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POINTS OF SYMMETRIC CONTINUITY OF REAL FUNCTIONS

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Abstract: We examine the set of points where a real function is symmetric or symmetrically continuous but not continuous. Among other things, we show that if G is a proper additive subgroup of the reals, then there exists a real function f with two-element range such that the set of points where f is symmetrically continuous but not continuous is the additive subgroup G. The above statement is not true if symmetrically continuous is replaced by symmetric. However, there exists a real function f with three-element range such that the set of points where f is symmetric but not continuous is the additive subgroup G. In both results, G can not be replaced by $\mathbb{R}$.

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1. Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be symmetric at $x \in \mathbb{R}$, if $\lim_{h\to 0}[f(x+h) + f(x-h) - 2f(x)] = 0$. f is said to be symmetrically continuous at $x \in \mathbb{R}$, if $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$. It is shown in [8] that if $f : \mathbb{R} \to \mathbb{R}$ is symmetrically continuous on a residual set R, then it is continuous at all

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points of R except on a set of first category. It follows from a theorem in [1] that if a function is symmetrically continuous on a measurable set E, then it is continuous at all points of E except on a set of measure zero. A special case of the above results is that if A is a residual set in $\mathbb{R}$, then there does not exist a function $f : \mathbb{R} \to \mathbb{R}$ such that the set of points where f is symmetrically continuous but not continuous is A. For the function f, defined by $f(x) = \frac{1}{q}$if $x = \frac{p}{q} \in \mathbb{Q}$ and $\gcd(p, q) = 1$, and $f(x) = 0$ if $x \in \mathbb{R}\backslash\mathbb{Q}$, the set of points where f is symmetrically continuous but not continuous is $\mathbb{Q}$. Note that $\mathbb{Q}$ is an additive subgroup of $\mathbb{R}$. We prove that, for any proper additive subgroup G of $\mathbb{R}$, there exists a function $f : \mathbb{R} \to \{0, 1\}$ such that the set of points where f is symmetrically continuous but not continuous is G. The above statement is not true if symmetrically continuous is replaced by symmetric. However, we prove that if G is a proper additive subgroup of $\mathbb{R}$, then there exists a function $f : \mathbb{R} \to \{0, 1, 2\}$ such that the set of points where f is symmetric but not continuous is G. We also prove that $\mathbb{R}$ is a union of countably many translates of a set that contains no arithmetic progression of length three, which implies that there exists a nowhere symmetric function with countable range.

2. Notations and Definition

$\mathbb{R}$ is the set of all real numbers and $\mathbb{Q}$ is the set of all rational numbers. If A and B are subsets of $\mathbb{R}$, then the symbols $A + B$ and $A\backslash B$ stand for the sets $\{a + b : a \in A \text{ and } b \in B\}$ and $\{a \in A : a \notin B\}$, respectively.

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

f is said to be symmetrically continuous at $x \in \mathbb{R}$, if $\lim_{h\to 0}[f(x + h) - f(x - h)] = 0$.

f is said to be symmetric at $x \in \mathbb{R}$, if $\lim_{h\to 0}[f(x+h)+f(x-h)-2f(x)] = 0$.

The symbols $SC(f)$ and $S(f)$ denote the set of all points where f is symmetrically continuous and the set of all points where f is symmetric, respectively. The set of all points where f is continuous is denoted by $C(f)$.

Definition 2.2. A basis for the vector space $\mathbb{R}$ over $\mathbb{Q}$ is called a Hamel basis.

Definition 2.3. A subset of the reals is of first category if it can be written as a countable union of nowhere dense sets. A set is of second category if it is not of first category. The complement of a first category set is called a residual set.

Definition 2.4. A function $f : \mathbb{R} \to \mathbb{R}$ is additive if $f(x+y) = f(x)+f(y)$ for all $x, y \in \mathbb{R}$.

3. Results

Theorem 1. *If G is a proper additive subgroup of the reals, then there exists a function $f : \mathbb{R} \to \{0,1\}$ such that the set of points where f is symmetrically continuous but not continuous in the ordinary sense is the set G. That is, $SC(f)\backslash C(f) = G$.*

First, we prove the following two lemmas.

