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DEFICIENCY OF FINITE SETS AND ZERO-DIMENSIONAL SCHEMES

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Abstract: Here we study zero-dimensional schemes $E \subset \mathbb{P}^r$ and finite sets $S \subset \mathbb{P}^r \setminus E_{red}$ such that $h^1(\mathcal{I}_{E\cup S}(d)) \ge e > 0$ and $\deg(E \cup S)$ is either small or $E = mP$ for some $m \ge 2$.

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1. Introduction

Let $G(1,r)$ be the set of all lines of $\mathbb{P}^r$. To study the minimum distance and the higher Hamming weights of the dual of evaluation codes it is quite useful to study the following problem. Fix integer $d > 0$, $e > 0$ and a zero-dimensional scheme $E \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_E(d)) = 0$. Find all finite sets $S \subset \mathbb{P}^r \setminus E_{red}$ such that $h^1(\mathcal{I}_{E\cup S}(d)) \ge e$ and $\sharp(S)$ is small (see [2] for the case $e = 1$ and $E = \emptyset$). We first prove the following result.

Theorem 1. *Fix integers $r \ge 2$, $d \ge 2$ and a zero-dimensional scheme $E \subset \mathbb{P}^r$ such that $\deg(E) \le 2d-1$ and $h^1(\mathcal{I}_E(d)) > 0$. Fix a finite set $B \subset \mathbb{P}^r \setminus E_{red}$. For any integer $e \ge 1$ let $\mathcal{S}(E,d,B,e)$ (resp. be $\mathcal{A}(E,d,B,e)$) denote the set of all $S \subseteq B$ such that $h^1(\mathcal{I}_{E\cup S}(d)) = e$ and $h^1(\mathcal{I}_{E\cup S'}(d)) < e$ for all $S' \subseteq S$ (resp. $h^1(\mathcal{I}_Z(d)) < e$ for any $Z \subsetneq E \cup S$).*

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(a) Let $\mathcal{G}$ be the set of all $L \in G(1,r)$ such that $\deg(L \cap (E \cup B)) \geq d+2$. If $\mathcal{G} = \emptyset$, then $\mathcal{S}(E,d,B,e)$ and $\mathcal{A}(E,d,B,e)$ have no element S such that $\sharp(S) \leq 2d + e - \deg(E)$.

(b) Let S be an element of $\mathcal{S}(E,d,B,e)$ such that $\sharp(S) \leq 2d + e - \deg(E)$. Then there is $L \in G(1,r)$ such that $S \subset L$ and $\deg(L \cap E) + \sharp(S) = d + 1 + e$.

It is easy to check that Theorem 1 implies the following result.

Corollary 1. *Fix integers $r \geq 2$, $d \geq 2$, $e_i > 0$, $1 \leq i \leq s$, lines $L_i \subset \mathbb{P}^r$, $1 \leq i \leq s$, zero-dimensional schemes $E_i \subset L_i$ such that $\deg(E_i) = e_i$ for all i and $E_i \cap L_j = \emptyset$ for all $i \neq j$. Set $E := E_1 \cup \cdots \cup E_s$ and $A := E_{red}$. Fix a finite set $B \subset \mathbb{P}^r$ such that $B \cap L_i = \emptyset$ for all i. Assume $e_1 + \cdots + e_s \leq 2d - 1$. If $\sharp(L \cap (B \cap A)) \leq d + 1$ for all $L \in G(1,r)$, then $h^1(\mathcal{I}_{E \cup S}(d)) = 0$ for all $S \subseteq B$ such that $\sharp(S) \leq 2d + 1 - \deg(E)$.*

The following one is the only case in which we are able to prove a result similar to Theorem 1 with $\deg(E) \gg d$. For any smooth point P of a variety V and any integer $m > 0$ let $\{mP, V\}$ be the $(m-1)$-th infinitesimal neighborhood of P in V, i.e. the closed subscheme of V with $(\mathcal{I}_P)^m$ as its ideal sheaf. We have $\{mP, V\}_{red} = \{P\}$ and $\deg(\{mP, V\}) = \binom{r+m-1}{r}$, where $r := \dim(V)$. If $V = \mathbb{P}^r$, then set $mP := \{mP, \mathbb{P}^r\}$.

