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EULER POLYNOMIALS, FOURIER SERIES AND ZETA NUMBERS

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Abstract: Fourier series for Euler polynomials is used to obtain information about values of the Riemann zeta function for integer arguments greater than one. If the argument is even we recover the well-known exact values, if the argument is odd we find integral representations and rapidly convergent series.

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1. Euler Polynomials and Fourier Series

The Euler polynomials $E_n(x)$ are for $x \in \mathbb{R}$ defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}\ , \quad |t| < \pi. \tag{1}$$

The first Euler polynomial is $E_0(x) = 1$, and the next four are as follows,

$E_1(x) = x - \frac{1}{2},\ E_2(x) = x^2 - x,\ E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4},\ E_4(x) = x^4 - 2x^3 + x.$

The next formulas are readily derivable from the definition

(E1) $E_n'(x) = nE_{n-1}(x)\ ,\ n \in \mathbb{N}.$

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(E2) $E_n(1-x) = (-1)^n E_n(x),\ n \in \mathbb{N}_0$.
(E3) $(-1)^{n+1} E_n(-x) = E_n(x) - 2x^n,\ n \in \mathbb{N}_0$.

For $n \in \mathbb{N}$ and $x \in [0,1]$ we have the Fourier series

$$E_{2n-1}(x) = 4(-1)^n (2n-1)! \sum_{m=0}^{\infty} \frac{\cos[(2m+1)\pi x]}{[(2m+1)\pi]^{2n}} \tag{2}$$

and

$$E_{2n}(x) = 4(-1)^n (2n)! \sum_{m=0}^{\infty} \frac{\sin[(2m+1)\pi x]}{[(2m+1)\pi]^{2n+1}}, \tag{3}$$

for example, see [6] . The periodic Euler functions $\tilde{E}_n$ is defined as

$$\tilde{E}_n(x) = E_n(x),\ 0 \le x < 1, \quad \tilde{E}_n(x+1) = -\tilde{E}_n(x),\ x \in \mathbb{R}. \tag{4}$$

The period is 2 because $\tilde{E}_n(x+2) = -\tilde{E}_n(x+1) = \tilde{E}_n(x),\ x \in \mathbb{R}$, and we can now generalize the formulas (2) and (3) to hold for $x \in \mathbb{R}$ by substituting $\tilde{E}_{2n-1}$ for E_{2n-1} in (2) and $\tilde{E}_{2n}$ for E_{2n} in (3).

2. Zeta, Lambda and Beta Numbers

The Riemann zeta function ζ and the lambda function λ are for $\Re s > 1$ defined by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \tag{5}$$

and

$$\lambda(s) = \sum_{m=0}^{\infty} \frac{1}{(2m+1)^s}\ , \tag{6}$$

see [4]. Setting $x = 0$ in (2) we get

$$\lambda(2n) = \frac{(-1)^n \pi^{2n}}{4(2n-1)!} E_{2n-1}(0),\ n \in \mathbb{N}$$

and using the fact that $E_{2n-1}(0) = -(4^n - 1)B_{2n}/n$, where $B_{2n},\ n \in \mathbb{N}$, are Bernoulli numbers, we get

$$\lambda(2n) = \frac{(-1)^{n+1}(4^n-1)\pi^{2n}}{4n(2n-1)!} B_{2n}\ ,\ n \in \mathbb{N}. \tag{7}$$

We have the following relation between the functions ζ and λ,

$$\zeta(s) = \frac{2^s}{2^s - 1}\lambda(s), \ \Re s > 1. \tag{8}$$

From (7) and (8) we get the well-known formula

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \ n \in \mathbb{N}. \tag{9}$$

The beta function is for $\Re s > 0$ defined by

$$\beta(s) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^s} \ , \tag{10}$$

see [4]. Setting $x = \frac{1}{2}$ in (3) we get

$$\beta(2n+1) = \frac{(-1)^n \pi^{2n+1}}{4(2n)!}E_{2n}(\tfrac{1}{2})$$

and using the fact that $E_{2n}(\frac{1}{2}) = 4^{-n}E_{2n}$, where $E_{2n}, \ n \in \mathbb{N}_0$, are Euler numbers, we get

$$\beta(2n+1) = \frac{(-1)^n \pi^{2n+1}}{4^{n+1}(2n)!}E_{2n}, \ n \in \mathbb{N}_0. \tag{11}$$