Lemma 1. *If there is a limit point of G, then every real number is a limit point of G.*

Proof. Suppose that a real number a is a limit point of G. Then there exists a sequence (g_n) of distinct elements of G such that $\lim_{n\to\infty} g_n = a$. Since the sequence $(g_{n+1} - g_n)$ converges to zero and $0 \neq g_{n+1} - g_n \in G$, 0 is a limit point of G.

Let $0 < r \in \mathbb{R}$ and let (b, c) be a nonempty open interval containing r. Then $0 \neq |g| < c - r$ for some $g \in G$. Let n be the smallest positive integer for which $n|g| \geq c$. Consequently, $c > (n-1)|g| = n|g| - |g| > c - (c-r) = r > b$ and $r \neq (n-1)|g| \in (b, c) \cap G$. This shows that every positive real number is a limit point of G. Since $G = -G$, every negative real number is a limit point of G. Thus, $G^{/} = \mathbb{R}$. □

Lemma 2. *Every real number is a limit point of $\mathbb{R}\backslash G$.*

Proof. By way of contradiction assume that r is not a limit point of $\mathbb{R}\backslash G$ for some real number r. Then there exists an open interval I containing r such that $I \cap (\mathbb{R}\backslash G) \subseteq \{r\}$. So, for some nonempty open interval $J \subset I$, we have $J \subset G$ and $\mathbb{R} = \bigcup_{n\in\mathbb{N}} n(J - J) \subseteq G$, which contradicts that G is a proper subgroup of $\mathbb{R}$. Thus, $(\mathbb{R}\backslash G)^{/} = \mathbb{R}$. □

Proof of Theorem 1. Case 1. $G^{/} \neq \emptyset$. Define $f : \mathbb{R} \to \{0,1\}$by $f(x) = 0$ if $x \in 2G = \{2g : g \in G\}$, and $f(x) = 1$if $x \in \mathbb{R}\backslash 2G$.

For $g \in G$ and $h \in \mathbb{R}$, $g + h \in 2G \Leftrightarrow -(g-h) = -2g + (g+h) \in 2G \Leftrightarrow g - h \in 2G$. This implies that f is symmetrically continuous at every point of G.

To see that f is not symmetrically continuous for any point $b \in \mathbb{R}\backslash G$, let (k_n) be a sequence of distinct elements of $2G$ converging to b (see Lemma 1).

Let $h_n = k_n - b$. Then (h_n) converges to 0, $b + h_n = k_n \in 2G$ and $b - h_n = 2b - k_n \notin 2G$. So, the function f is not symmetrically continuous at every point of $\mathbb{R}\backslash G$.

By Lemmas 1 and 2, $(2G)^{/} = \mathbb{R}$ and $(\mathbb{R}\backslash 2G)^{/} = \mathbb{R}$. This implies that f is discontinuous at every point of $\mathbb{R}$.

Case 2. $G^{/} = \emptyset$. Define $f : \mathbb{R} \to \{0, 1\}$by $f(x) = 0$ if $x \in G$ and $f(x) = 1$if $x \in \mathbb{R}\backslash G$.

For every $x \in \mathbb{R}\backslash G$, $\exists h > 0$such that $(x - h, x + h) \cap G = \emptyset$ and hence f is continuous at every point of $\mathbb{R}\backslash G$.

For $x \in G$, $\exists h > 0$ such that $(x - h, x + h) \cap G = \{x\}$. This implies that f is symmetrically continuous at x and discontinuous at x.

Thus, the set of points where f is symmetrically continuous but not continuous is the set G. □

Remark 1. The conclusion of the above theorem is not true if symmetrically continuous is replaced by symmetric. For, if $f : \mathbb{R} \to \{0, 1\}$, then $S(f) = C(f)$, because $f(x + h) + f(x - h) - 2f(x) = 0$ if and only if $f(x + h) = f(x - h) = f(x)$.

However, we have the following result.

