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ON 'USEFUL' R-NORM RELATIVE INFORMATION AND J-DIVERGENCE MEASURES

Satish Kumar[1], Gurdas Ram[2 §], Vishal Gupta[3]
[1]Department of Mathematics
Geeta Institute of Management and Technology
Kanipla, (Kurukshetra), INDIA
[2]Department of Applied Sciences
M.M. University
Solan (H.P.), INDIA
[3]Department of Mathematics
M.M. University
Mullana (Ambala), INDIA

Abstract: In this paper some new generalized R-Norm measures of useful relative information have been defined and their particular cases have been studied. From these measures new useful R-Norm information measures have also been derived. We have obtained J-divergence corresponding to each measure of useful relative R-norm information. in the end, an equality satisfied by useful J-divergence of type β has been proved.

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1. Introduction

Let $P = \{(p_1,p_2,...,p_n), 0 \leq p_i \leq 1\}$, $Q = \{(q_1,q_2,...,q_n), 0 \leq q_i \leq 1\}$be two posterior and priori distributions of a random variable having utility distribution

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§Correspondence author

$U = (u_1, u_2, ..., u_n)$,where $u_i > 0$ is the importance or usefulness of an event E_i. A measure of directed divergence $D(P : Q; U)$ of $(P; U)$ to $(Q; U)$ has to satisfy the following conditions:

(i) $D(P : Q; U) \geq 0$;

(ii) $D(P : Q; U) = 0$ iff $p_i = q_i$ for each i.

(iii) $D(P : Q; U)$ is a convex or pseudo-convex function of $p_1, p_2, ..., p_n$ as well as $q_1, q_2, ..., q_n$.

Bhaker and Hooda[1] defined and characterized 'useful' directed divergence measure given as below.

$$D(P : Q; U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \tag{1}$$

This measure satisfies all three conditions (i) - (iii) provided $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$ we do come across some practical problem where we are to minimize $D(P : Q; U)$ with constraints $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$ e.g. A manufacturer makes n different types of articles in proportions $q_1, q_2, ..., q_n$ having profit of $u_1, u_2, ..., u_n$ respectively.

On the basis of availability of labour and raw material he plans to change the proportions of articles to $p_1, p_2, ..., p_n$; however, close to $q_1, q_2, ..., q_n$ such that the average profit may not decrease i.e. $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$

To obtain p_i's we minimize

$$D(P : Q; U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i}$$

subject to

$$\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$$

It may be noted that $D(P : Q; U)$ is not a metric as it does not satisfy symmetric and triangle inequality. However $D(P : Q; U)$ satisfy convexity property (iii) which we need to minimize $D(P : Q; U)$ as a function of P or Q subject to $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$.

2. Some Requisite Results

(a) For all probability distribution P and Q having attached with Utility distribution U,

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \gtreqless 1$$

according to $R \gtreqless 1$ provided $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$

Proof. By Holder's inequality, for $R > 1$.

$$\sum_{i=1}^{n} u_i p_i^R q_i^{1-R} \geq \left[\sum_{i=1}^{n} \left(u_i^R p_i^R\right)^{\frac{1}{R}}\right]^R \left[\sum_{i=1}^{n} (u_i^{1-R} q_i^{1-R})^{\frac{1}{1-R}}\right]^{1-R}$$

$$\Rightarrow \sum_{i=1}^{n} u_i p_i^R q_i^{1-R} \geq \left[\sum_{i=1}^{n} u_i p_i\right]^R \left[\sum_{i=1}^{n} u_i q_i\right]^{1-R} \geq \sum_{i=1}^{n} u_i p_i,$$

since $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$

$$\Rightarrow \quad \frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i} \geq 1$$

$$\Rightarrow \left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \geq 1$$

For $R = 1$, $\left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} = 1$ for all probability distributions.

Also if $R \neq 1$ and $p_i = q_i$ for each i, i.e. $P = Q$, we have

$$\left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} = 1$$

and in other cases $\left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} \neq 1.$

(b) $\frac{R}{1-R} \left(\frac{\sum_{i=1}^{n} u_i p_i^R q_i^{1-R}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}}$ is a convex function of P and Q.

