^t Qx, \\ A^t y + z - Qx = c, \\ z \geq 0, \quad y \in \mathbb{R}^m. \end{cases}$$

We introduce the following assumptions:

(H1) $K_{int} = \{x \in \mathbb{R}^n /\ Ax = b,\ x > 0\}$ a strictly feasible point of ($PCQP$) is non-empty,

(H2) $T_{int} = \left\{y \in \mathbb{R}^m, z \in \mathbb{R}^n / A^t y + z - Qx = c,\ z > 0\right\}$ a strictly feasible point of ($DCQD$) is non-empty.

(H3) The vectors of matrices A are linearly independent.

These assumptions are often used to develop the interior points methods.

In order to introduce an interior point method to solve (1), we associate with it the following barrier minimization problem

$$(PCQP_\mu)\begin{cases} min\ c^t x + \frac{1}{2}x^t Qx - \mu\sum_{i=1}^{n} r_i ln\ x_i = f_\mu(x), \\ Ax = b, \\ x > 0, \end{cases} \tag{2}$$

where $\mu > 0$ be the barrier parameter and $r = (r_1, r_2, ..., r_n) \in IR^n_{++}$ is a weighted vector introduced to ensure that the initial point (x^0, z^0, μ^0) verified $\delta(x^0 z^0, \mu^0) = 0 < 1$ (proximity measure define bellow), if $r_i = 1 \forall i$ then the weighted central path coincides with the classical one. Hence, this approach can be seen as a generalization of central path methods.

The resolution of $(PCQP_\mu)$ is equivalent at that of $(PCQP)$ with that if $x^*(\mu)$ is an optimal solution of $(PCQP_\mu)$ then $x^* = \lim_{\mu\to 0} x^*(\mu)$ is an optimal solution of $(PCQP)$.

The problem (2) is a convex optimization problem and then its first order optimality conditions are:

$$\begin{cases} c - z - A^t y + Qx = 0,\ x > 0,\ z > 0, \\ Ax = b, \\ xz = \mu r, \end{cases} \tag{3}$$

where xs denotes the coordinatewise product of the vectors x and s, hence $xs = (x_1 z_{1,} x_2 z_2, ..., x_n z_n)^T$.

Under the assumption that A has full rank and the existe feasible points, this system has a unique solution [6].

3. A New Search Directions

The basic idea behind this approach is to replace the non linear equation:

$$xz = \mu r,$$

in (3) by an equivalent equation

$$\psi(xz) = \psi(\mu r),$$

where ψ is a real valued function on $[0, +\infty)$ and differentiable on $(0, +\infty)$ such that $\psi(t)$ and $\psi\prime(t) > 0$, for all $t > 0$. Then the system (3) can be written as the following equivalent form:

$$\begin{cases} c - z - A^t y + Qx = 0,\ x > 0,\ z > 0, \\ Ax = b, \\ \psi(xz) = \psi(\mu r). \end{cases} \tag{4}$$

Applying Newton's method for the system (4), we obtain a new class of search directions:

$$\begin{cases} -Q\Delta x + A^t \Delta y + \Delta z = 0, \\ A\Delta x \ = 0, \\ z\psi\prime(xz)\Delta x + x\psi\prime(xz)\Delta z = \psi(\mu r) - \psi(xz). \end{cases} \tag{5}$$

Now, the following notations are useful for studying the complexity of the proposed algorithm.

Let (x, z) be a pair of primal-dual interior feasible solutions, we introduce the scaled vectors v and d as follows:

$$v = \sqrt{xz}, \quad d = \sqrt{\frac{x}{z}}.$$

Using d we can rescale both x and z to the same vector;

$$d^{-1}x = dz = v,$$

we also use d to rescale Δx and Δz :

$$p_x = d^{-1}\Delta x, \quad p_z = d\Delta z.$$

Now we may write

$$x\Delta z + z\Delta x = xd^{-1}d\Delta z + zdd^{-1}\Delta x = v(\ p_x +\ p_z).$$

Hence, Newton's direction is determined by the following linear system:

$$\begin{cases} -DQDp_x + (AD)^t\Delta y + p_z = 0 \\ \quad ADp_x \ = 0 \\ \quad p_x + p_z = p_v \end{cases}$$

$$\iff \begin{cases} -\bar{Q}p_x + (\bar{A})^t\Delta y + p_z = 0 \\ \bar{A}p_x \ = 0 \\ p_x + p_z = p_v \end{cases} \tag{6}$$

where $\bar{Q} = DQD$, $\bar{A} = AD$ with $D = diag(d)$ and $p_v = \frac{\psi(\mu r) - \psi(xz)}{v\psi\prime(v^2)}$

As in [7], we shall consider the following function:

$$\psi(t) = \sqrt{t},$$

with $\psi\prime(t) = \frac{1}{2\sqrt{t}} > 0$ for all $t > 0$.