Theorem 2. *If G is a proper additive subgroup of the reals, then there exists a function $f : \mathbb{R} \to \{0, 1, 2\}$ such that the set of points where f is symmetric but not continuous in the ordinary sense is the set G. That is*

$$S(f)\backslash C(f) = G.$$

Proof. An equivalence relation ˜ on $\mathbb{R}\backslash G$ is defined by $a\tilde{}b$ if $a - b \in 2G$. Distinct equivalence classes are disjoint. Pick an element from each distinct equivalence class and let $K = \{k_i : 1 \leq i < \xi\}$ be the collection of all such elements. Then $\mathbb{R}\backslash G = \bigcup_{1 \leq i < \xi} (2G + k_i)$. Define $f : \mathbb{R} \to \mathbb{R}$ as follows.

(1) $f(x) = 1 \forall x \in G$.

(2) By the definition of K, $k \notin G$ for every $k \in K$. For each $k_i \in K$, there exists a unique $k_j \in K$ such that $-k_i \in (2G + k_j)$ and $k_i \neq k_j$. So, K can be written as a disjoint union of sets of the form $\{k_i, k_j\}$, where $k_i + k_j \in 2G$ and $k_i \neq k_j$.

To each such pair $\{k_i, k_j\}$ and for each $x \in G$, define f as $\{f(2x + k_i), f(2x + k_j)\} = \{0, 2\}$.

Claim 1. *For $x \in G$, f is symmetric and discontinuous at x.*

Proof of Claim 1. For $x \in G$ and $h \in G$, $f(x+h) = f(x-h) = f(x) = 1$.

For $x \in G$ and $h \notin G$, there exist unique k_i, k_j in K such that $x+h \in 2G+k_i$ and $x - h \in 2G + k_j$. Consequently, $k_i + k_j \in 2G$, $k_i \neq k_j$, $\{f(x+h), f(x-h)\} = \{0, 2\}$ and $f(x) = 1$. In both cases $f(x+h)+f(x-h)-2f(x) = 0$, which implies that f is symmetric at each $x \in G$.

By Lemma 2, there exists a sequence (x_n) of distinct elements of $\mathbb{R}\backslash G$ such that $\lim_{n\to\infty} x_n = x$, where $x \in G$. Since $f(x_n) \in \{0, 2\}$and $f(x) = 1$, f is discontinuous at each $x \in G$. □

Claim 2. *For $x \in \mathbb{R}\backslash G$, f is symmetric at x if and only if f is continuous at x.*

Proof of Claim 2. For $x \in \mathbb{R}\backslash G$, we have $f(x) \in \{0, 2\}$ and, since $f(\mathbb{R}) = \{0,1, 2\}$, $f(x+h) + f(x-h) - 2f(x) = 0 \Longleftrightarrow f(x+h) = f(x-h) = f(x)$. □

By Claims 1 and 2, $S(f)\backslash C(f) = G$. This concludes the proof of the theorem. □

Remark 2. It follows from the following theorem that Theorem 1 is not true if G is replaced by $\mathbb{R}$.

Theorem 3. (see [8]) *For any function $f : \mathbb{R} \to \mathbb{R}$, $SC(f)\backslash C(f)$ is not residual in $\mathbb{R}$. In particular, $SC(f)\backslash C(f) \neq \mathbb{R}$. In other words, there does not exist an everywhere symmetrically continuous function that is nowhere continuous.*

The following example shows that Theorem 3 is not true if $SC(f)$ is replaced by $S(f)$.

Example 1. There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $S(f)\backslash C(f) = \mathbb{R}$. In other words, there exists an everywhere symmetric function that is nowhere continuous.

Proof. It is well known and easy to see that there exists an additive function that is nowhere continuous. For, let H be a Hamel basis. Then every real number x has a unique representation $x = \sum_{1 \le i \le n} q_i h_i$, where $q_i \in \mathbb{Q}$ and $h_i \in H$. Let $f(x) = \sum_{1 \le i \le n} q_i$. Then f is additive, $S(f) = \mathbb{R}$, and $C(f) = \emptyset$. □

Remark 3. We prove the following theorem to show that Theorem 2 is not true if G is replaced by $\mathbb{R}$.