For example we have $E_0 = 1$ and $E_2 = -1$ leading to the wellknown results

$$\beta(1) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)} = \frac{\pi}{4}$$

and

$$\beta(3) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^3} = \frac{\pi^3}{32} \ .$$

Setting x to a particular value in (3) only gives us information about $\beta(2n+1)$ but not about $\zeta(2n+1)$.

3. Integral Representations of Zeta(2n+1)

From (3) and (4) we get

$$\tilde{E}_{2n}(x) = 4(-1)^n(2n)!\sum_{m=0}^{\infty}\frac{\sin\left[(2m+1)\pi x\right]}{\left[(2m+1)\pi\right]^{2n+1}},\ n\in\mathbb{N},\ x\in\mathbb{R}. \tag{12}$$

If $x>0$ we can obtain an uniformly convergent series for $\tilde{E}_{2n}(x)/x$ from (12), and integration from zero to infinity now gives us information about $\zeta(2n+1)$ using the well-known formula

$$\int_0^{\infty}\frac{\sin(ux)}{x}\,dx = \frac{\pi}{2}\ ,\ u>0. \tag{13}$$

Indeed, for $n\in\mathbb{N}$ we get

$$\begin{aligned}\int_0^{\infty}\frac{\tilde{E}_{2n}(x)}{x}\,dx &= \frac{4(-1)^n(2n)!}{\pi^{2n+1}}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^{2n+1}}\cdot\frac{\pi}{2}\\ &= (-1)^n\frac{2(2n)!}{\pi^{2n}}\lambda(2n+1)\\ &= (-1)^n\frac{(2-2^{-2n})(2n)!}{\pi^{2n}}\zeta(2n+1),\end{aligned} \tag{14}$$

and then

$$\zeta(2n+1) = \frac{(-1)^n\pi^{2n}}{(2-2^{-2n})(2n)!}\int_0^{\infty}\frac{\tilde{E}_{2n}(x)}{x}\,dx. \tag{15}$$

Using the definition of $\tilde{E}_{2n}(x)$ we can write the integral on the right hand side of (15) in a form involving only $E_{2n}(x),\ x\in[0,1]$. For $n\in\mathbb{N}$ we have

$$\begin{aligned}\int_0^{\infty}\frac{\tilde{E}_{2n}(x)}{x}\,dx &= \int_0^{1}\frac{E_{2n}(x)}{x}\,dx + \sum_{k=1}^{\infty}\int_{2k-1}^{2k+1}\frac{\tilde{E}_{2n}(x)}{x}\,dx\\ &= \int_0^{1}\frac{E_{2n}(x)}{x}\,dx + \sum_{k=1}^{\infty}\int_{-1}^{1}\frac{\tilde{E}_{2n}(t)}{t+2k}\,dt,\end{aligned}$$

because $\tilde{E}_{2n}$ is periodic with period 2, and therefore $\tilde{E}_{2n}(t+2k) = \tilde{E}_{2n}(t)\ ,\ k\in\mathbb{N}$. We can simplify the integral in the summation by

$$\int_{-1}^{1}\frac{\tilde{E}_{2n}(t)}{t+2k}\,dt = \int_{-1}^{0}\frac{\tilde{E}_{2n}(t)}{t+2k}\,dt + \int_0^{1}\frac{E_{2n}(t)}{t+2k}\,dt$$

$$= \int_0^1 \frac{-E_{2n}(u)}{u+2k-1}\,du + \int_0^1 \frac{E_{2n}(u)}{u+2k}\,du$$
$$= \int_0^1 \frac{-E_{2n}(u)}{(u+2k)(u+2k-1)}\,du,$$

and then

$$\int_0^\infty \frac{\tilde{E}_{2n}(x)}{x}\,dx = \int_0^1 \frac{E_{2n}(x)}{x}\,dx - \sum_{k=1}^{\infty}\int_0^1 \frac{E_{2n}(x)}{(x+2k)(x+2k-1)}dx. \tag{16}$$