Hence, the Newton directions in (5) is

$$\begin{cases} -Q\Delta x + A^t\Delta y + \Delta z = 0, \\ A\Delta x = 0, \\ \sqrt{\frac{z}{x}}\Delta x + \sqrt{\frac{x}{z}}\Delta z = 2(\sqrt{\mu r} - v), \end{cases} \tag{7}$$

with $p_v = 2(\sqrt{\mu r} - \sqrt{xz}) = 2(\sqrt{\mu r} - v)$, and we define for all vector v the following proximity measure by:

$$\delta(xz,\mu) = \delta(v,\mu) = \frac{\|p_v\|}{2\min(\sqrt{\mu r})} = \frac{\left\|\sqrt{\mu r} - v\right\|}{\min(\sqrt{\mu r})},$$

were $\|\|$ is the Euclidean norm (l_2 norm) and $\min(x) = \min\{x_1, x_2, ..., x_n\}$.

We introduce another measure $\sigma_c(r) = \frac{\max(r)}{\min(r)}$.

Now, we get the short-step primal-dual algorithm to solve (CQP) :

Algorithm for convex quadratic programming

- **Input:** $(x^{(0)}, y^{(0)}, z^{(0)})$ where $x^{(0)}$ is a strictly feasible solution of (1), $(y^{(0)}, z^{(0)})$ is a strictly feasible solution of (2), $\mu^{(0)} > 0$ an initial barrier parameter and ε is the accuracy parameter.
- compute: $r^0 = \frac{x^{(0)}z^{(0)}}{\mu^0}$
- **begin:**
  - $x = x^{(0)}, z = z^{(0)}, v = \sqrt{xz}, \mu = \mu^{(0)}$
  - **while** $x^tz > \varepsilon$ **do**
  - Solve the Newton system of equations in (7);
  - compute $x = x + \Delta x$, $y = y + \Delta y$, $z = z + \Delta z$ and $\mu = (1-\theta)\mu$.
- **end.**

Remark. By construction, to guarantee that the next Newton iterate $\hat{x} = x + \alpha_x\Delta x > 0$ and $\hat{z} = z + \alpha_z\Delta z$ for any $\alpha \in IR$, it suffices to set.

$$\alpha_x = \begin{cases} min\,(-x_i/\Delta x_i) & si\ \Delta x_i \ \langle\ 0 \\ 1 & si\ \Delta x_i \succeq 0 \end{cases}$$
$$\alpha_z = \begin{cases} min\,(-z_i/\Delta z_i) & si\ \Delta z_i \ \langle\ 0 \\ 1 & si\ \Delta z_i \succeq 0 \end{cases}$$

In the next section, we give some results concerning the complexity analysis of the algorithm. We notice that all the results stated later are relatively simple and they are deduced straightforwardly from the analysis used for the linear programming and lineary constrained convex programming cases. For more details the reader consults the reference [1,2].

4. Complexity Analysis

Let

$$q_v = p_x - p_z.$$

We have

$$p_x = \frac{1}{2}(p_v + q_v), \quad p_z = \frac{1}{2}(p_v - q_v),$$

whence

$$p_x p_z = \frac{1}{4}(p_v^2 - q_v^2) \text{ and } \|q_v\| \leq \|p_v\|.$$

This last result follows directly from the equality

$$\|p_v\|^2 = \|q_v\|^2 + 4p_x^T p_z,$$

since

$$p_x^T p_z = p_x^T \bar{Q} p_x \geq 0,$$

because $\bar{Q}$ is positive semidefinite. We have have

$$\delta(v,\mu) \geq \frac{\|q_v\|}{2\min \mu r}.$$

In the following lemma, we state a condition which ensures the feasibility of the full Newton step. Let $\hat{x} = x + \Delta x$ and $\hat{z} = z + \Delta z$, be the new iterate after a full Newton step.

Lemma 1. *Let $\delta = \delta(v,\mu) < 1$.Then the full Newton step is strictly feasible, hence: $\hat{x} > 0$ and $\hat{z} > 0$, see* [1].