Theorem 2. *Fix integers $r \geq 2$, $e > 0$, and $d > m > 0$ and $d \geq 2m - 3$. Fix $P \in \mathbb{P}^r$ and a finite set $S \subset \mathbb{P}^r \setminus \{P\}$ such that $= \sharp(S) \leq 2d - 2m + 2e - 1$. Set $E := mP$. We have $h^1(\mathcal{I}_{E \cup S}(d)) \geq e$ if and only if there is $L \in G(1,r)$ such that either $P \in L$ and $\sharp(S \cap L) \geq d + 1 + e - m$ or $P \notin L$ and $\sharp(S \cap L) \geq d + 1 + e$.*

We work over an algebraically closed base field $\mathbb{K}$.

2. The Proofs

Lemma 1. *Fix integers $d \geq 2$, $r \geq 2$, $L \in G(1,e)$, a zero-dimensional scheme $F \subset L$ and a zero-dimensional scheme $G \subset \mathbb{P}^r \setminus L$ such that $\deg(G) \leq d$. We have $h^1(\mathcal{I}_F(d)) = \max\{0, \deg(F) - d - 1\}$ and $h^1(\mathcal{I}_{F \cup G}(d)) = h^1(\mathcal{I}_F(d))$.*

Proof. Since the first equality is obvious, it is sufficient to prove the second one. Since $h^1(\mathcal{I}_{F \cup G}(d)) \geq h^1(\mathcal{I}_F(d))$, it is sufficient to prove $h^1(\mathcal{I}_{F \cup G}(d)) \leq h^1(\mathcal{I}_F(d))$. Set $Z := F \cup G$. For any hyperplane $H \subset \mathbb{P}^r$ we have the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(Z)}}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z \cap H, H}(d) \to 0 \tag{1}$$

First assume $r = 2$. Take $H := L$. We have $\mathrm{res}_H(F \cup G) = G$. Since $\deg(G) \leq d$, we have $h^1(\mathcal{I}_G(d-1)) = 0$. Hence (1) gives $h^1(\mathcal{I}_{F\cup G}(d)) \leq h^1(\mathcal{I}_F(d))$. Now assume $r \geq 3$ and that the lemma is true in $\mathbb{P}^{r-1}$. Let H be any hyperplane containing L. Since $Z \cap H = F \cup (G \cap H)$ and $\deg(G \cap H) \leq \deg(G) \leq d$, the inductive assumption gives $h^1(H, \mathcal{I}_{Z\cap H,H}(d)) = h^1(\mathcal{I}_F(d))$. Since $\mathrm{Res}_H(Z) = \mathrm{Res}_H(G) \subseteq G$, we have $\deg(\mathrm{Res}_H(Z)) \leq d$. Hence $h^1(\mathcal{I}_G(d-1)) = 0$. Apply again (1). □

Lemma 2. *Fix an integer* $e > 0$. *Let* $Z \subset \mathbb{P}^r$ *be a zero-dimensional scheme such that* $\deg(Z) \leq 2d + e$ *and* $h^1(\mathcal{I}_Z(d)) \geq e$. *Then* $\deg(Z) \geq d + 1 + e$ *and there is a line* $L \subset \mathbb{P}^r$ *such that* $\deg(L) \geq d + 1 + e$.

Proof. If $e = 1$, then this lemma is [1], Lemma 34. Hence we may assume $e \geq 2$ and that the lemma is true for the integer $e' := e - 1$. We may also use induction on the integer $\deg(Z)$, the case $\deg(Z) = 0$, being trivial. Let Z' be any zero-dimensional subscheme of Z such that $\deg(Z') = \deg(Z) - 1$. We have $e \geq h^1(\mathcal{I}_{Z'}(d)) \geq e - 1$. If $h^1(\mathcal{I}_{Z'}(d)) = e$, then we are done by the inductive assumption Hence we may assume $h^1(\mathcal{I}_{Z'}(d)) = e - 1$. By the inductive assumption of e there is $L \in G(1, r)$ such that $\deg(Z' \cap L) \geq d + e$. Since $h^1(\mathcal{I}_{Z'}(d)) = e - 1$, we have $\deg(Z' \cap L) = d + e$. Since $\deg(Z) - \deg(Z' \cap L) \leq d$, Lemma 1 gives $\deg(Z \cap L) \geq d + 1 + e$. □