For the right side of (16) we now change the order of summation and integration, and then use the Digamma function ψ to find the sum of the series, wich will reduce (16) to

$$\begin{aligned}\int_0^\infty \frac{\tilde{E}_{2n}(x)}{x}\,dx &= \int_0^1 E_{2n}(x)\left(\frac{1}{x} - \sum_{k=1}^{\infty}\frac{1}{(x+2k)(x+2k-1)}\right)dx\\ &= \int_0^1 E_{2n}(x)\left(\frac{1}{x} - \tfrac{1}{2}\psi(1+\tfrac{1}{2}x) + \tfrac{1}{2}\psi(\tfrac{1}{2}+\tfrac{1}{2}x)\right)dx\\ &= \int_0^1 \tfrac{1}{2}E_{2n}(x)\left(\psi(\tfrac{1}{2}+\tfrac{1}{2}x) - \psi(\tfrac{1}{2}x)\right)dx. \end{aligned} \tag{17}$$

Then we have for $n \in \mathbb{N}$ from (15) and (17) the integral formula

$$\zeta(2n+1) = \frac{(-1)^n\pi^{2n}}{(2-2^{-2n})(2n)!}\int_0^1 \tfrac{1}{2}E_{2n}(x)\left(\psi(\tfrac{1}{2}+\tfrac{1}{2}x) - \psi(\tfrac{1}{2}x)\right)dx. \tag{18}$$

Using $\psi(x) = \Gamma'(x)/\Gamma(x)$ and $E'_{2n}(x) = 2nE_{2n-1}(x)$ we can reduce (18) by partial integration to an integral formula involving the Gamma function and $E_{2n-1}(x)$ instead of $E_{2n}(x)$. The result is

$$\zeta(2n+1) = \frac{(-1)^{n+1}\pi^{2n}}{(2-2^{-2n})(2n-1)!}\int_0^1 E_{2n-1}(x)\ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}x)}{\Gamma(\frac{1}{2}x)}dx, \tag{19}$$

which holds for $n \in \mathbb{N}$. Integral representations for $\zeta(2n+1)$ involving Euler polynomials, Bernoulli polynomials and trigonometric function can be found in [2]. One of these representations is

$$\zeta(2n+1) = \frac{(-1)^n\pi^{2n+1}}{4(1-2^{-2n-1})(2n)!}\int_0^1 E_{2n}(x)\csc(\pi x)dx. \tag{20}$$

Partial integration on the integral in (20) leads to

$$\int_0^1 E_{2n}(x)\csc(\pi x)dx = -\frac{2n}{\pi}\int_0^1 E_{2n-1}(x)\ln\tan\left(\frac{\pi x}{2}\right)dx,$$

because $E_{2n}(x) = O(x)$ and $\tan(\pi x/2) = O(x)$ as $x \to 0$, so

$$E_{2n}(x)\ln\tan\left(\frac{\pi x}{2}\right) = O(x)\ln O(x) = O(x) \text{ as } x \to 0^+,$$

and analogous

$$E_{2n}(x)\ln\tan\left(\frac{\pi x}{2}\right) = -O(1-x)\ln O(1-x) = O(1-x) \text{ as } x \to 1^-.$$

Using

$$\begin{aligned}
\ln\tan\left(\frac{\pi x}{2}\right) &= \ln\sin\left(\frac{\pi x}{2}\right) - \ln\sin\left(\frac{\pi(1-x)}{2}\right)\\
&= \ln\frac{\Gamma(\frac{1}{2}-\frac{1}{2}x)\Gamma(\frac{1}{2}+\frac{1}{2}x)}{\Gamma(\frac{1}{2}x)\Gamma(1-\frac{1}{2}x)}\\
&= \ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}x)}{\Gamma(\frac{1}{2}x)} - \ln\frac{\Gamma(1-\frac{1}{2}x)}{\Gamma(\frac{1}{2}-\frac{1}{2}x)}.
\end{aligned}$$

