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KK-ISOMORPHISM AND ITS PROPERTIES

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Abstract: In this paper, we introduce homomorphisms of KK-algebra and investigate its properties. Moreover, the relations between quotient KK-algebra and isomorphism also provided.

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1. Introduction

In [1], S. Asawasamrit and A. Sudprasert introduced a new algebraic structure which is called KK-algebras. And we described the relation between ideals and congruences. Furthermore, we define quotient KK-algebra and study its properties.

Several authors [2, 7] have studied homomorphism of BCI-algebras, BCK-algebras and binary algebra. In this paper, we apply the notion of homomorphism of BCI-algebras, BCK-algebras and binary algebra theory to KK-algebras, and as a result. We introduce a new concept, called *KK-isomorphism.* Using this concept as defined we investigated some of its properties. The purpose of this paper is to derive some straightforward consequences of the relations between quotient KK-algebras and isomorphisms and also investigate some of its properties.

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2. Preliminaries

In this section we introduced an algebraic structure called a *KK-algebra* which is an algebra $(X, *, 0)$ with a binary operation $*$ and a nullary operation 0 such that for all $x, y, z \in X$, satisfies the following properties:

(KK-1) $(x * y) * ((y * z) * (x * z)) = 0$;

(KK-2) $0 * x = x$;

(KK-3) $x * y = 0$ and $y * x = 0$ if and only if $x = y$.

On KK-algebra $(X, *, 0)$. We define a binary relation $\leq$ on X by putting $x \leq y$ if and only if $y * x = 0$. Then $(X, \leq)$ is a partially ordered set. It is easy to show that the following properties are true for a KK-algebra. For all x, y, z in X :

(P-1) $x * ((x * y) * y) = 0$;

(P-2) $x * x = 0$;

(P-3) $x * (y * z) = y * (x * z)$;

(P-4) $((x * y) * y) * y = x * y$;

(P-5) $(x * y) * 0 = (x * 0) * (y * 0)$;

(P-6) $(x * y) * ((z * x) * (z * y)) = 0$;

(P-7) $x \leq y$ implies $y * z \leq x * z$;

(P-8) $x \leq y$ implies $z * x \leq z * y$.

A subset A of a KK-algebra X is called *closed* of X if $x * y \in A$ whenever $x, y \in A$. A non-empty subset A of a KK-algebra X is called an *ideal* of X if it satisfies the following conditions:

(I-1) $0 \in A$;

(I-2) for any $x, y \in X$, $x * y \in A$ and $x \in A$ imply $y \in A$.

Assume I is an ideal of KK-algebra X. Define the relation $\sim$ on X by $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. Then the relation $\sim$ is an equivalence relation on X.

Assume $\sim$ is an equivalence relation on a KK-algebra X and let I be an ideal of X. Define $[x]_I$ by $[x]_I = \{y \in X \mid x \sim y\} = \{y \in X \mid x * y \in I$ and $y * x \in I\}$. Then the family $\{[x]_I : x \in X\}$ gives a partition of X which is denoted by X/I. For any $x, y \in X$, we define $[x]_I \circ [y]_I = [x * y]_I$, then the binary operation $\circ$ is a mapping from $X/I \times X/I$ to X/I. It is easily checked that $(X/I, \circ, [0]_I)$ is a KK-algebra. Moreover, the set X/I is called the *quotient KK-algebra.* And $[0]_I = \{x \in X \mid x \sim 0\}$ is an ideal of X. If I is a closed ideal of KK-algebra X, then it is clear that $[a]_I = I$ for all a in I.

3. Results

In this section, first we will define KK-homomorphism and next we can describe properties of KK-homomorphism.

Definition 3.1. Let $(X, *, 0_X)$ and $(Y, \circ, 0_Y)$ be KK-algebras. A *KK-homomorphism* is a map $f : X \to Y$ satisfying $f(x * y) = f(x) \circ f(y)$ for all $x, y \in X$.