In the next lemma we show that $\delta < 1$ is sufficient for the quadratic convergence of the Newton process.

Lemma 2. *Let $\hat{x} = x + \Delta x$ and $\hat{z} = z + \Delta z$ be the iteration obtained after a full Newton step with $v = \sqrt{xz}$ and $\hat{v} = \sqrt{\hat{x}\hat{z}}$.*

Suppose $\delta = \delta(v, \mu) < 1$. Then $\delta(\hat{v}, \mu) \leq \frac{\delta^2}{1+\sqrt{1-\delta^2}}$, thus $\delta(\hat{v}, \mu) < \delta^2(v, \mu)$, which means quadratic convergence of the Newton step.

Proof. We have:

$$(\hat{v})^2 = \hat{x}\hat{z} = (x + \Delta x)(z + \Delta z) = v^2 + vp_v + \frac{p_v^2}{4} - \frac{q_v^2}{4}$$
$$= \mu r - \frac{p_v^2}{4} + \frac{p_v^2}{4} - \frac{q_v^2}{4} = \mu r - \frac{q_v^2}{4},$$

we obtain

$$\min{(\hat{v})}^2 \geq \min{(\mu r)} - \frac{\left\|q_v^2\right\|_\infty}{4} \geq \min{(\mu r)} - \frac{\|q_v\|^2}{4} \geq \min{(\mu r)}(1 - \delta^2)$$

and this relation yields:

$$\min{(\hat{v})} \geq \min{(\sqrt{\mu r})}(\sqrt{1-\delta^2}).$$

Furthermore

$$\begin{aligned}
\delta(\hat{v}, \mu) =& \frac{1}{\min\sqrt{\mu r}} \left\| \frac{\mu r - \hat{v}}{\sqrt{\mu r} + \hat{v}} \right\| \\
\leq& \frac{\|\mu r - v^+\|}{\min\sqrt{\mu r}(\min(\sqrt{\mu r} + \hat{v}))} \\
\leq& \frac{\|\mu r - v^+\|}{(\min\sqrt{\mu r})^2(1 + \sqrt{1-\delta^2})} \\
\leq& \frac{\left\|q_v^2\right\|}{(\min\sqrt{\mu r})^2(1 + \sqrt{1-\delta^2})} \\
\leq& \frac{\delta^2}{1 + \sqrt{1-\delta^2}}.
\end{aligned}$$

□

In the next lemma we state an upper bound for the duality gap obtained after a full Newton step.

Lemma 3. *Let $\hat{x} = x + \Delta x$ and $\hat{z} = z + \Delta z$. Then the duality gap is:*

$$(\hat{x})^T \hat{z} = \mu \left\|\sqrt{r}\right\|^2 - \frac{\|q_v\|^2}{4},$$

hence

$$(\hat{x})^T\hat{z} \leq \mu \left\| \sqrt{\frac{x^0z^0}{\mu^0}} \right\|^2.$$

Proof. From

$$(\hat{v})^2 = \mu r - \frac{q_v^2}{4}$$

we have

$$\hat{x}\hat{z} = \mu r - \frac{q_v^2}{4}.$$

We obtain

$$(\hat{x})^T\hat{z} = e^T(\hat{x}\hat{z}) = \mu e^T r - \frac{e^T q_v^2}{4} = \mu \left\|\sqrt{r}\right\|^2 - \frac{\|q_v\|^2}{4}.$$

This relation yields

$$(\hat{x})^T\hat{z} \leq \mu \left\|\sqrt{r}\right\|^2 = \mu \left\| \sqrt{\frac{x^0z^0}{\mu^0}} \right\|^2. \qquad \square$$

The next Lemma discusses the influence on the proximity measure of the Newton process followed by a long the central path.We assume the parameter μ will be reduced by a constant factor $(1-\theta)$.