and $E_{2n-1}(x) = -E_{2n-1}(1-x)$ now leads to

$$\begin{aligned}
\int_0^1 E_{2n}(x)\csc(\pi x)dx &= -\frac{2n}{\pi}\int_0^1 E_{2n-1}(x)\ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}x)}{\Gamma(\frac{1}{2}x)}dx\\
&\quad +\frac{2n}{\pi}\int_1^0 E_{2n-1}(t)\ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}t)}{\Gamma(\frac{1}{2}t)}(-1)dt\\
&= -\frac{4n}{\pi}\int_0^1 E_{2n-1}(x)\ln\frac{\Gamma(\frac{1}{2}+\frac{1}{2}x)}{\Gamma(\frac{1}{2}x)}dx,
\end{aligned}$$

so the integral representations (19) and (20) are equivalent.

4. Series Representations of Zeta(2n+1)

Another way to go for the right side of (16) is to use it directly in (15) to give

$\zeta(2n+1)$

$$= \frac{(-1)^n \pi^{2n}}{(2-2^{-2n})(2n)!} \left(\int_0^1 \frac{E_{2n}(x)}{x}\, dx - \sum_{k=1}^{\infty} \int_0^1 \frac{E_{2n}(x)}{(x+2k)(x+2k-1)} dx \right). \quad (21)$$

Using decomposition we find

$$\begin{aligned} \frac{E_{2n}(x)}{(x+2k)(x+2k-1)} &= p_{2n-2}(x,k) + \frac{E_{2n}(1-2k)}{x+2k-1} - \frac{E_{2n}(-2k)}{x+2k} \\ &= p_{2n-2}(x,k) + \frac{E_{2n}(2k)}{x+2k-1} + \frac{E_{2n}(2k) - 2(2k)^{2n}}{x+2k} , \end{aligned}$$

where p_{2n-2} is a polynomial of degree $2n-2$ in x and k, fx. we have $p_0(x,k) = 1$ and $p_2(x,k) = x^2 + 12k^2 - 4kx - x + 2k - 1$.
Integration from 0 to 1 leads to

$$\begin{aligned} &\int_0^1 \frac{E_{2n}(x)}{(x+2k)(x+2k-1)} dx \\ = &\int_0^1 p_{2n-2}(x,k)dx + E_{2n}(2k) \ln \frac{2k}{2k-1} + \left[E_{2n}(2k) - 2(2k)^{2n}\right] \ln \frac{2k+1}{2k} \\ = &\ P_{2n-2}(k) + E_{2n}(2k) \ln \frac{2k+1}{2k-1} - 2(2k)^{2n} \ln \frac{2k+1}{2k} , \end{aligned}$$

where P_{2n-2} is a polynomial of degree $2n-2$ in k, fx. we have $P_0(k) = 1$ and $P_2(k) = 12k^2 - \frac{7}{6}$. Substitution in (21) now gives

$$\begin{aligned} \zeta(2n+1) = &\frac{(-1)^n \pi^{2n}}{(2-2^{-2n})(2n)!} \left[\int_0^1 \frac{E_{2n}(x)}{x}\, dx \right. \qquad (22) \\ &\left. - \sum_{k=1}^{\infty} \left(P_{2n-2}(k) + E_{2n}(2k) \ln \frac{2k+1}{2k-1} - 2(2k)^{2n} \ln \frac{2k+1}{2k} \right) \right]. \end{aligned}$$

Setting $n = 1$ and $n = 2$ in (22) we get

$$\zeta(3) = \frac{2\pi^2}{7} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(1 + (4k^2 - 2k) \ln \frac{2k+1}{2k-1} - 8k^2 \ln \frac{2k+1}{2k} \right) \right] \qquad (23)$$

and

$$\begin{aligned} \zeta(5) = &\frac{2\pi^4}{93} \\ \times &\left[\frac{7}{12} - \sum_{k=1}^{\infty} \left(12k^2 - \frac{7}{6} + (16k^4 - 16k^3 + 2k) \ln \frac{2k+1}{2k-1} - 32k^4 \ln \frac{2k+1}{2k} \right) \right]. \end{aligned} \qquad (24)$$