Proof of Theorem 1. Since $\mathcal{A}(E, d, B, e) \subseteq \mathcal{S}(E, d, B, e)$, it is sufficient to prove part (b). Fix $S \in \mathcal{S}(E, d, B, e)$ such that $\deg(E) + \sharp(S) \leq 2d + 1 + e$. Lemma 2 gives the existence of a line L such that $\deg(E \cap L) + \sharp(S \cap L) \geq d + 1 + e$. Since $h^1(\mathcal{I}_{E\cup S}(d)) \geq h^1(\mathcal{I}_{E\cup S}(d))$, we also get $\deg(E \cap L) + \sharp(S \cap L) = d + 1 + e$. Since $h^1({}_{E\cup S}(d)) \geq h^1(\mathcal{I}_{E\cup(S\cap L}(d))$ and $S \in \mathcal{S}(E, d, B, e)$, we get $S = S \cap L$. □

Proof of Theorem 2. Since the " if " part is obvious, we will only prove the " only if " part. Assume $h^1(\mathcal{I}_{E\cup S}(d) \geq e$. Since $m \leq d + e$, we have $h^1(\mathcal{I}_E(d) < e$. Hence $S \neq \emptyset$. Until step (d) we assume $e = 1$. Set $Z := E \cup S$.

(a) Assume $r = 2$.

(a1) Notice that if μ is a positive integer, L is a line through P and A is a finite subset of $L \setminus \{P\}$, then $h^1(\mathcal{I}_{\mu P\cup A}(x)) = \max\{0, \sharp(A) + \mu - x - 1\}$ for every integer $x \geq \mu - 1$ (use that $h^2(\mathcal{I}_W(x-1)) = 0$ for every zero-dimensional scheme $W \subset \mathbb{P}^2$, that $\mathrm{Res}_L(\mu P \cup A) = (\mu - 1)P$ and hence $h^1(\mathcal{I}_{(\mu-1)P}(x-1)) = 0$ for every $x \geq \mu - 1$).

(a2) If there is a line L through P such that $\sharp(S \cap L) \geq d + 2 - m$, then we are done. Hence we may assume the non-existence of any such a line. By (a1) we get the non-existence of a line L through P such that $S \subset L$. Hence there are at least two points of S spanning a line not containing P. Set $S_0 := S$. Let R_1 be a line not though P and such that $e_1 := \sharp(R_1 \cap S_0)$ is maximal. Set $S_1 := S \backslash R \cap S_0$. Fix an integer $i \geq 2$ and assume for the moment defined for each $j \in \{1, \dots, j-1\}$ the lines R_j, $1 \leq j \leq i-1$, the integer $e_j := \sharp(S_{j-1} \cap R_j) \geq 2$ and the set $S_j := S_{j-1} \setminus S_{j-1} \cap R_j$. If S_{i-1} is contained in a line through P, then we say that we stop at the integer $i-1$. If S_{i-1} is not contained in a line through P, then we take a line R_i not through P such that the integer $e_i := \sharp(S_{i-1} \cap R_i)$ is maximal and set $S_i := S_{i-1} \setminus S_{i-1} \cap R_i$. Notice that in the latter case we have $e_1 \geq \cdots \geq e_i \geq 2$ and $e_1 + \cdots e_i \leq \sharp(S)$. Hence the construction stop at an integer $c+1$ for some c such that $1 \leq c \leq d-m$. We have $\sharp(S) = e_1 + \cdots + e_c + \sharp(S_{c+1})$ and there is a line R through P containing S_{c+1} (the line R is unique if $S_{c+1} \neq \emptyset$). We have $\mathrm{Res}_{R_1 \cup \cdots \cup R_c}(Z) = mP \cup S_{c+1}$. Look at the the residual exact sequences

$$0 \to \mathcal{I}_{mP \cup S_i}(d-i) \to \mathcal{I}_{mP \cup S_{i-1}}(d-i+1) \to \mathcal{I}_{R_i \cap S_{i-1}, R_i}(d-i+1) \to 0 \quad (2)$$