Proof. We prove this in the following steps:

Step 1. Let $S = \frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}$ if we differentiate S partially w.r.t. p_i taking all q_i and u_i fixed, then $\sum_{i=1}^n u_i q_i$ is fixed and thus $\sum_{i=1}^n p_i u_i \geq \sum_{i=1}^n q_i u_i$ is constant. Hence we can write

$S = C \sum_{i=1}^n u_i p_i^R q_i^{1-R}$, where $\frac{1}{C} = \sum_{i=1}^n p_i u_i \geq \sum_{i=1}^n q_i u_i > 0$

$$\Rightarrow \frac{\partial S}{\partial p_i} = CRu_i p_i^{R-1} q_i^{1-R} \tag{2}$$

and

$$\frac{\partial^2 S}{\partial p_i^2} = R(R-1)Cu_i p_i^{R-2} q_i^{1-R}. \tag{3}$$

For $R > 1$, (3) is positive therefore $\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}$ is a convex function of P. and hence $\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}$ is concave function of P.

For $0 < R < 1$, (3) is negative therefore $\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}$ is a concave function of P and hence $\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}$ is convex function of P.

Similarly, we can prove that $\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}$ is concave function of Q for $R > 1$ and is convex function of Q for $0 < R < 1$.

Step 2. Since $\frac{R}{1-R} < 0$ for $R > 1$ and > 0 for $R < 1$, therefore,

$$\frac{R}{1-R} \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}$$

is a convex function of P and Q for all $R > 0$ provided $\sum_{i=1}^n p_i u_i \geq \sum_{i=1}^n q_i u_i$.

3. A Generalized Measure of 'Useful' R-Norm Relative Information

We consider the function

$$D_R(P : Q; U) = \frac{R}{1-R} \left[\phi(1) - \phi \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}} \right]$$

where $\phi(x)$ is a monotonic increasing convex function of x, then

$$R > 1 \Rightarrow \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}} \geq 1 \Rightarrow \phi \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}} \geq \phi(1)$$

$$\Rightarrow D_R(P:Q;U) \geq 0$$

$$0 < R < 1 \Rightarrow \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}} \leq 1 \Rightarrow \phi\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}} \leq \phi(1)$$

$$\Rightarrow D_R(P:Q;U) \geq 0$$

As $R \to 1$, $D_R(P:Q;U) \to \phi'(1)\frac{\sum_{i=1}^n u_i p_i \log\left(\frac{p_i}{q_i}\right)}{\sum_{i=1}^n u_i p_i}$.

Also for $p_i = q_i$ for each i , $D_R(P:Q;U) = 0$

$$\Rightarrow \frac{R}{1-R}\left[\phi(1) - \phi\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{1}{R}}\right] = 0$$

$$\Rightarrow \phi\left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right) = \phi(1) \Rightarrow \frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i} = 1$$

$\Rightarrow$either $R = 1$ or $p_i = q_i$ for each i.

Since an increasing convex function of a convex function is a convex function and $\phi(x)$ is a monotonic increasing convex function and $R > 1$ therefore, $D_R(P:Q;U)$is a convex function of P and Q.

Similarly, if $\phi(x)$ is a concave function and $0 < R < 1$, then by the facts that an increasing concave function of a concave function is a concave and the negative of a concave function is a convex function, $D_R(P:Q;U)$ is a convex function of P and Q. Thus we can use $D_R(P:Q;U)$as a measure of useful relative information if $R > 1$ and $\phi(x)$ is any monotonic increasing twice differentiable convex function of x and if $0 < R < 1$ and $\phi(x)$ is any monotonic decreasing twice differentiable concave function of x.

4. Special Cases

$$(i)\quad D_{R,j}(P:Q;U) = \frac{R}{1-R}\left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i}\right)^{\frac{j}{R}}\right], \tag{4}$$

where $R > 1$ and $j \geq 1$ or $0 < R < 1$ and $0 < j \leq 1$.

(ii) When j=1 , (4) gives

$$D_R(P:Q;U) = \frac{R}{1-R}\left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^R q_i^{1-R}}{\sum_{i=1}^n u_i p_i},\right)^{\frac{1}{R}}\right] \tag{5}$$

which is generalized R-Norm useful relative information measure. If u_i=1 for each i, then

$$D_{R,j}(P:Q)=\frac{R}{1-R}\left[1-\left(\sum_{i=1}^{n}p_i^R q_i^{1-R}\right)^{\frac{1}{R}}\right], \tag{6}$$

which is R-Norm relative information measure for Boekee and Lubbe[2]

$$(iii) \qquad \underset{R\to 1}{Lim}\, D_{R,j}(P:Q;U)=j\frac{\sum_{i=1}^{n}u_i p_i \log\frac{p_i}{q_i}}{\sum_{i=1}^{n}u_i p_i}, \tag{7}$$

which is j multiple useful relative information measure characterized and studied by Bhaker and Hooda[1].

