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SOLVING RICCATI EQUATION USING ADOMIAN DECOMPOSITION METHOD

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Abstract: Riccati equation with variable coefficient and constant coefficient is considered. The numerical method based on the Adomian decomposition is introduced to obtained results and compared with the exact solution to show that the Adomian decomposition method is a powerful method for the solution of nonlinear differential equations. The results obtained are presented and show only few term are required to obtain an approximate solution.

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1. Introduction

The decomposition method has been shown [1-2] to solve effectively, easily and accurately a large class of linear and nonlinear equations with approximate solutions which converge rapidly to accurate solutions. The Adomian decomposion method has succeefully been applied to many situations for example D.J. Evans

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and K.R. Raslan [3] solve delay differential equation problems using Adomian decomposition method. The Adomian decomposition method has been applied for solving fully non-linear Sine-Gordon equation by Yuchun Wang et al [4]. The Adomian decomposition has beent applied for finding the solution to nonlinear vibrations of multiwalled carbon nanotubes by Weilem et al [5] Adomian decomposition method is a powerful approach in nonlinear differential equations and accuracy of it depends on number of used partial solutions. Also, solution of this method has a fast convergence to exact solution generally. Recently, some modification on this method have been presented [7,8]. Modification of method is in quality of computation on convergence of method. In this paper, we apply Adomian method to Riccati equation and find solution. Also two example will be presented and method will be applied to them and solutions will be compared.

2. Analysis of the Method

Let the general form of a differential equation be

$$Q_Y = k, \tag{1}$$

where Q is the non-linear differential operator with linear and non-linear terms. The linear term is decomposed as

$$Q = L + J, \tag{2}$$

where L is easily invertible operator and J is the remainder of the linear operator. The equation may be written as

$$L_Y + J_Y + N_Y = k, \tag{3}$$

where N_Y corresponds to the non-linear terms.Solving L_Y from (3) we have

$$L_Y = k - R_Y - N_Y.$$

Since L is invertible, the equivalent expression is

$$L^{-1}(L_Y) = L_k^{-1} - L^{-1}(R_Y) - L^{-1}(N_Y). \tag{4}$$

If L is a second order operation, then L^{-1} is a two fold integration operation

$$L^{-1} = \int\int(\#)dx_1dx_2,$$

and

$$L^Y = Y(x) - Y(0) - xY^1(0).$$

Then equation (4) for Y yields

$$Y(x) = Y(0) + xY^1(0) + L^{-1}(g) - L^{-1}(RY) - L^{-1}(NY). \tag{5}$$

Therefore, Y can be presented as a series

$$Y(x) = \sum_{n=0}^{\infty} Yn, \tag{6}$$

with Y_0 identified as

$$Y(x_0) + xY^1(x_0) + L^{-1}(k) \text{ and } Y_n(n > 0)$$

is to be determined. The non-linear term N(Y) will be decomposed by the infinity series of Adomian polynomials

$$N(Y) = \sum_{n=0}^{\infty} Fn, \tag{7}$$

where $F_{n's}$ are obtained by writting

$$\gamma(\lambda) = \sum_{n=0}^{\infty} \lambda^n Y_n, \tag{8}$$

$$N(\gamma(\lambda)) = \sum_{n=0}^{\infty} \lambda^n F_n, \tag{9}$$

where λ is a parameter introduced for convenience, from equation (8) and (9) we have

$$F_n = \frac{1}{n!}\frac{d^n}{d\lambda}[N(\sum_{n=0}^{\infty} \lambda^n Y_n)]_\lambda = 0(10)$$

substitute (6) and (9) into (5), we have

$$\sum_{n=0}^{\infty} Y_n = Y0 - L^{-1}[J(\sum_{n=0}^{\infty} Y_n - L^{-1}(\sum_{n=0}^{\infty} F_n)].$$

Then we can write

$$Y0 = Y0 - (x_0) + xY^1(x_0) + L^{-1}(k)$$

$$
\begin{aligned}
Y1 &= -L^{-1}J(Y_0) - L^{-1}(F_0) \\
Y2 &= -L^{-1}J(Y_1) - L^{-1}(F_1) \\
Y3 &= -L^{-1}J(Y_2) - L^{-1}(F_2) \\
Y4 &= -L^{-1}J(Y_3) - L^{-1}(F_3) \\
&\vdots \\
Y_{n+1} &= -L^{-1}J(Y_n) - L^{-1}(F_n),
\end{aligned} \tag{11}
$$

we considered the solution $Y(x)$ as

$$Y = \lim_{n\to\infty} \Psi_n,$$

where the (n+1) term approximation of the solution is defined in the following from

$$\Psi_n = \sum_{n=0}^{\infty} Y_n(x), \quad n > 0$$

In many problems, the practical solutionΨ_n the n-term approximation is converging and accurate for low valves of n. We apply this method to many numerical examples and results we obtained have shown a high degree of accuracy.

3. Application

In this section, we presented some numerical and analytical solutions for the Riccati differential equation [10]

$$
\begin{aligned}
\frac{dY}{dx} =& M(x)Y + J(x)Y^2 + H(x), \\
Y(0) =& G(x),
\end{aligned} \tag{12}
$$

$M(x)$, $J(x)$, $H(x)$ and $U(x)$ are scale.