First assume the existence of an integer $i \in \{1, \dots, c\}$ such that $e_i \geq d+3-i$ and call x the minimal integer with this property. We have $e_i \geq e_x \geq d+3-x$ if $1 \leq i \leq x$ and $e_i \geq 2$ if $x+1 \leq x \leq c$. Hence $\sharp(S) \geq e_1 + \cdots + e_c \geq x(d+3-x) + 2(c-x)$. If $x = 1$, then we are done, because $\sharp(S \cap R_1) = e_1 \geq d+2$. Now assume $x \geq 2$. The function $y \mapsto y(d+3-y)$ is increasing if $y \leq (d+3)/2$ and decreasing if $y > (d+3)/2$. Hence if $2 \leq x \leq (d+3)/2$, then $\sharp(S) \geq 2d$, a contradiction. Now assume $x > (d+3)/2$. We get $\sharp(S) \geq c(d+3-c)$. Since $\sharp(S) \leq 2d-2m+1$ and $d \geq 2m-3$, we get $c \geq (2d-2m+2)/2$ and hence $\sharp(S) \geq 2d-2m+2$, a contradiction.

Now assume $e_i \leq d+2-i$ for all $i \in \{1, \dots, c\}$. From the exact sequences (2) with $i \in \{1, \dots, c\}$ we get $h^1(\mathcal{I}_{mP \cup S}(d)) \leq h^1(\mathcal{I}_{mP \cup S_{c+1}}(d-c)) > 0$. Step (a1) gives $\sharp(S_{c+1}) \geq d-c-m+3$.

Let L_1 be the line through P such that $f_1 := \sharp(L_1 \cap S)$ is maximal. Set $S(1) := S \setminus S \cap L_1$. For each integer $i \geq 2$ define inductively the integer f_i, the line L_i and the set $S(i)$ in the following way. Let L_i be any line through P such that $f_i := \sharp(S(i-1) \cap L_i)$ is maximal. Set $S_i := S_{i-1} \setminus S_{i-1} \cap L_i$. Hence $\sum_{i \geq 1} f_i = \sharp(S)$. Let $z \geq 1$ be the largest integer such that $f_z \geq 2$. First assume the existence of an integer $i \leq z$ such that $f_i \geq d+3-m$ and call y the minimal such an integer. By assumption we have $y \geq 2$. We have $\sharp(S) \geq y(d+3-m) > 2d+2m+1$, absurd. Now assume $f_i \leq d+2-m$ for all $i \in \{1, \dots, z\}$. Until step (a3) we assume $z \leq m$. We have $\mathrm{Res}_{L_i}((m-i+1)(P) \cup S(i-1)) = (m-i)P \cup S(i)$.

From the exact sequences

$$0 \to \mathcal{I}_{(m-i)P\cup S(i)}(d-i) \to \mathcal{I}_{(m-i+1)P\cup S(i)}(d-i+1) \to \mathcal{I}_{\{m-i+1)P_i\cup(S(i-1)\cap L_i,L_i}(d-i+1) \to 0 \quad (3)$$

for $i = 1, \dots, z$, we get $h^1(\mathcal{I}_{(m-z)P\cup S(z)}(d+1-z)) > 0$. Since $f_{z+1} \le 1$, induction on $d-m$ gives the existence of a line T such that $P \notin T$ and $\sharp(T \cap S(z)) \ge d+3-z$. Hence $e_1 \ge d+3-z$. Hence $e_1+\cdots+e_c \ge d+3-z+2c-2$. Since $\sharp(S_{c+1}) \ge d-c-m+3$, we get $\sharp(S) \ge d-c-m+3+d+3-z+2c-2 > 2d-2m+1$, a contradiction!

(a3) Now we assume $z > m$ and $f_i \le d+2-m$ for all $i \le z$. In this case we get $h^1(\mathcal{I}_{S(z)}(d+1-z)) > 0$. Hence $\sharp(S(z)) \ge d+3-z$ and either $\sharp(S(z)) \ge 2d+4-2z$ or there is a line T' such that $\sharp(S\cap T') \ge d+3-z$ (see [1], Lemma 34). In the latter case we get $P \notin T'$, because $f_{z+1} \le 1$ and conclude as above. If $\sharp(S(z)) \ge 2d+4-2z$, then $\sharp(S) \ge 2d$, because $f_i \ge 2$ for all $i \le z$.