(iv) If $u_i = 1$ for each i , then(7) , gives $j\sum_{i=1}^{n}p_i\log\left(\frac{p_i}{q_i}\right)$, which is j-multiple of Kullback and Leibler[5] measure of directed divergence.

5. Measure of 'Useful' Information

Let $C=\left(\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right)$ be uniform distribution. Then:

$(a) D(P:C;U)=\frac{\sum_{i=1}^{n}u_i p_i\log np_i}{\sum_{i=1}^{n}u_i p_i}\geq 0$, since $np_i \geq 1$;

$(b) D(P:C;U)=0$, if $P=C$;

$(c) D(P:C;U)$ is a convex function of P.

We see that $D(P:C;U)=\log n+\frac{\sum_{i=1}^{n}u_i p_i\log p_i}{\sum_{i=1}^{n}u_i p_i}$

$$=D(C;U)-D(P;U),$$

where $D(P;U)=-\frac{\sum_{i=1}^{n}u_i p_i\log p_i}{\sum_{i=1}^{n}u_i p_i}$ is useful information measure characterized by Bhaker and Hooda[1].

So to minimize $D(P:C;U)$ we maximize $D(P;U)$.

$(d)\;\; D(C;U)\geq D(P;U)$

$(e)\;\; D(C;U)=D(P;U)$ if $P=C$

(f) $D(P;U)$ is a concave function of P. Thus $D(P;U)$ is useful information measure of P and U corresponding to 'useful' directed divergence measure $D(P:Q;U)$.

Next we define

$$H_{R,\phi}(P;U)=\frac{R}{1-R}\left[\phi\left(\frac{\sum_{i=1}^{n}u_i p_i^R n^{R-1}}{\sum_{i=1}^{n}u_i p_i}\right)^{\frac{1}{R}}-\phi(n^{\frac{R-1}{R}})\right] \tag{8}$$

so that

$$H_{R,\phi}(C;U) = \frac{R}{1-R}\left[\phi\left(1\right) - \phi(n^{\frac{R-1}{R}})\right] \tag{9}$$

and

$$\underset{R\to 1}{Lim}\, H_{1,\phi}(C;U) = \phi'(1)\log n \tag{10}$$

If $\phi(x) = x^j, (j \geq 1)$, we have

$$H_{R,j}(P;U) = \frac{R}{1-R}\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^R n^{R-1}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{j}{R}} - (n^{R-1})^{\frac{j}{R}}\right]$$

$$= \left(n^{R-1}\right)^{\frac{j}{R}} \frac{R}{1-R}\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{j}{R}} - 1\right] \tag{11}$$

For $j = 1$, we get

$$H_{R,1}(P;U) = n^{\frac{R-1}{R}}\left(\frac{R}{1-R}\right)\left[\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} - 1\right] \tag{12}$$

When $R \to 1$, the measure (8), (11) and (12) respectively reduces to

$$H_{1,\phi}(P;U) = -\phi'(1)\left(\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}\right) \tag{13}$$

$$H_{1,j}(P;U) = -j\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{14}$$

$$H_{1,1}(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} \tag{15}$$

If we take $\phi(x) = \log x$ in (8), we get

$$H_{R,\phi}(P;U) = \frac{R}{1-R}\left[\log\left(\frac{\sum_{i=1}^{n} u_i p_i^R n^{R-1}}{\sum_{i=1}^{n} u_i p_i}\right)^{\frac{1}{R}} - \log(n^{\frac{R-1}{R}})\right]$$

$$H_{R,\phi} = \frac{1}{1-R}\log\left(\frac{\sum_{i=1}^{n} u_i p_i^R}{\sum_{i=1}^{n} u_i p_i}\right), \tag{16}$$

which is Bhaker and Hooda’s[1] measure of useful information of order R.

If $u_i = 1$ for each i, then from (16), we have

$$H_{R,\phi}(P;U) = \frac{1}{1-R}\log\sum_{i=1}^{n} p_i^R, \tag{17}$$

which is Renyi's entropy[7] of order R.