Exact values of the series of the right side of (23) and (24) cannot be evaluated, and the series also converge slowly. For numerical computation we can rewrite the series in (23) and (24) to rapidly convergent series using

$$\ln\frac{2k+1}{2k}=\ln(1+\frac{1}{2k})=\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left(\frac{1}{2k}\right)^{m+1},\ k\geq 1, \tag{25}$$

and

$$\ln\frac{2k+1}{2k-1}=\ln\frac{1+\dfrac{1}{2k}}{1-\dfrac{1}{2k}}=\sum_{m=0}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m+1},\ k\geq 1. \tag{26}$$

5. Rapidly Convergent Series for Zeta(3)

Using (25) and (26) we find for the general terms of the series in (23)

$$\begin{aligned}
& 1+(4k^2-2k)\sum_{m=0}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m+1}-8k^2\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left(\frac{1}{2k}\right)^{m+1}\\
= \; & 1+4k-2+\sum_{m=1}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m-1}-\sum_{m=1}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m}\\
& -4k+1+\sum_{m=2}^{\infty}\frac{2(-1)^{m+1}}{m+1}\left(\frac{1}{2k}\right)^{m-1}\\
= \; & \sum_{m=1}^{\infty}\left(\frac{1}{m+1}-\frac{2}{2m+1}\right)\left(\frac{1}{2k}\right)^{2m}=-\sum_{m=1}^{\infty}\frac{1}{(m+1)(2m+1)4^m}\cdot\frac{1}{k^{2m}}.
\end{aligned}$$

Inserting this in (23) and changing the order of summation we find

$$\begin{aligned}
\zeta(3) &= \frac{2\pi^2}{7}\left[\frac{1}{2}-\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+1)(2m+1)4^m}\cdot\frac{1}{k^{2m}}\right]\\
&= \frac{2\pi^2}{7}\left[\frac{1}{2}-\sum_{m=1}^{\infty}\frac{1}{(m+1)(2m+1)4^m}\cdot\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\right]\\
&= \frac{2\pi^2}{7}\left[\frac{1}{2}-\sum_{m=1}^{\infty}\frac{\zeta(2m)}{(m+1)(2m+1)4^m}\right].
\end{aligned}$$

If we use $\zeta(0) = -\frac{1}{2}$ this can be simplified to

$$\zeta(3) = -\frac{2\pi^2}{7}\sum_{m=0}^{\infty}\frac{\zeta(2m)}{(m+1)(2m+1)4^m}. \tag{27}$$

The series representations (27) was contained in a 1772 paper by Euler [1] and later rediscovered by others [5]. A series representation analogous to (27) has been evaluated in [3] using Fourier series for an odd periodic function $f(x)$, and where $f(x)/x$ is integrated in the same way as $\tilde{E}_{2n}(x)/x$ is integrated in (14). From (27) we can compute $\zeta(3)$ approximately by

$$\zeta(3) \cong -\frac{2\pi^2}{7}\sum_{m=0}^{M}\frac{\zeta(2m)}{(m+1)(2m+1)4^m} \tag{28}$$

with an error $R_M < 0$ satisfying

$$\begin{aligned} |R_M| &< \frac{2\pi^2}{7}\frac{\zeta(2M+2)}{(M+2)(2M+3)}\sum_{m=M+1}^{\infty}\frac{1}{4^m} \\ &< \frac{2\pi^2}{21}\frac{1}{(M+2)(2M+3)(4^M-\frac{1}{2})}. \end{aligned} \tag{29}$$

Setting $M = 25$ in (28) and (29) we get the approximately value $\zeta(3) = 1.20205\,69031\,59594\,285$ with an error bound $R_{25} < 6 \cdot 10^{-19}$.