Theorem 4. *For every function $f : \mathbb{R} \to \{0, 1, 2\}$, $S(f)\backslash C(f)$ contains no nonempty open interval. In particular, $S(f)\backslash C(f) \neq \mathbb{R}$.*

Proof. To prove, let us assume the opposite that, for some function $f : \mathbb{R} \to \{0, 1, 2\}$, there exists a nonempty open interval I such that $I \subseteq S(f)\backslash C(f)$. Since $S(f) \cap SC(f) = C(f)$ and $I \subseteq S(f)\backslash C(f)$, for each $x \in I$, we have $x \in S(f)$ and $x \notin SC(f)$. So, there exists a sequence (h_n) converging to zero such that $f(x+h_n) \neq f(x-h_n)$ and $f(x+h_n)+f(x-h_n)-2f(x) = 0$ for every n. Hence, since the range of f is $\{0, 1, 2\}$, we have $f(x) = 1$ and $\{f(x + h_n), f(x - h_n)\} = \{0, 2\}$. That is $f(x) = 1$ for every $x \in I$, which contradicts that $I \cap C(f) = \emptyset$. □

The continuum hypothesis is equivalent to the statement that $\mathbb{R}$ is a countable union of sets P_n such that each P_n has distinct distances (see [5]). (That is, if x, y, s, t belong to P_n and $|x - y| = |s - t|$, then $\{x, y\} = \{s, t\}$.) It follows from [6, Thm 2] that If P is a subset of $\mathbb{R}$ that has distinct distances and C is a subset of $\mathbb{R}$ such that $|C| < |\mathbb{R}|$, then $P + C = \{p + c : p \in P$ and $c \in C\}$ is not residual in $\mathbb{R}$. That is, the complement of a union of fewer than continuum many translates of the set P is of second category. Note that if a set has distinct distances, then it has no arithmetic progression of length three. The converse is not true. Using the fact that $\mathbb{R}$ is a countable union of sets such that each set contains no arithmetic progression of length three [2, Thm 1.1], it is shown in [3] that there exists a function $f : \mathbb{R} \to \mathbb{N}$ such that $\{h > 0 : f(x + h) = f(x - h) = f(x)\} = \emptyset$ for each x in $\mathbb{R}$. We show that there exists a subset A of $\mathbb{R}$ such that A contains no arithmetic progression of length three and $A + C = \mathbb{R}$ for some countable set C.

The following question remains open.

Problem 1. Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that the range of f is finite and $\{h > 0 : f(x + h) = f(x - h) = f(x)\} = \emptyset$ for each x in $\mathbb{R}$.

Theorem 5. $\mathbb{R}$ is a union of countably many translates of a set that contains no arithmetic progression of length three.

Proof. Let H be a Hamel basis and $h \in H$. Let L be the set of all finite linear combinations from elements of $H\backslash\{h\}$ with rational coefficients. It is known that $\mathbb{R}$ is a countable union of sets such that each set contains no arithmetic progression of length three [2, Thm 1.1]. Hence $L = \bigcup_{n\in\mathbb{N}} L_n$, where each L_n contains no arithmetic progression of length three. Let $A = \bigcup_{n\in\mathbb{N}} (L_n + 2^n h)$. First, we show that A has no arithmetic progression of length three. Assume, to the contrary, that A has an arithmetic progression of length three. Then there exist distinct elements a, b, c from A such that $a+c-2b = 0$, $a = a_i + 2^i h$, $b = b_j + 2^j h$, and $c = c_k + 2^k h$ for some $a_i \in L_i$, $b_j \in L_j$, and

$c_k \in L_k$. Since L is an additive group not containing any nonzero rational multiple of h and $2^i h + 2^k h - 2^{j+1} h = 2b_j - a_i - c_k \in L$, we have $2^i + 2^k - 2^{j+1} = 0$. This implies that $i = j = k$ and $a_i + c_i - 2b_i = 0$, where a_i, b_i, and c_i are distinct elements of L_i, which contradicts that L_i contains no arithmetic progression of length three. So, A has no arithmetic progression of length three and $\mathbb{R} = A + Qh$. □

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