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OSCILLATION OF THIRD ORDER HALF LINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract: This paper is concerned with the oscillatory behavior of third order neutral differential equation

$$[a(t)([x(t)+p(t)x(\delta(t))]'')^{\alpha}]' + q(t)x^{\alpha}(\sigma(t)) = 0, \quad t \geq t_0, \qquad (E)$$

where $a(t), p(t)$, and $q(t)$ are positive functions, $\alpha > 0$ is a quotient of odd positive integers, and $\sigma(t) \leq t$, $\delta(t) \leq t$.

Some new oscillation criteria for equation (E) are established. Examples illustrating the main results are included.

AMS Subject Classification: 34K11
Key Words: third order, half-linear, neutral delay differential equation, oscillation, nonoscillation

1. Introduction

This paper is concerned with the oscillatory behavior of third order neutral differential equation of the form

$$(a(t)([x(t)+p(t)x(\delta(t))]'')^{\alpha})' + q(t)x^{\alpha}(\sigma(t)) = 0, t \geq t_0 \qquad (1.1)$$

subject to the following conditions:

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(C_1) $a(t), p(t), q(t), \sigma(t), and\ \delta(t) \in C([t_0, \infty)), a(t), q(t), \sigma(t), and$ $\delta(t)$ are positive functions;

(C_2) α is a ratio of odd positive integers, $0 \leq p(t) \leq p < 1$, $\delta(t) \leq t, \sigma(t) \leq t,$ and $\lim_{t\to\infty} \sigma(t) = \lim_{t\to\infty} \delta(t) = \infty$;

(C_3) $\displaystyle\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)} dt = \infty.$

We put $z(t) = x(t) + p(t)x(\delta(t))$. By a solution of equation (1.1), we mean a function $x(t) \in \mathcal{C}^1[T_x, \infty), T_x \geq t_0$ which has the property $a(t)(z''(t))^\alpha \in \mathcal{C}^1[T_x, \infty)$ and satisfies equation (1.1), on $[T_x, \infty)$. We consider only those solutions $x(t)$ of equation (1.1), which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that equation (1.1) possesses such a solution. A solution of equation (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and otherwise it is called nonoscillatory.

Recently,great attention has been devoted to oscillation theory of neutral differential equations,see for example [1-8], and the references cited therein. In particular Baculikova and Dzurina [2] studied the oscillatory behavior of equation (1.1) when $a(t)$ is nondecreasing and $\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)} dt = \infty$.

Motivated by the above observation, in this paper,we establish some new oscillation criteria for equation(1.1) when $a(t)$ is nondecreasing and $\int_{t_0}^{\infty} \frac{1}{a^{\frac{1}{\alpha}}(t)} dt < \infty$. Examples are provided to illustrate the main results. In what follows, all functional inequalities considered in this paper are assumed to hold eventually,that is,they are satisfied for all t large enough.

2. Oscillation Theorems

In this section, we establish some new oscillation criteria for equation (1.1). We begin with some useful lemmas, which we intend to use later.

Lemma 2.1. *Let $x(t)$ be a positive solution of equation* (1.1)*. Then there are only the following three cases for $z(t)$:*

(I) $z(t) > 0, z'(t) > 0, z''(t) > 0, (a(t)(z''(t))^\alpha)' \leq 0;$

(II) $z(t) > 0, z'(t) < 0, z''(t) > 0, (a(t)(z''(t))^\alpha)' \leq 0;$

(III) $z(t) > 0, z'(t) > 0, z''(t) < 0, (a(t)(z''(t))^\alpha)' \leq 0$

for $t \geq t_1, where\ t_1$ *is sufficiently large.*

Proof. Assume that $x(t)$ is a positive solution of equation(1.1) on $[t_0, \infty)$. We see that $z(t) > 0$, and

$$[a(t)([z(t)]'')^{\alpha}]' = -q(t)x^{\alpha}(\sigma(t)) < 0. \tag{2.1}$$

Thus $a(t)[z''(t)]^{\alpha}$ is nonincreasing and of one sign. Therefore $z''(t)$ is also of one sign, and so we have either $z''(t) > 0$ or $z''(t) < 0$ for $t \geq t_1$. If $z''(t) < 0$, then $z'(t)$ is nonincreasing, and of one sign. Let $z''(t) < 0$ and $z'(t) < 0$ for $t \geq t_1$. Then one can easily see that $z(t) \to$ -∞ as $t \to \infty$, which is a contradiction. This contradiction proves that we have only three cases for $z(t)$. The proof is now complete. □