For example, the zero mapping $g : X \to Y$ where $g(x) = 0_Y$, for any $x \in X$, then g is a KK-algebra. In general a KK-homomorphism $f : X \to Y$ may not be surjective or injective. An injective KK-homomorphism is called *monomorphism*, a surjective KK-homomorphism is called *epimorphism* and a bijective KK-homomorphism is called *isomorphism*. Moreover, we say X is isomorphic to Y, symbolically, $X \cong Y$. The *kernel* of the KK-homomorphism f, denoted by Ker$f$, is the set of elements of X that map to 0_Y in Y.

Definition 3.2. Let f be a mapping of a KK-algebra X into a KK-algebra Y, and let $I \subseteq X$ and $A \subseteq Y$. The image of I in X under f is

$$f(I) = \{f(x) \mid x \in I\}$$

and the inverse image of A in Y is

$$f^{-1}(A) = \{x \in X \mid f(x) \in A\}.$$

Next, the basic properties of KK-homomorphism are considered as the following theorem.

Theorem 3.3. *Let f be a KK-homomorphism of a KK-algebra X into a KK-algebra Y. Then:*

(1) $f(0_X) = 0_Y$.

(2) If 0_X *is the identity in* X*, then* $f(0_X)$ *is the identity in* Y*.*

(3) f *is injective if and only if* $\mathrm{Ker}f = \{0_X\}$.

(4) $x \leq_X y$ *implies* $f(x) \leq_Y f(y)$.

Proof. Assume that $f : X \to Y$ is a KK-homomorphism.

(1) Since $0_X * 0_X = 0_X$, then $f(0_X) = f(0_X * 0_X) = f(0_X) \circ f(0_X) = 0_Y$.

(2) Assume that 0_X is the identity in X and 0_Y is the identity in Y. From KK-2, $0_Y \circ f(0_X) = 0_Y$ and $f(0_X) \circ 0_Y = f(0_X) \circ [f(0_X) \circ f(0_X)] = f(0_X) \circ f(0_X * 0_X) = f(0_X) \circ f(0_X) = 0_Y$. By KK-3, we get that $f(0_X) = 0_Y$. This show that $f(0_X)$ is the identity in Y.

(3) Suppose that f is injective and $x \in \mathrm{Ker}f$. It follows that $f(x) = 0_Y$. Since $f(0_X) = 0_Y$, so $f(x) = f(0_X)$. By assumption, $x = 0_X$. Thus $\mathrm{Ker}f = \{0_X\}$.

Conversely, suppose that $\mathrm{Ker}f = \{0_X\}$. Let $x, y \in X$ be such that $f(x) = f(y)$. Then we get that $f(x*y) = f(x) \circ f(y) = 0_Y$ and $f(y*x) = f(y) \circ f(x) = 0_Y$, thus $x * y, y * x \in \mathrm{Ker}f$, this means that $x * y = 0_X = y * x$. From KK-3, $x = y$, and shows that f is injective.

(4) Let $x \leq_X y$. It follows that $y * x = 0_X$. So, (1) implies $f(y) \circ f(x) = f(y * x) = f(0_X) = 0_Y$. Hence $f(x) \leq_Y f(y)$. □

Theorem 3.4. *Let* $f : X \to Y$ *be a KK-homomorphism. Then:*

(1) If I *is an ideal of* X*, then* $f(I)$ *is an ideal of* Y*.*

(2) If I *is a closed of* X*, then* $f(I)$ *is a closed of* Y*.*

(3) If A *is an ideal in* Y*, then* $f^{-1}(A)$ *is an ideal in* X*.*

(4) If A *is a closed of* Y*, then* $f^{-1}(A)$ *is a closed of* X*.*

(5) $\mathrm{Ker}f$ *is a closed ideal of* X*.*

(6) $\mathrm{Im}f$ *is a closed of* Y*.*

Proof. Assume that $f : X \to Y$ is a KK-homomorphism.

