International Journal of Pure and Applied Mathematics
Volume 78 No. 1 2012, 97-103
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

EXISTENCE OF SOLUTIONS OF NONLINEAR OPERATOR IMPULSE DIFFERENTIAL EQUATIONS WITH GENERALIZED DICHOTOMOUS LINEAR PART IN A BANACH SPACE

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Abstract: By the help of a generalized dichotomy we introduce and consider nonlinear operator impulse differential equations in an arbitrary Banach space.

AMS Subject Classification: 34G10, 34H20, 47H10
Key Words: nonlinear impulse differential equations, generalized dichotomy, Banach fixed point principle

1. Introduction

This paper marks the beginning of the consideration of nonlinear operator impulse differential equations in an arbitrary Banach space. By help of the fixed point principle of Banach we find sufficient conditions for the existence of a

2. Problem Statement

Let X be an arbitrary Banach space with identity I and norm $|.|$. Let $L(X)$ be the space of all linear bounded operators acting in X with norm $||.||$. Let be $\mathbb{R}_+ = [0, \infty)$. Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of points in $\mathbb{R}_+$ such that $\lim_{n\to\infty} t_n = \infty$.

Received: March 21, 2012

We consider the nonlinear operator impulse differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x + F(t, x, Tx) \quad \text{for } t \neq t_n \tag{1}$$

$$x(t_n^+) = Q_n x(t_n) + h_n(x(t_n)) \quad \text{for } n = 1, 2, ..., \tag{2}$$

where $A(.) : \mathbb{R}_+ \to L(X)$ is continuous operator function , $\{Q_n\}_{n=1}^{\infty}$ is a sequence of operators such that $Q_n \in L(X)$ $(n = 1, 2, ...)$ and the operator $T \in L(X)$.

Furthermore, we assume that all considered functions are left continous.

Let $V(t, s)$ $(0 \leq s < t < \infty)$ is the Cauchy operator of the linear impulse differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x \quad \text{for } t \neq t_n \tag{3}$$

$$x(t_n^+) = Q_n x(t_n) \quad \text{for } n = 1, 2, ..., \tag{4}$$

We set $V(t) = V(t, 0)$ $(t \in \mathbb{R}_+)$. Let the operators Q_n $(n = 1, 2, ...)$ have bounded inverse ones, then there exists $V^{-1}(t)$ and $V(t, s) = V(t)V^{-1}(s)$.

Remark 1. Integral equivalence between the equations (1), (2) and (3), (4) is considered in [3].

Let $\varphi_1(.), \varphi_2(.), \psi_1(.), \psi_2(.) : \mathbb{R}_+ \to \mathbb{R}_+$ be positive continuous functions.

Definition 1. The linear impulse differential equation (3), (4) is said to be (φ, ψ, R)- dichotomous if there exists an linear operator family $R(t) : X \to X$ for any $t \in \mathbb{R}_+$ such that

$$||V(t)R(s)V^{-1}(s)|| \leq \varphi_1(t)\varphi_2(s) \quad for\ t \geq s \geq 0$$

$$||V(t)(I - R(s))V^{-1}(s)|| \leq \psi_1(t)\psi_2(s) \quad for\ 0 \leq t \leq s$$

Remark 2. (φ, ψ, R)- dichotomy for differential equations without impulse is introduced and considered in [1],[2].

Lemma 1. *Let the impulse differential equation (3), (4) be* (φ, ψ, R)-

dichotomous. Then the solutions of the integral equation

$$x(t) = V(t)x(0) + \int_0^t V(t)R(s)V^{-1}(s)F(s,x(s),Tx(s))ds -$$
$$- \int_t^\infty V(t)(I-R(s))V^{-1}(s)F(s,x(s),Tx(s))ds +$$
$$+ \sum_{0<t_n<t} V(t)R(t_n^+)V^{-1}(t_n^+)h_n(x(t_n)) -$$
$$- \sum_{t_n>t} V(t)(I-R(t_n^+))V^{-1}(t_n^+)h_n(x(t_n))$$

are solutions of the nonlinear operator impulse differential equation (1), (2).