Lemma 2.2. *Let $x(t)$ be a positive solution of equation* (1.1), *and the corresponding $z(t)$ satisfies Case (II) of Lemma 2.1. If*

$$\int_{t_0}^{\infty}\int_{s}^{\infty}\left[\frac{1}{a(u)}\int_{u}^{\infty}q(v)dv\right]^{\frac{1}{\alpha}} duds = \infty, \tag{2.2}$$

then $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0.$

Proof. The proof can be found in [2]. □

Lemma 2.3. *Assume that $u(t) > 0, u'(t) > 0, u''(t) \leq 0$ on $[t_0, \infty)$. Then for each $\ell \in (0, 1)$ there exists a $T_\ell \geq t_0$ such that*

$$\frac{u(\sigma(t))}{\sigma(t))} \geq \ell\frac{u(t)}{t} \ for\ t \geq T_\ell, \tag{2.3}$$

and

$$u(t) \geq (t - T_\ell)u'(t)\ for\ t \geq T_\ell. \tag{2.4}$$

Proof. From the monotone property of $u'(t)$, we have

$$u(t) - u(\sigma(t)) \leq u'(\sigma(t))(t - \sigma(t)).$$

or

$$\frac{u(t)}{u(\sigma(t))} \leq 1 + \frac{u'(\sigma(t))}{u(\sigma(t)}(t - \sigma(t)). \tag{2.5}$$

Further

$$u(\sigma(t)) \geq u(\sigma(t)) - u(t_0) \geq u'(\sigma(t))(\sigma(t) - t_0).$$

So for each $\ell \in (0,1)$ there is a $T_\ell \geq t_0$ such that

$$\frac{u(\sigma(t))}{u'(\sigma(t))} \geq \ell\sigma(t), t \geq T_\ell. \tag{2.6}$$

Combining (2.5) *with* (2.6), we obtain

$$\frac{u(t)}{u(\sigma(t))} \leq 1 + \frac{1}{\ell\sigma(t)}(t - \sigma(t)) \leq \frac{t}{\ell\sigma(t)}$$

and the inequality (2.3) follows.
Again from the monotone property of $u'(t)$,we have

$$u(t) - u(T_\ell) = \int_{T_\ell}^{t} u'(s)ds \geq (t - T_\ell)u'(t), \tag{2.7}$$

and the inequality (2.4) follows from inequality (2.7). This completes the proof. □

Next , we present the oscillation results for equation (1.1). For simplicity,we introduce the following notations:

$$P_1 = \lim_{t\to\infty} \inf \frac{t^\alpha}{a(t)} \int_t^\infty P_\ell(s)ds, \ P_2 = \lim_{t\to\infty} \sup \frac{1}{t} \int_{t_0}^{t} \frac{s^{1+\alpha}}{a(s)} P_\ell(s)ds$$

$$A(t) = \int_t^\infty \frac{1}{a^{\frac{1}{\alpha}}(s)}ds, \ B(t) = q(t)(1-p)^\alpha \ell^\alpha \left(\frac{\sigma(t)}{t}\right)^\alpha (t - T_\ell)^\alpha$$

where

$$P_\ell(t) = \ell^\alpha (1-p)^\alpha q(t) \left(\frac{\sigma(t)}{t}\right)^\alpha \left(\frac{\sigma(t) - T_\ell}{2}\right)^\alpha \textit{with } \ell \in (0,1)$$

arbitrarily chosen and T_ℓ large enough. Further for $z(t)$ satisfying case (I),we define

$$w(t) = a(t)\left(\frac{z''(t)}{z'(t)}\right)^\alpha, \tag{2.8}$$

and

$$r = \lim_{t\to\infty} \inf \frac{t^\alpha w(t)}{a(t)} \textit{ and } R = \lim_{t\to\infty} sup \frac{t^\alpha w(t)}{a(t)}. \tag{2.9}$$