(1) Let I be an ideal of X. We see that $0_X \in I$, and by theorem 3.3(1), $0_Y = f(0_X) \in f(I)$, so $0_Y \in f(I)$. Now, assume that $f(x) \circ f(y) \in f(I)$ and $f(x) \in f(I)$, it follows that $f(x * y) \in f(I)$, so $x * y, x \in I$. Since I is an ideal of X, $y \in I$, it follows that $f(y) \in f(I)$. Hence $f(I)$ is an ideal of Y.

(2) Let I be a closed of X and $x, y \in f(I)$. Then there exist $a, b \in I$ such that $x = f(a)$ and $y = f(b)$. Since $x \circ y = f(a) \circ f(b) = f(a * b) \in f(I)$. Thus $f(I)$ is a closed of Y.

(3) Let A be an ideal in Y. Then $0_Y \in A$, we get that $0_X = f^{-1}(0_Y) \in f^{-1}(A)$. For any $x, y \in X$, let $x * y \in f^{-1}(A)$ and $x \in f^{-1}(A)$. It follows that $f(x) \circ f(y) = f(x * y) \in A$ and $f(x) \in A$. Since A is an ideal of Y, we obtain that $f(y) \in A$. Consequently $y \in f^{-1}(A)$, proving that $f^{-1}(A)$ is an ideal of X.

(4) Let A be a closed of Y and $x, y \in f^{-1}(A)$. Then $f(x) = a$ and $f(y) = b$ for some $a, b \in A$. Thus $f(x * y) = f(x) \circ f(y) = a * b \in A$, as A is a closed. Hence $x * y \in f^{-1}(A)$.

(5) It is clear that $\mathrm{Ker}f \subseteq X$. Since $f(0_X) = 0_Y$, so $0_X \in \mathrm{Ker}f$. It follows that $\mathrm{Ker}f \neq \emptyset$. Let $x * y \in \mathrm{Ker}f$ and $x \in \mathrm{Ker}f$. We get that $f(y) = 0_Y \circ f(y) = f(x) \circ f(y) = f(x * y) = 0_Y$. Thus $y \in \mathrm{Ker}f$. Now, we will show $\mathrm{Ker}f$ is closed of X. Let $x, y \in \mathrm{Ker}f$. Then $f(x * y) = f(x) \circ f(y) = 0_Y \circ 0_Y = 0_Y$, these imply that $x * y \in \mathrm{Ker}f$. Therefore $\mathrm{Ker}f$ is a closed ideal of X.

(6) Let $a, b \in \mathrm{Im}(f)$, then there exist $x, y \in X$ such that $a = f(x)$ and $b = f(y)$, so $a \circ b = f(x) \circ f(y) = f(x * y) \in \mathrm{Im}(f)$. This proves that $\mathrm{Im}f$ is a closed of Y. □

In general, $\mathrm{Im}(f)$ may not be an ideal.

Example 3.5. Let $X = \{0, 1, 2\}$. Define an operation $*$ on X by

*	0	1	2
0	0	1	2
1	0	0	2
2	0	0	0

Then, it can be easily show that $(X, *, 0)$ is a KK-algebras. Now, let f be the mapping from X to itself such that $f(0) = 0, f(1) = 0$ and $f(2) = 2$, then we see that $\mathrm{Im}(f) = \{0, 2\}$. So $\mathrm{Im}(f)$ is not an ideal of X, since $2 \in \mathrm{Im}(f)$ and $2 * 1 = 0 \in \mathrm{Im}(f)$, but $1 \notin \mathrm{Im}(f)$.

The next proposition holds, whose verification is routine and omitted.

Proposition 3.6. *Let f be a KK-homomorphism from a KK-algebra X to a KK-algebra Y. Then:*

(1) f is an epimorphism if and only if $Im(f) = Y$.

(2) f is an isomorphism if and only if the inverse mapping f^{-1} is an isomorphism.