For the independent distributions P and Q having U and V respectively as utility distributions from (11), we have

$$H_{\alpha,j}(P*Q\ :\ U*V) = \frac{R}{1-R}n^{\frac{(R-1)j}{R}}\left[\left(\frac{\sum_{i=1}^{n}u_ip_i^R}{\sum_{i=1}^{n}u_ip_i}\frac{\sum_{i=1}^{m}v_jq_j^R}{\sum_{i=1}^{n}v_jq_j}\right)^{\frac{j}{R}} - 1\right]$$

$$= \frac{R}{1-R}n^{\frac{(R-1)j}{R}}\left[\left(\frac{\sum_{i=1}^{n}u_ip_i^R}{\sum_{i=1}^{n}u_ip_i}\right)^{\frac{j}{R}}\left(\frac{\sum_{i=1}^{m}v_jq_j^R}{\sum_{i=1}^{m}v_jq_j}\right)^{\frac{j}{R}} - 1\right]$$

$$= \frac{R}{1-R}n^{\frac{(R-1)j}{R}}\left[\left\{\frac{1-R}{Rn^{\frac{(R-1)j}{R}}}H_{R,j}(P;U)+1\right\}\left\{\frac{1-R}{Rn^{\frac{(R-1)j}{R}}}H_{R,j}(Q;V)+1\right\} - 1\right]$$

$$= \frac{1-R}{Rn^{\frac{(R-1)j}{R}}}H_{R,j}(P;U)H_{R,j}(Q;V) + H_{R,j}(P;U) + H_{R,j}(Q;V) \tag{18}$$

which is a well known functional equation and proves that measure (11) is non additive.

6. Measure of Useful J-Divergence of type β

Now we define a measure of symmetric divergence called J-divergence of type β.

$$J_\beta(P:Q;\,U) = D_\beta(P:Q;U) + D_\beta(Q:P;U)$$

$$= \frac{\sum_{i=1}^{n}u_i(p_i^\beta - q_i^\beta)\log(\frac{p_i}{q_i})}{\sum_{i=1}^{n}u_ip_i^\beta}, \quad \beta > 0. \tag{19}$$

In case utilities are ignored i.e. $u_i = 1$ for each i , (18) reduce to

$$J_\beta(P;Q) = \frac{\sum_{i=1}^{n}(p_i^\beta - q_i^\beta)\log(\frac{p_i}{q_i})}{\sum_{i=1}^{n}p_i^\beta}.$$

Theorem. *1.* $J_\beta(P:Q;U)$ *is not a homogeneous function in* U. *However, if* $U^\alpha = (U_1^\alpha, U_2^\alpha, ...U_n^\alpha))$, $\alpha > 0$ *be the* α*-power utility distribution , then the following inequality holds:*

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} J_\beta(P:Q;U^\alpha) = \frac{\alpha\, J_\beta(P:Q;U^\alpha)}{E(P)},$$

where $E(P) = \sum_{i=1}^{n} u_i^\alpha p_i^\beta$ *=constant.*

Proof. Let $J_\beta(P:Q;U)$ be a homogeneous function of degree α in utility distribution. Then by Euler's theorem, we have

$$J_\beta(P:Q;\lambda U) = \lambda^\alpha J_\beta(P:Q;U) = \frac{\lambda^\alpha \sum_{i=1}^{n} u_i(p_i^\beta - q_i^\beta)\log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i^\beta}$$
$$= \lambda^{\alpha-1} J_\beta(P:Q;\lambda U)$$

It implies $\lambda^{\alpha-1} = 1$ or $\alpha = 1$

Hence $J_\beta(P:Q;U)$ is not a homogeneous function in U.

From (19), we have

$$J_\beta(P:Q;U^\alpha) = \frac{\sum_{i=1}^{n} u_i^\alpha(p_i^\beta - q_i^\beta)\log(p_i/q_i)}{\sum_{i=1}^{n} u_i^\alpha p_i^\beta}$$
$$= \frac{\sum_{i=1}^{n} u_i^\alpha(p_i^\beta - q_i^\beta)\log(p_i/q_i)}{E(P)} \qquad (20)$$

Differentiating (20) with respect to u_i and multiplying by u_i on both sides and taking summation over i, we get

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} J_\beta(P:Q;U^\alpha) = \frac{\alpha\, J_\beta(P:Q;U^\alpha)}{E(P)},$$

Hence theorem is proved.

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