6. Rapidly Convergent Series for Zeta(5)

Using (25) and (26) we find for the general terms of the series in (24)

$$\begin{aligned} & 12k^2 - \frac{7}{6} + (16k^4 - 16k^3 + 2k)\sum_{m=0}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m+1} \\ & -32k^4\sum_{m=0}^{\infty}\frac{(-1)^m}{m+1}\left(\frac{1}{2k}\right)^{m+1} \\ = \; & 12k^2 - \frac{7}{6} - \sum_{m=0}^{\infty}\frac{4}{2m+1}\left(\frac{1}{2k}\right)^{2m-2} + \sum_{m=0}^{\infty}\frac{2}{2m+1}\left(\frac{1}{2k}\right)^{2m} \\ & + \sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{2k}\right)^{2m-4} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=1}^{\infty}\left(-\frac{4}{2m+3}+\frac{2}{2m+1}+\frac{1}{m+2}\right)\left(\frac{1}{2k}\right)^{2m} \\
&= \sum_{m=1}^{\infty}\frac{2m+7}{(m+2)(2m+1)(2m+3)4^m}\cdot\frac{1}{k^{2m}}
\end{aligned}
$$

Inserting this in (24) and changing the order of summation we find

$$
\begin{aligned}
\zeta(5) &= \frac{2\pi^4}{93}\left[\frac{7}{12}-\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\frac{2m+7}{(m+2)(2m+1)(2m+3)4^m}\cdot\frac{1}{k^{2m}}\right] \\
&= \frac{2\pi^4}{93}\left[\frac{7}{12}-\sum_{m=1}^{\infty}\frac{2m+7}{(m+2)(2m+1)(2m+3)4^m}\cdot\sum_{k=1}^{\infty}\frac{1}{k^{2m}}\right] \\
&= \frac{2\pi^4}{93}\left[\frac{7}{12}-\sum_{m=1}^{\infty}\frac{(2m+7)\zeta(2m)}{(m+2)(2m+1)(2m+3)4^m}\right],
\end{aligned}
$$

wich can be simplified to

$$
\zeta(5) = -\frac{2\pi^4}{93}\sum_{m=0}^{\infty}\frac{(2m+7)\zeta(2m)}{(m+2)(2m+1)(2m+3)4^m}. \tag{30}
$$

Other series representations for $\zeta(3)$ and $\zeta(5)$, together with a systematic investigation of series representations of $\zeta(2n+1)$, $n \in \mathbb{N}$, can be found in [5]. From (30) we can compute $\zeta(5)$ approximately by

$$
\zeta(5) \cong -\frac{2\pi^4}{93}\sum_{m=0}^{M}\frac{(2m+7)\zeta(2m)}{(m+2)(2m+1)(2m+3)4^m} \tag{31}
$$

with an error $R_M < 0$ satisfying

$$
\begin{aligned}
|R_N| &< \frac{2\pi^4}{93}\frac{(2M+9)\zeta(2M+2)}{(M+3)(2M+3)(2M+5)}\sum_{m=M+1}^{\infty}\frac{1}{4^m} \\
&< \frac{2\pi^4}{279}\frac{2M+9}{(M+3)(2M+3)(2M+5)(4^M-\frac{1}{2})}.
\end{aligned} \tag{32}
$$

Setting $M = 25$ in (31) and (32) we get the approximately value $\zeta(5) = 1.03692\,77551\,43369\,926$ with an error bound $R_{25} < 5\cdot 10^{-19}$.

7. Conclusion

Fourier series for odd-indexed Euler polynomials are leading to well-known explicit formulas for lambda numbers $\lambda(2n)$, $n \in \mathbb{N}$, and zeta numbers $\zeta(2n)$, $n \in \mathbb{N}$. Fourier series for even-indexed Euler polynomials are leading to well-known explicit formulas for beta numbers $\beta(2n+1)$, $n \in \mathbb{N}_0$, but no analoguos closed evaluation for $\zeta(2n+1)$, $n \in \mathbb{N}$, can be found in the same way. Using Fourier series for periodic Euler functions $\tilde{E}_{2n}(x)$ and integration of the series for $\tilde{E}_{2n}(x)/x$ term by term leads to integral representations and series representations of $\zeta(2n+1)$, $n \in \mathbb{N}$. The series representations are used to evaluate rapidly convergent series for $\zeta(3)$ and $\zeta(5)$.

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