Theorem 3.7. *Let I be a closed ideal of KK-algebra X. Defined the map $f : X \to X/I$ by $f(x) = [x]_I$, for all $x \in X$. Then f is epimorphism, we call f is the natural KK-homomorphism of X onto X/I. Furthermore,* $\mathrm{Ker}f = I$.

Proof. Let I be a closed ideal of KK-algebra X and $x, y \in X$. Since $f(x*y) = [x*y]_I = [x]_I \circ [y]_I = f(x) \circ f(y)$, proving that f is a KK-homomorphism. Next we will show f is surjective, let $[x]_I \in X/I$ and $x \in X$. Then $f(x) = [x]_I$, so f is surjective. Finally, to show that $\mathrm{Ker}f = I$, let $x \in \mathrm{Ker}f$. We get that $[x]_I = f(x) = [0]_I$, then $x \sim 0$. It follows that $x * 0 \in I$ and $0 * x \in I$. By hypothesis, $0 \in I$. Hence, $x \in I$, this mean $\mathrm{Ker}f \subseteq I$. To show that $I \subseteq \mathrm{Ker}f$, let $x \in I$. Since I is a closed ideal of X, we have $0 \in I$. Thus $x * 0 \in I$ and $0 * x \in I$. It follows that $x \sim 0$, so $[x]_I = [0]_I$. Since $f(x) = [x]_I = [0]_I$, then $x \in \mathrm{Ker}f$. Accordingly, $\mathrm{Ker}f = I$. □

Theorem 3.8. *Let f be a KK-homomorphism of a KK-algebra $(X, *, 0_X)$ onto a KK-algebra $(Y, \cdot, 0_Y)$ and I be an ideal of X contain in* $\mathrm{Ker}f$. *Let g be the natural KK-homomorphism of X onto X/I then there exists a unique KK-homomorphism h of X/I onto Y such that $f = hog$. Furthermore, h is an injective if and only if* $I = \mathrm{Ker}f$.

Proof. Define the map $h : X/I \to Y$ by $h([a]_I) = f(a)$ for all $[a]_I \in X/I$.

We first show that, h is well-defined, let $[a]_I, [b]_I \in X/I$ be such that $[a]_I = [b]_I$. We get that $a \sim b$, so $a * b \in I$ and $b * a \in I$. Since $I \subseteq \mathrm{Ker}f$, $a * b \in \mathrm{Ker}f$ and $b*a \in \mathrm{Ker}f$. Thus $f(a) \cdot f(b) = f(a*b) = 0_Y$ and $f(b) \cdot f(a) = f(b*a) = 0_Y$. From KK-3, $f(a) = f(b)$. Hence h is well-defined.

We will show that h is KK-homomorphism. Let $[a]_I, [b]_I \in X/I$. Then $h([a]_I \circ [b]_I) = h([a * b]_I) = f(a * b) = f(a) \cdot f(b) = h([a]_I) \cdot h([b]_I)$, proving that h is a KK-homomorphism.

Next, to show that $f = hog$. For any $a \in X$, then $(hog)(a) = h(g(a)) = h([a]_I) = f(a)$. Hence $hog = f$.

Finally, if $h' : X/I \to Y$ is another function such that $f = h'og$. Let $[a]_I \in X/I$. The equation $h([a]_I) = f(a) = (h'og)(a) = h'(g(a)) = h'([a]_I)$. Thus $h([a]_I) = h'([a]_I)$, for all $[a]_I \in X/I$.

Now, we will show that h is injective if and only if $I = \mathrm{Ker}f$. Suppose firstly that h is injective and $a \in \mathrm{Ker}f$. Then $h([0_X]_I) = 0_Y = f(a) = h([a]_I)$ and since h is an injective, thus $[0_X]_I = [a]_I$. It follows that $0_X \sim a$, then $0_X * a \in I$ and $a * 0_X \in I$. By hypothesis, $0_X \in I$. Hence, $a \in I$, this mean $\mathrm{Ker}f \subseteq I$. This show that $\mathrm{Ker}f = I$.