Lemma 2.4. *Assume that a(t) is nondecreasing.Let x(t) be a positive solution of equation*(1.1)*.*

(*a*) *Let* $P_1 < \infty$ *and* $P_2 < \infty$*.Suppose that the corresponding* $z(t)$ *satisfies the case* (I)*. Then* $P_1 \leq r - r^{1+\frac{1}{\alpha}}$ *and* $P_1 + P_2 \leq 1$*.*

(*b*) *If* $P_1 = \infty$ *or* $P_2 = \infty$*, then* $z(t)$*does not have the case* (I)*.*

Proof. The proof is similar to that of Lemma 6 of [2] and hence the details are omitted. □

Next,we present the oscillation criteria for equation(1.1).

Theorem 2.5. *Assume that condition* (2.2) *holds, and* $a(t)$ *is nondecreasing. If*

$$P_1 = \liminf_{t\to\infty} \int_t^{\infty} P_\ell(s)ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\ , \tag{2.10}$$

and

$$\lim_{t\to\infty} sup \int_{t_0}^{t} [A^\alpha(s)B(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{a^{\frac{1}{\alpha}}(s)A(s)}]ds = \infty, \tag{2.11}$$

then every solution of equation(1.1) *is either oscillatory or tends to zero as* $t \to \infty$*.*

Proof. Suppose that $x(t)$ is a positive solution of equation(1.1). If $P_1 = \infty$, then by Lemma 2.4, $z(t)$ does not have case(I) , that is, $z(t)$ has to satisfy (II) and (III) of Lemma 2.1. Assume that case (III) holds. From $(a(t)(z''(t)^\alpha)' \leq 0$, we have

$$a(s)(z''(s))^\alpha \leq a(t)(z''(t))^\alpha, s \geq t \geq t_1.$$

Dividing the inequality by $a(s)$, and integrating it from t to ℓ, we obtain

$$z'(\ell) \leq z'(t) + a^{\frac{1}{\alpha}}(t)z''(t) \int_t^{\ell} \frac{1}{a^{\frac{1}{\alpha}}}ds.$$

Letting $\ell \to \infty$, we have

$$0 \leq z'(t) + a^{\frac{1}{\alpha}}(t)z''(t)A(t)$$

or

$$-A(t)\frac{a^{\frac{1}{\alpha}}(t)z''(t)}{z'(t)} \le 1. \tag{2.12}$$

Define

$$\phi(t) = a(t)\left(\frac{z''(t)}{z'(t)}\right)^{\alpha}, t \ge t_1. \tag{2.13}$$

Then $\phi(t) < 0$ for all $t \ge t_1$. From (2.12) and (2.13) , we obtain

$$-A^{\alpha}(t)\phi(t) \le 1. \tag{2.14}$$

Differentiating (2.13) , we obtain

$$\phi'(t) = \frac{(a(t)(z''(t))^{\alpha})'}{(z'(t))^{\alpha}} - \alpha\frac{a(t)(z''(t))^{\alpha+1}}{(z'(t))^{\alpha+1}}. \tag{2.15}$$

From the property of $z'(t) > 0$, *we have* $x(t) \ge (1-p)z(t)$. From equation (1.1), we have

$$\phi'(t) \ge -q(t)(1-p)^{\alpha}\frac{z^{\alpha}(\sigma(t))}{(z'(t))^{\alpha}} - \alpha\frac{\phi^{1+\frac{1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}. \tag{2.16}$$

Using (2.3) and (2.4) in (2.16), we obtain

$$\phi'(t) \le -B(t) - \alpha\frac{\phi^{1+\frac{1}{\alpha}}(t)}{a^{\frac{1}{\alpha}}(t)}.$$