Lemma 4. *Let $\delta = \delta(xz,\mu) < 1$ and $\mu^+ = (1-\theta)\mu$, where $0 < \theta < 1$. Then*

$$\delta(\hat{v},\hat{\mu}) \leq \frac{\theta}{1-\theta}\sqrt{\sigma_c(r)} + \frac{1}{1-\theta}\ \delta(\hat{v},\mu).$$

Furthermore, if $\delta \leq \frac{1}{2}$, $\theta = \frac{2}{5\sqrt{\sigma_{\mathbf{c}}(r)n}}$ and $n \geq 4$ then we get

$$\delta(\hat{v},\hat{\mu}) \leq \frac{1}{2}.$$

Proof. We have

$$\begin{aligned}
\delta(\hat{v},\hat{\mu}) =& \frac{\left\|\sqrt{\hat{\mu}r} - \hat{v}\right\|}{\min\sqrt{\mu^+ r}} = \frac{\left\|\sqrt{\hat{\mu}r} - \sqrt{\mu r} + \sqrt{\mu r} - \hat{v}\right\|}{\min\sqrt{\hat{\mu}r}} \\
\leq& \frac{\left\|\sqrt{\hat{\mu}r} - \sqrt{\mu r}\right\|}{\min\sqrt{\mu^+ r}} + \frac{\left\|\sqrt{\mu r} - \hat{v}\right\|}{\min\sqrt{\mu^+ r}} \\
=& \frac{1-\sqrt{1-\theta}}{\sqrt{1-\theta}}\left(\frac{\|\mu r\|}{\min(\sqrt{\mu r})}\right) + \frac{1}{\sqrt{1-\theta}}\ \delta(\hat{v},\mu)
\end{aligned}$$

$$\leq \frac{1-\sqrt{1-\theta}}{\sqrt{1-\theta}}\sqrt{n\sigma_c(r)} + \frac{1}{\sqrt{1-\theta}}\ \delta(\hat{v},\mu).$$

Now let $\theta = \frac{2}{5\sqrt{n\sigma_{\mathbf{c}}(r)}}$, oberve that $\sigma_c(r) \geq 1$ and for $n \geq 4$ we obtain $\theta \leq \frac{2}{10}$ if $\delta(v,\mu) \leq \frac{1}{2}$ then from Lemma 2 we deduce $\delta(\hat{v},\mu) \leq \frac{1}{4}$. Finally, the obove relation yields: $\delta(\hat{v},\hat{\mu}) \leq \frac{1}{2}$. □

In the next lemma we calculate an upper bound for the total number of iterations performed by the algorithm.

Lemma 5. *Assume that x^0 and z^0 are strictly feasible, $\mu^0 = \frac{(x^0)^{\mathsf{T}} z^0}{n} > 0$. Moreover, let x^k and z^k be the vectors obtained after k iterations. Then the inequality $(x^k)^T z^k \leq \varepsilon$ is satisfied for $k \geq \left\lceil \frac{1}{\theta}\log\frac{(x^0)^{\mathsf{T}} z^0}{\epsilon}\right\rceil$*

Proof. After k ierations, we get $\mu^k = (1-\theta)^k\mu^0$. Using Lemma 3 we find that

$$(x^k)^T z^k \leq \mu^k \left\|\sqrt{r}\right\|^2 = (1-\theta)^k\mu^0\left\|\sqrt{r}\right\|^2 = (1-\theta)^k\left\|\sqrt{\mu^0 r}\right\|^2$$
$$= (1-\theta)^k\left\|\sqrt{x^0 z^0}\right\|^2 = (1-\theta)^k(x^0)^T z^0.$$

Hence $(x^k)^T z^k \leq \varepsilon$ holds if $(1-\theta)^k (x^0)^T z^0 \leq \varepsilon$.

Taking logarithms, we obtain

$$k\log\left(1-\theta\right) + \log\left(x^0\right)^T z^0 \leq \log\epsilon.$$

Using the inequality $-log(1-\theta) \geq \theta$ we deduce that the obove relation holds if

$$k\theta \geq \log\frac{(x^0)^T z^0}{\varepsilon} \Rightarrow k \geq \frac{1}{\theta}\log\frac{(x^0)^T z^0}{\varepsilon}.$$ □

For the default $\theta = \frac{2}{5\sqrt{\sigma_{\mathbf{c}}(r)n}}$, we obtain the following theorem.

Theorem 6. *Suppose that $x^0 \in K_{int}$, $z^0 \in T_{int}$, and let $\mu^0 = \frac{(x^0)^{\mathsf{T}} z^0}{n}$. If $\theta = \frac{2}{5\sqrt{\sigma_{\mathbf{c}}(r)n}}$, then the algorithm requires at most*

$$\left\lceil \frac{5}{2}\sqrt{\sigma_c(r)n}\log\frac{(x^0)^T z^0}{\epsilon}\right\rceil$$

iterations.

For the resulting vectors we have $(x^k)^T z^k \leq \varepsilon$.