On the other hand, suppose that $\mathrm{Ker}f = I$ and $[a]_I, [b]_I \in X/I$ such that $h([a]_I) = h([b]_I)$. Then $f(a) = f(b)$, it follows that $f(a * b) = f(a) \cdot f(b) = 0_Y$. Thus $a * b \in \mathrm{Ker}f$. Since $Kerf = I$, so $a * b \in I$. Similarly, $b * a \in I$. Hence

$a \sim b$, proving that $[a]_I = [b]_I$. This show that h is injective. This completes the proof. □

Next, we state the first isomorphism of KK-algebras as the following theorem.

Theorem 3.9. (First Isomorphism Theorem) *If f be a KK-homomorphism of a KK-algebra $(X, *, 0_X)$ into a KK-algebra $(Y, \cdot, 0_Y)$, then the quotient KK-algebra $X/\mathrm{Ker}(\phi)$ is isomorphic to $\phi(X)$.*

Proof. Let $\phi : X \to Y$ be a KK-homomorphism and let $K = \mathrm{Ker}(\phi) = \{a \in X : \phi(a) = 0_Y\}$. We get that $X/K = \{[a]_K : a \in X\}$, where $[a]_K = \{b \in X : a \sim b\}$. From theorem 3.4(5), we have $\mathrm{Ker}(\phi)$ is an ideal of X. Thus $(X/K, \circ, [0]_K)$ is a KK-algebra and $\phi(X) = \{\phi(a) : a \in X\}$.

Assume that $f : X/K \to \phi(X)$ defined by $f([a]_K) = \phi(a)$, where $[a]_K \in X/K$.

Let $[a]_K, [b]_K \in X/K$ be such that $[a]_K = [b]_K$. Then $a \sim b$, it follows that $a * b \in K$ and $b * a \in K$. Thus $\phi(a) \cdot \phi(b) = 0_Y = \phi(b) \cdot \phi(a)$. By KK-3, we get that $\phi(a) = \phi(b)$. Hence f is well-defined.

Let $[a]_K, [b]_K \in X/K$. We get that $f([a]_K \circ [b]_K) = f([a * b]_K) = \phi(a * b) = \phi(a) \cdot \phi(b) = f([a]_K) \cdot f([b]_K)$. This show that f is a KK-homomorphism.

Let $[a]_K, [b]_K \in X/K$ be such that $f([a]_K) = f([b]_K)$. Then $\phi(a) = \phi(b)$, it follows that $\phi(a * b) = \phi(a) \cdot \phi(b) = 0_Y$. Thus $a * b \in \mathrm{Ker}(\phi) = K$. Similarly, $b * a \in K$. We see that $a \sim b$, this mean $[a]_K = [b]_K$. Hence f is an injective.

Let $a \in \phi(X)$. Then there exists $b \in X$ such that $a = \phi(b)$ and $[b]_K \in X/K$. Thus $f([b]_K) = \phi(b) = a$. Therefore f is a surjective, proving our theorem. □

It is easy to check that if A is an ideal of KK-algebra X and B is an ideal of A, then B is an ideal of X. So, it follows that B is an ideal of $A \cup B$ and $A \cap B$ is an ideal of A.

Theorem 3.10. (Second Isomorphism Theorem) *Let X be a KK-algebra and A, B be ideals of X. If $A \cup B$ is a KK-algebra, then the quotient KK-algebras $A/(A \cap B)$ and $(A \cup B)/B$ are isomorphic.*

Proof. Let $\phi : A \to (A \cup B)/B$ be a map by $\phi(x) = [x]_B$ for all $x \in A$. It is obvious that ϕ is well defined. Let $[x]_B \in (A \cup B)/B$. If $x \in A$, then $[x]_B = \phi(x)$. If $x \in B$, then $[x]_B = [0]_B = \phi(0)$. Thus ϕ is onto $(A \cup B)/B$. Consider the equation

$$\begin{aligned} \phi(x * y) &= [x * y]_B \\ &= [x]_B \circ [y]_B \\ &= \phi(x) \circ \phi(y). \end{aligned}$$

Shows that ϕ is a KK-homomorphism.