4. Numerical Examples

Example 4.1. Consider the following example

$$
\begin{aligned}
\frac{dY}{dx} &= Y^2 + 5, \\
Y(0) &= 0.
\end{aligned} \tag{13}
$$

Adomian polynomial can be derived as follows

$$\begin{aligned} F_0 &= Y0^2 \\ F_1 &= 2Y0Y1 \\ F_2 &= 2Y0Y2 + Y1^2 \\ F_3 &= 2Y0Y3 + 2Y1Y2 \\ F_4 &= 2Y0Y4 + 2Y1Y2 + Y1^2 \\ &\vdots \end{aligned}$$

Using equation (11) the rest of the polynomial con be constructed and we have

$$\begin{aligned} Y0 &= 5x \\ Y1 &= \frac{25}{3}x^3 \\ Y2 &= \frac{50}{3}x^5 \\ Y3 &= \frac{2125}{63}x^7 \\ Y4 &= \frac{38750}{567}x^9 \\ Y5 &= \frac{387500}{6237}x^{11} \\ Y6 &= \frac{53975000}{6237}x^{13}. \end{aligned}$$

Hence, we have

$$\begin{aligned} Y(x) = 5x + \frac{25}{3}x^3 + \frac{50}{3}x^5 + \frac{2125}{62}x^7 + \frac{38750}{567}x^9 \\ + \frac{387500}{6237}x^{11} + \frac{53975000}{6237}x^{13} + \cdots . \end{aligned} \tag{14}$$

Example 4.2. Consider the following example

$$\begin{aligned} \frac{dY}{dx} =& x^5 Y(x)^2 - 2x^6 Y(x) + x^7 + 1, \\ Y(0) =& 0, \end{aligned} \tag{15}$$

x	Approximate solution	Exact Solution	Absolute error
0	1.000000000	1.000000000	$0 * 10^0$
0.01	1.010000000	1.010000000	$0 * 10^0$
0.02	1.020000000	1.020000000	$0 * 10^0$
0.03	1.030000000	1.030000000	$0 * 10^0$
0.04	1.040000000	1.040000001	1.10^{-9}
0.05	1.050000000	1.050000003	$3 * 10^{-9}$
0.06	1.060000000	1.060000008	$8 * 10^{-9}$
0.07	1.070000000	1.070000020	$2.0 * 10^{-8}$
0.08	1.080000000	1.080000044	$4.4 * 10^{-9}$
0.09	1.090000001	1.090000088	$8.7 * 10^{-8}$
0.10	1.100000001	1.100000167	$1.66 * 10^{-7}$

Table 1: Approximation solution for Example 4.1

$$\begin{aligned} M(x) &= -2x^6 \\ J(x) &= x^5 \\ H(x) &= x^7 + 1 \\ U(x) &= 1. \end{aligned}$$

Use the ADM, we have

$$Y(x) = Y(1) + L^{-1}(x^5Y^2) - 2L^{-1}(x6Y)L^{-1} + L^{-1}(x7+1), \tag{16}$$

where

$$L^{-1}(\#) = \int_0^x (\#)dx. \tag{17}$$

Let

$$Y(x) = \sum_{n=0}^{\infty} Y_n, \qquad Y^2 = \sum_{n=0}^{\infty} F_n$$

The recurrent scheme of Adomian decomposition method is written as

$$\begin{aligned} Y0 =& Y(1) + L^{-1}(x^{-1}+1), \\ Y_{n+1} =& L^{-1}(xF_n) - 2L^{-1}(x^6Y_n), \end{aligned} \qquad n \geq 0. \tag{18}$$

The ADM polynomials are calculated by (12) then we have

$$Y0 = 1 + \frac{1}{8}x^8 + x,$$

x	Approximate solution	Exact Solution	Absolute error
0	0.000000000	0.000000000	$1 * 10^{-11}$
0.01	0.05000833500	0.05000833499	$0 * 10^{0}$
0.02	0.1000667200	0.1000667200	$0 * 10^{0}$
0.03	0.1502254057	0.1502254057	$0 * 10^{0}$
0.04	0.2005350455	0.2005350455	1.10^{0}
0.05	0.2510469015	0.2510469013	$2.0 * 10^{-10}$
0.06	0.3018130551	0.3018130549	$2.0 * 10^{-10}$
0.07	0.3528866256	0.3528866254	$2.0 * 10^{-10}$
0.08	0.4043219967	0.4043219966	$1.1 * 10^{-10}$
0.09	0.4561750550	0.4561750548	$2.0 * 10^{-10}$
0.10	0.5085034419	0.5085034425	$6.0 * 10^{-10}$

Table 2: Approximation solution for Example 4.2

implies that

$$F_0 = Y0^2 = (1 + \frac{1}{8}x^8 + x)^2.$$

By equation (4) we have

$$Y1 = L^{-1}(x^5 F_0) - 2L^{-1}(x^6 Y0). \tag{19}$$

Then we have

$$\begin{aligned}
Y1 &= -\frac{1}{1408}x^{22}, \\
Y2 &= \frac{1}{202750}x^{36}, \\
Y3 &= -\frac{31}{892108800}x^{50}, \\
Y4 &= \frac{7}{28547481600}x^{64},
\end{aligned}$$

we use $\Psi_n(x)$ as the approximating of $Y(x)$ for example $n = 4$ to obtain:

$$\begin{aligned}
Y(x) = 1 + x + \frac{1}{8}x^8 - \frac{1}{1408}x^{22} + + \frac{1}{202750}x^{36} - \frac{31}{892108800}x^{50} \\
+ \frac{7}{28547481600}x^{64} + \cdots .
\end{aligned} \tag{20}$$

5. Conclution

We introduced a new technique, Adomian decomposition method, to numerically solve the Riccati equation is presented. All the numerical results obtained by using the Adomian decomposition method described earlier show very good agreement with the exact solutions for only a few terms. Comparing the decomposition method with several other methods that have been advanced for solving Riccati equation shows that the new technique is reliable, powerful and promissing. The computations associated with the examples in this paper have been performed using Maple 14.

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