Multiplying the last inequality by $A^{\alpha}(t)$,and integrating it from $t_1 \to t$, we have $\phi(t)A^{\alpha}(t) - \phi(t_1)A^{\alpha}(t_1) + \int_{t_1}^{t} A^{\alpha}(s)B(s)ds + \int_{t_1}^{t} \alpha\frac{A^{\alpha-1}(s)\phi(s)}{a^{\frac{1}{\alpha}}(s)}ds$

$$+\int_{t_1}^{t} \alpha\frac{A^{\alpha}(s)\phi^{1+\frac{1}{\alpha}}(s)}{a^{\frac{1}{\alpha}}(s)}ds \le 0.$$

or

$\phi(t)A^{\alpha}(t) - \phi(t_1)A^{\alpha}(t_1) + \int_{t_1}^{t} A^{\alpha}(s)B(s)ds$

$$+\int_{t_1}^{t} \alpha\frac{A^{\alpha}(s)}{a^{\frac{1}{\alpha}}(s)}\left[\frac{\phi(s)}{A(s)} + \phi^{1+\frac{1}{\alpha}}(s)\right]ds \le 0.$$

Since

$$\frac{\phi(s)}{A(s)} + \phi^{1+\frac{1}{\alpha}}(s) \geq -\frac{1}{A^{\alpha+1}(s)} \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}$$

we have

$$\int_{t_1}^{t} \left[A^{\alpha}(s)B(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{a^{\frac{1}{\alpha}}(s)A(s)} \right] ds$$

$$\leq -\phi(t)A^{\alpha}(t) + \phi(t_1)A^{\alpha}(t_1). \tag{2.17}$$

Using (2.14) in (2.17), and then taking $t \to \infty$, we obtain

$$\int_{t_1}^{\infty} \left[A^{\alpha}(s)B(s) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{1}{a^{\frac{1}{\alpha}}(s)A(s)} \right] ds \leq 1 + \phi(t_1)A^{\alpha}(t_1)$$

a contradiction to (2.11).

If $z(t)$ satisfies case (II), then from Lemma 2.2 we see that $\lim_{t\to\infty} x(t) = 0$.

Next we assume that $P_1 < \infty$.We shall discuss three possibilities. If case(III) holds then as before we obtain a contradiction to (2.11). If case(II) holds then exactly as above we are led by Lemma 2.2 to $\lim_{t\to\infty} x(t) = 0$.

Now we assume that for $z(t)$ case(I) holds.Let w *and* r be defined by (2.8) and (2.9), respectively.Then from Lemma 2.4 we see that r satisfies the inequality

$$P_1 \leq r - r^{1+\frac{1}{\alpha}}.$$

Using the inequality

$$Bu - Au^{1+\frac{1}{\alpha}} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{B^{\alpha+1}}{A^{\alpha}}$$

with A= B = 1, we get that

$$P_1 \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}$$

which contradicts (2.10). This completes the proof. □

Theorem 2.6. *Assume that conditions* (2.2) *and* (2.11) *hold and a(t) is nondecreasing. If*

$$P_1 + P_2 > 1,$$

then every solution of equation (1.1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Assume that $x(t)$ is a positive solution of equation(1.1). If $P_1 = \infty$ *or* $P_2 = \infty$, then by Lemma 2.4, $z(t)$ does not have case (I),that is,$z(t)$ has to satisfy case (II) and case (III) of Lemma 2.1 If $z(t)$ satisfies case(III), then as in the proof of Theorem 2.5 we obtain a contradiction to condition (2.2). If $z(t)$ satisfies case (II), then by Lemma 2.2, we see that $\lim_{t\to\infty} x(t) = 0$.

Next,we assume that $P_1 < \infty$ *and* $P_2 < \infty$. We shall discuss three possibilities. If case (III) holds, then exactly as above we are led to contradiction to (2.11). In case (II) holds, then from Lemma 2.2, we see that $\lim_{t\to\infty} x(t) = 0$.

Now we assume that for $z(t)$ case(I) holds. Let w *and* r be defined by (2.8) and (2.9), respectively. Then from Lemma 2.4,we see that P_1 *and* P_2 satisfy the inequality$P_1 + P_2 \leq 1$. which is a contradiction. This completes the proof. □

Based on Corollaries 1 and 2 of [2], we obtain the following results as a consequence of Theorems 2.5 and 2.6.