Now let $x \in \operatorname{Ker}(\phi)$. Then we get $\phi(x) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Since $\operatorname{Ker}(\phi) \subseteq A$, so $x \in A \cap B$. Hence $\operatorname{Ker}(\phi) \subseteq A \cap B$. On the other hand, let $x \in A \cap B$. Then $x \in B$. Thus $\phi(x) = [x]_B = [0]_B$, so $x \in \operatorname{Ker}(\phi)$. Hence $A \cap B \subseteq \operatorname{Ker}(\phi)$. Therefore, $\operatorname{Ker}(\phi) = A \cap B$. From theorem 3.9, immediately gives us that $A/(A \cap B) \cong (A \cup B)/B$. □

Next, we state the third isomorphism theorem of KK-algebras.

Theorem 3.11. (Third Isomorphism Theorem) *Let X be a KK-algebra and A, B be ideals of X, with $A \subseteq B \subseteq X$. Then:*

(1) the quotient B/A is an ideal of the quotient X/A, and

(2) the quotient KU-algebra $(X/A)/(B/A)$ is isomorphic to X/B.

Proof. (1) To show that B/A is an ideal of X/A. It is clear that $B/A \subseteq X/A$ and $[0]_I \in B/A$. Let $[x]_A \circ [y]_A \in B/A$ and $[x]_A \in B/A$. Then $x * y \in B$ and $x \in B$. Since B is an ideal of X, $y \in B$, so $[y]_A \in B/A$. Therefore, B/A is an ideal of X/A.

(2) Let $\phi : X/A \to X/B$ defined by $\phi([x]_A) = [x]_B$. Assume that $[x]_A = [y]_A$. Then $x \sim y$ determined by A, that is $x*y, y*x \in A$. Since $A \subseteq B$, $x*y, y*x \in B$. Thus $x \sim y$ determined by B, and hence $[x]_B = [y]_B$. Then $\phi([x]_A) = \phi([y]_A)$. Therefore, ϕ is well defined. Next, to show that ϕ is onto X/B, let $[x]_B \in X/B$. If $x \in X$ and $x \notin B$, then $[x]_B = \phi([x]_A)$. If $x \in B$, then $[x]_B = [0]_B = \phi([0]_B$. Hence ϕ is onto. Consider the equation

$$\begin{aligned}
\phi([x]_A \circ [y]_A) &= \phi([x * y]_A) \\
&= [x * y]_B \\
&= [x]_B \circ [y]_B \\
&= \phi([x]_A) \circ \phi([y]_A).
\end{aligned}$$

Shows that ϕ is a KK-homomorphism.

Finally, to show that $\operatorname{Ker}(\phi) = B/A$, let $[x]_A \in \operatorname{Ker}(\phi)$. Then $\phi([x]_A) = [0]_B$, so $[x]_B = [0]_B$. It follows that $x \in B$. Now we have $[x]_A \in B/A$. Hence $\operatorname{Ker}(\phi) \subseteq B/A$. Going the other hand, let $[x]_A \in B/A$. We get that $\phi([x]_A) = [x]_B = [0]_B$, since $x \in B$. Thus $[x]_A \in \operatorname{Ker}(\phi)$, and hence $A/B \subseteq \operatorname{Ker}(\phi)$. Consequently, $\operatorname{Ker}(\phi) = B/A$. By theorem 3.9, $(X/A)/(B/A)$ is isomorphic to X/B. □

It turns out that an analogous result of the third isomorphism theorem for groups is also true for KK-algebras.

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