3. Main Results

We denote

$$|||x||| = \sup_{t\in\mathbb{R}_+} |x(t)|.$$

Let $\rho > 0$ and $B_\rho = \{x \in X : |x| \le \rho\}$.

Theorem 1. *Let the following conditions hold:*

1. The linear impulse differential equation (3), (4) is (φ,ψ,R)*-dichotomious with* $R(t) \in L(X)$ $(t \in \mathbb{R}_+)$.

2. There exists a positive constant $K > 0$*, such that*

$$||V(t)|| \le K \quad \text{for } t \in \mathbb{R}_+$$

3. There exists a positive constant $\mu > 0$*, such that*

$$\sup_{t\in\mathbb{R}_+} \int_0^\infty g(t,s)ds < \mu$$

and

$$\sup_{t\in\mathbb{R}_+} \sum_{t_n>0} g(t,t_n^+) < \mu,$$

where

$$g(t,s) = \begin{cases} \varphi_1(t)\varphi_2(s) & ,t \ge s \ge 0 \\ \psi_1(t)\psi_2(s) & ,0 \le t < s \end{cases}$$

4. For any $\rho > 0$, there exist a positive constant M such that

$$|Tx| \leq M|x| \quad \text{for every } x \in B_\rho$$

5. For any $\rho > 0$, there exist positive constants C, Q for which

$$C < (1 + M) < \mu^{-1}$$

and

$$|F(t, x, Tx)| \leq C(|x| + |Tx|) + Q \quad \text{for any} \quad x \in B_\rho \quad \text{and} \quad t \in \mathbb{R}_+$$

6. For any $t \in \mathbb{R}_+$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ there exists a cositive constant L such that

$$|F(t, x_1, y_1) - F(t, x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$$

7. For any $\rho > 0$ there exist positive constants C_n, Q_n $(n = 1, 2, ...)$ with

$$\tilde{C} = \max_{n \in \mathbb{N}} C_n \text{ and } \tilde{C} < \mu^{-1} \quad \text{for which}$$

$$|h_n(x)| < C_n|x| + Q_n \quad \text{for every } x \in B_\rho$$

8. For any $t \in \mathbb{R}_+$ and $x_1, x_2 \in X$ there exist positive constants $L_n (n = 1, 2, ...)$ for which

$$|h_n(x_1) - h_n(x_2)| \leq L_n|x_1 - x_2|$$

for each

$$\tilde{L} = \max_{n \in \mathbb{N}} L_n < \infty.$$

9. The constants $K, \mu, L, \tilde{L}$ fulfilled the following inequality

$$K + L\mu(1 + M) + \tilde{L}\mu < 1$$

Then there exists a positive constant ρ such that for every $|x(0)| < \frac{\rho}{3K}$, the nonlinear operator impulse differential equation (1),(2) has a solution $x(t) \in B_\rho$ $(t \in \mathbb{R}_+)$.

*Proof.*We consider the operators

$$(Q_1 x)(t) = \int_0^t V(t)R(s)V^{-1}(s)F(s, x(s), Tx(s))ds -$$

$$- \int_t^\infty V(t)(I - R(s))V^{-1}(s)F(s, x(s), Tx(s))ds$$

and

$$(Q_2x)(t) = \sum_{0<t_n<t} V(t)R(t_n^+)V^{-1}(t_n^+)h_n(x(t_n)) -$$
$$- \sum_{t_n>t} V(t)(I - R(t_n^+))V^{-1}(t_n^+)h_n(x(t_n))$$

We shall proof that there exists a positive constant ρ, such that the operators $Q_1, Q_2 : B_{\frac{\rho}{3}} \to B_{\frac{\rho}{3}}$.

Let suppose that for any $k \in \mathbb{N}$ there exists $x_k \in B_k$ such that $|(Q_1x_k)(t)| > k$ and $|(Q_2x_k)(t)| > k$.