Corollary 2.7. *Assume that* (2.2) *and* (2.11) *hold, and a(t) is nondecreasing. If*

$$\lim_{t\to\infty} \inf \frac{t^\alpha}{a(t)} \int_t^\infty q(s) \frac{(\sigma(s))^{2\alpha}}{s^\alpha} ds > \frac{(2\alpha)^\alpha}{(\alpha+1)^{\alpha+1}(1-p)^\alpha}$$

then every solution of equation(1.1) *is either oscillatory or tending to zero as* $t \to \infty$.

Corollary 2.8. *Assume that* (2.2) *and* (2.11) *hold, and* $a(t)$ *is nondecreasing . If*

$$P_2 = \lim_{t\to\infty} \sup \frac{1}{t} \int_{t_0}^t \frac{s^{\alpha+1}}{a(s)} P_\ell(s) ds > 1$$

then every solution of equation(1.1) *is either oscillatory or tends to zero as* $t \to \infty$

Next we consider $\alpha = 1, \delta(t) = t - \delta$ *and* $\sigma(t) = t - \sigma$ in equation (1.1). where δ *and* σ are nonnegative constants with $\sigma > \delta \geq 0$. In the following theorem we establish a sufficient condition for the oscillation of all solutions of equation (1.1)

Theorem 2.9. *Let a(t) be nondecreasing. If for* $\ell \in (0,1)$ *and* $T_\ell \geq t_0$

$$\lim_{t\to\infty} \inf \frac{t}{a(t)} \int_t^\infty q(s) \left(\frac{s-\sigma}{s}\right) (s - \sigma - T_\ell) ds > \frac{1}{2\ell(1-p)} \tag{2.18}$$

$$\limsup_{t\to\infty} \int_{t}^{t+\sigma-\delta} q(s) \left[\int_{t}^{s} \left(\int_{u}^{s} \frac{1}{a(v)} dv \right) du \right] ds > 1+p \tag{2.19}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[A(s)q(s) \left(\frac{s-\sigma}{s} \right) (s-T_\ell) - \frac{1}{4a(s)A(s)} \right] ds = \infty \tag{2.20}$$

then every solution of equation(1.1) *is oscillatory.*

Proof. Let $x(t)$ be a positive solution of equation(1.1). Then z(t) satisfies three cases of Lemma 2.1. Assume that case(I) holds. Then as in the proof of Theorem 2.5.We obtain a contradiction to condition (2.18). If case(II) holds then using the proof of [6,Theorem1,case(II)] we can get a contradiction due to condition (2.19). Finally if case(III) holds then as in the proof of Theorem 2.5,we obtain a contradiction to (2.20). This completes the proof. □

3. Examples

Here we present some examples to illustrate the main results.

Example 3.1. Consider the neutral differential equation

$$\left(e^t \left[\left(x(t) + \frac{1}{3} x\left(\frac{t}{2}\right) \right)'' \right]^3 \right)' + \frac{e^{2t}}{t^5} x^3 \left(\frac{t}{2}\right) = 0. \tag{3.1}$$

It is easy to see that all conditions of corollary 2.8 are satisfied and hence every solution of equation(3.1) is either oscillatory or tends to zero as $t \to \infty$.

Example 3.2. Consider the third order neutral differential equation

$$\left(e^t \left(x(t) + \frac{1}{2} x(t-2\pi) \right)'' \right)' + \frac{3\sqrt{2}}{2} e^t x \left(t - \frac{89\pi}{4} \right) = 0, \quad t \geq 1. \tag{3.2}$$

All conditions of Theorem 2.9 are satisfied and so every solution of equation (3.2) is oscillatory. In fact one such solution is $x(t) = \cos t$.

We conclude this paper with the following remarks.

If we relax condition (2.2) in all the results then the assertion of these results should be reformulated as: every solution of equation (1.1) is either oscillatory or bounded.Further the results presented here complement to that of in [2].

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