5. Numerical Implementation

In this section, we deal with the numerical implementation of this algorithm applied to a problem CQP. Here we used $x^*$the optimal solution of CQP and $Iter$ means the iterations number produced by the algorithm. The implementation is manipulated in Turbo-Pascal on a Pentium 4. Our tolerance is $\epsilon = 10^{-6}$. For the update parameter we have used a practical strategy which is $\mu^0 = 0.1$ and $\theta \in \{0.2,\ 0.5,\ 0.7,\ 0.9\}$.

We present some numerical results. We consider the following example:

$$A = \begin{pmatrix} 1 & -1 & 1.9 & 1.25 & 1.2 & 0.4 & -0.7 & 1.06 & 1.5 & 1.05 \\ 1.3 & 1.2 & 0.15 & 2.15 & 1.25 & 1.5 & 0.4 & 1.52 & 1.3 & 1 \\ 1.5 & -1.1 & 3.5 & 1.25 & 1.8 & 2 & 1.95 & 1.2 & 1 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 30 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 21 & 0 & 1 & -1 & 1 & 0 & 1 & 0.5 & 1 \\ 1 & 0 & 15 & -0.5 & -2 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & -0.5 & 30 & 3 & -1 & 1 & -1 & 0.5 & 1 \\ 1 & -1 & -2 & 3 & 27 & 1 & 0.5 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 16 & -0.5 & 0.5 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0.5 & -0.5 & 8 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 0.5 & 1 & 24 & 1 & 1 \\ 1 & 0.5 & 1 & 0.5 & 1 & 0 & 1 & 1 & 39 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 11 \end{pmatrix},$$

$$b = (11.651, 16.672, 21.295)^T,$$

$$c = (-0.5, -1, 0, 0, -0.5, 0, 0, -1, -0.5, -1)^T.$$

The optimal solutions of the primal-dual problem are obtained as follows:

$$x^* = (0.963886, 0.509607, 1.739953, 1.905056, 1.243511, 2.626820, 1.322918, 1.617087, 0.824013, 0.897582)^T,$$

$$y^* = (4.243380, 22.362785, 5.192083)^T,$$

$$z^* = (15.410001, 36.395541, -10.259604, 24.140452, 12.485220, 32.238247, 33.095797, 16.542761, 6.549879, 29.830593)^T.$$

Numbers of iterations for several choices of θ.

θ	0.2	0.5	0.7	0.9
iter	37	32	29	24

The numerical results show that the number of iterations of the Algorithm depends on the values of the parameter θ.It is quite susprising that $\theta = 0.9$ gives the lowest iteration.

6. Conclusion

In this paper we have developed a new primal-dual target-following algorithm for solving the convexe quadratique problems (CQP). Our approach is a generalization of [1,2] for linear optimization (LO) and for lineary constrained convex optimisation (LCO). we have defined a new class of search directions by applying Newton's method. we have developed a new primal-dual target-following algorithm, and we have proved that this algorithm performs no more than $(\frac{5}{2}\sqrt{\sigma_c(r)n}\log\frac{(x^0)^{\mathsf{T}}z^0}{\epsilon})$. Our numerical results are acceptable whereas getting a starting feasible centered point for these algorithms is an uneasy task. Finally, we point out that the implementation with the update parameter θ reduces significantly the number of iterations produced by this algorithm and leads this algorithm to reach their real numerical performances.

References

[1] Zsolt Daravay, A weited-path following method for linear optimisation, *Studia Univ. Babes-Bolyai, Informatica*, **XLVII**, No. 2 (2002).

[2] Zsolt Daravay, A predictor-corrector algorithm for lineary constrained convex optimisatin, *Studia Univ. Babes-Bolyai, Informatica*, **LIV**, No. 2 (2009).

[3] B. Jansen, C. Roos, T. Terlaky, J.-Ph. Vial, Long-step primal-dual target-following algorithms for linear programming, *Reports of the Faculty of Technical Mathematics and Informatics*, No. 94-46 Delft, The Netherlands (May 26, 1994).

[4] B. Jansen, C. Roos, T. Terlaky, J.-Ph. Vial, Primal-dual target-following algorithms for linear programming, *Reports of the Faculty of Technical*

Mathematics and Informatics, No. 93-107 Delft, The Netherlands (November 23, 1993).

[5] N.K. Karmarkar, A new polynomial-time algorithm for linear programming, *Combinatorica*, **4** (1984), 373-395.

[6] M.H. Wright, Interior methods for constrained optimization, *Acta Numerica*, **1** (1992), 341-407.