(b) Here we assume $r > 2$ but that there is a plane $M \subset \mathbb{P}^r$ such that $\{P\}\cup S \subset M$. If $r \ge 4$ we also use induction on r (under the further assumption of the existence of a plane M as above). Fix any hyperplane $H \subset \mathbb{P}^r$ containing M (hence $H = M$ if $r = 3$). We have $\mathrm{Res}_H(Z) = (m-1)P$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(d-1)) = 0$. By the residual exact sequence (1) we get $h^1(H, \mathcal{I}_{Z\cap H,H}(d)) > 0$. the inductive assumption gives the existence of a line. $L \subset H$ such that either $P \in L$ and $\sharp(S\cap L) \ge d+2-m$ or $P \notin L$ and $\sharp(S\cap L) \ge d+2$.

(c) Here we assume $r > 2$. Fix a general $(r-3)$-dimensional linear subspace V of $\mathbb{P}^r$. Since V is general we may assume that V is disjoint from any line spanned by two points of $S\cup\{P\}$ and from any plane spanned by 3 points of $S\cup\{P\}$ (if there is at least one such a plane). Hence the linear projection $\pi_V : \mathbb{P}^r \setminus V \to \mathbb{P}^2$ induces an injective map from $S\cup\{P\}$ into a finite subset of $\mathbb{P}^2$. Choose homogeneous coordinates $x_0, \dots, x_r$ of $\mathbb{P}^r$ such that V has $x_i = 0$, $3 \le i \le r$, as its equations. For each $\lambda \in \mathbb{K}\setminus\{0\}$ the map $h_\lambda : \mathbb{P}^r \to \mathbb{P}^r$ defined by the formula $h_\lambda(x_0, x_1, x_2, x_3, \dots, x_r) = (x_0, x_1, x_2, \lambda x_3, \dots, \lambda x_3)$ is an automorphism and hence $h^1(\mathcal{I}_{h_\lambda(Z)}(d)) = h^1(\mathcal{I}_{E\cup S}(d)) > 0$ for all $\lambda \ne 0$. When λ goes to 0, h_λ tends to π_V, in which we see $\mathbb{P}^2$ as a plane of $\mathbb{P}^r$. By semicontinuity we get $h^1(\mathbb{P}^r, \mathcal{I}_{m\pi_V(P)\cup\pi_V(S)}(d)) > 0$. Apply part (b) to $m\pi_V(P)\cup\pi_V(S)$. We get the existence of a line $L' \subset \mathbb{P}^2$ such that either $\pi_V(P) \in L'$ and $\sharp(\pi_V(S)\cap L') \ge d+2-m$ or $\sharp(\pi_V(S)\cap L') \ge d+2$. Since V is disjoint from any plane spanned by 3 points of $S\cup\{P\}$, there is a unique line

$L \subset \mathbb{P}^3$ containing two of the points mapped onto $L' \cap \pi_V(S \cup \{P\})$ and for this line L either $P \in L$ and $\sharp(S \cap L) \geq d+2-m$ or $P \notin L$ and $\sharp(S \cap L) \geq d+2$.

(d) Now assume $e \geq 2$ and that Proposition 2 is true for all integers $e' \in \{1, \dots, e-1\}$. We also use induction on the integer $\sharp(S)$. Fix any $S' \subset S$ such that $\sharp(S') = \sharp(S) - 1$. We have $h^1(\mathcal{I}_{E \cup S}(d)) \geq h^1(\mathcal{I}_{E \cup S'}(d)) \geq h^1(\mathcal{I}_{E \cup S}(d)) - 1$. By the inductive assumption there is $L_{S'} \in G(1, r)$ such that either $P \in L_{S'}$ and $\sharp(S' \cap L) \geq d + e - m$ or $P \notin L_{S'}$ and $\sharp(S' \cap L_{S'}) \geq d + e$. In both cases $\sharp(S') \geq 2$ and hence $\sharp(S) \geq 3$. Fix $S', S_1 \subset S$ such that $\sharp(S') = \sharp(S_1) = \sharp(S) - 1$ and S_1 does not contain one of the points of $S' \cap L_{S'}$. If $L_{S_1} = L_{S'}$, then we get $\sharp(S \cap L) = \sharp(S' \cap L) + 1$, concluding the proof in this case. If $L_{S_1} \neq L_{S'}$, then easily we get $\sharp(S \cap (L_{S'} \cup L_{S_1}) \geq 2d + 2e - 2m$. $\square$

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