By condition 5 of Theorem 1 there exists $N_1 \in \mathbb{N}$ sufficiently large such that if $k \geq N_1$ then

$$\frac{|F(t, x_k, Tx_k)|}{k} \leq C(1 + M) + \frac{Q}{k} < \mu^{-1}$$

and

$$\frac{|(Q_1x_k)(t)|}{k} \leq \frac{1}{k} \int_0^t \varphi_1(t)\varphi_2(s)|F(s, x_k(s), Tx_k(s)|ds +$$
$$+ \frac{1}{k} \int_0^t \psi_1(t)\psi_2(s)|F(s, x_k(s), Tx_k(s)|ds \leq$$
$$\leq (C(1 + M) + \frac{Q}{k}) \int_0^\infty g(t, s)ds \leq$$
$$\leq (C(1 + M) + \frac{Q}{k})\mu < 1$$

So $\lim\limits_{k\to\infty} \sup \frac{|(Q_1x_k)(t)|}{k} < 1$ contradict $|(Q_1x_k)(t)| > k$. Hence there exists $N_1 \in \mathbb{N}$ such that $Q_1 : B_{N_1} \to B_{N_1}$.

By condition 7 of Theorem 1 there exists $N_2 \in \mathbb{N}$ sufficiently large such that if $k \geq N_2$ then

$$\frac{|h_n(x_k(t_n))|}{k} < \tilde{C} + \frac{\tilde{Q}}{k} < \mu^{-1}$$

where

$$\tilde{Q} = \max_{n\in\mathbb{N}} Q_n$$

and

$$\frac{|(Q_2x_k)(t)|}{k} \leq \frac{1}{k} \sum_{0<t_n<t} \varphi_1(t)\varphi_2(t_n^+)|h_n(x_k(t_n))| +$$

$$+\frac{1}{k}\sum_{t_n>t}\psi_1(t)\psi_2(t_n^+)|h_n(x_k(t_n))| \le$$
$$\le (\tilde{C}+\frac{\tilde{Q}}{k})\sum_{0<t_n} g(t,t_n^+) \le (\tilde{C}+\frac{\tilde{Q}}{k})\mu < 1.$$

So $\lim_{k\to\infty}\sup\frac{|(Q_2x_k)(t)|}{k} < 1$ contradict $|(Q_2x_k)(t)| > k$. Hence there exists $N_2 \in \mathbb{N}$ such that $Q_2 : B_{N_2} \to B_{N_2}$.

Let be $\frac{\rho}{3} = \min\{N_1, N_2\}$. Then

$$Q_1, Q_2 : B_{\frac{\rho}{3}} \to B_{\frac{\rho}{3}}. \tag{5}$$

We consider the operator

$$(Qx)(t) = V(t)x(0) + (Q_1x)(t) + (Q_2x)(t)$$

From condition 2 of Theorem 1 and (5) for every $|x(0)| < \frac{\rho}{3k}$ we obtain for the operator$Q$ that $Q : B_\rho \to B_\rho$.

We shall proof that Q is a contracting map in the B_ρ.

Let $x_1, x_2 \in B_\rho$. Then by conditions 4,6 and 8 of Theorem 1 we obtain

$$\begin{aligned}
|(Qx_1)(t) - (Qx_2)(t)| \le\ & ||V(t)|||x_1(0) - x_2(0)|+ \\
& + \int_0^\infty g(t,s)|F(s,x_1(s),Tx_1(s)) - F(s,x_2(s),Tx_2(s))|ds+ \\
& + \sum_{0<t_n} g(t,t_n^+)|h_n(x_1(t_n)) - h_n(x_2(t_n))| \le \\
& \le K|||x_1 - x_2||| + \int_0^\infty g(t,s)L|x_1(s) - x_2(s)|(1+M)ds+ \\
& + \sum_{0<t_n} g(t,t_n^+)L_n|x_1(t_n) - x_2(t_n)| \le \\
& K|||x_1 - x_2||| + L(1+M)|||x_1 - x_2|||\int_0^\infty g(t,s)ds+ \\
& + \tilde{L}|||x_1 - x_2||| + \sum_{0<t_n} g(t,t_n^+) \le \\
& \le |||x_1 - x_2|||(K + L\mu(1+M) + \tilde{L}\mu).
\end{aligned}$$

By condition 9 of Theoren 1 it follows that the operator Q is a contraction in the ball B_ρ. □

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