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OSCILLATION AND NONOSCILLATION FOR CERTAIN CLASS OF FIRST AND SECOND ORDER GENERALIZED $\alpha-$DIFFERENCE EQUATIONS

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Abstract: In this paper, the authors discuss the oscillation and nonoscillation of the solutions of the generalized $\alpha-$difference equations

$$\Delta_{\alpha(\ell)}u(k) + \delta \sum_{i=1}^{m} f_i(k)F_i(u(g_i(k))) = 0, \ k \in [0, \infty), \ \delta = \pm 1 \tag{0.1}$$

$$\text{and } \Delta^2_{\alpha(\ell)}u(k) = f(k, u(k), \Delta_{\beta(\ell)}u(k)), \ k \in [0, \infty) \tag{0.2}$$

where α, β and ℓ are positive real fixed constants, f_i, F_i are defined on $\mathbb{R}$, for each $i, 1 \le i \le m$, $\{g_i(k)\} \subseteq [0, \infty)$ and f is defined on $[0, \infty) \times \mathbb{R}^2$.

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1. Introduction

The basic theory of difference equations is based on the operator Δ defined as $\Delta u(k) = u(k+1) - u(k),\ k \in \mathbb{N} = \{0, 1, 2, 3, \cdots\}$. Eventhough many authors ([1], [9]-[10]) have suggested the definition of Δ as $\Delta u(k) = u(k+\ell) - u(k),\ k \in \mathbb{R},\ \ell \in \mathbb{R} - \{0\},$ (3) no significant progress has taken place on this line. But recently, E. Thandapani, M.M.S. Manuel and G.B.A.Xavier [5] considered the definition of Δ as given in (3) and developed the theory of difference equations in a different direction. For convenience, the authors labeled the operator Δ defined by (3) as Δ_ℓ and by defining its inverse Δ_ℓ^{-1}, many interesting results and applications in number theory (see [5],[6]-[8]) were obtained. By extending the study related to the sequences of complex numbers and ℓ to be real, some new qualitative properties of the solutions like rotatory, expanding, shrinking, spiral and weblike of difference equations involving Δ_ℓ were obtained. The results obtained using Δ_ℓ can be found in ([5]-[8]). Jerzy Popenda, et all [2, 3] worked on equations (0.1) and (0.2) when $\ell = 1$. In this paper, we introduce the operator $\Delta_{\alpha(\ell)}$ and discuss the oscillation and nonoscillation for the $\alpha-$difference equations (0.1) and (0.2).

Throughout this paper, we use the following notations:

(i) $f_i(k) \geq 0$ for all $k \in [0, \infty),\ 1 \leq i \leq m$,

(ii) $\lim\limits_{k\to\infty} g_i(k) = \infty,\ 1 \leq i \leq m,\ uF_i(u) > 0$ for $u \neq 0,\ 1 \leq i \leq m$,

(iii) $[k]$ and $\lceil x \rceil$ denotes the integer and upper integer part of x,

(iv) $\mathbb{N}_\ell(j) = \{j, j+\ell, j+2\ell, \ldots\}$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$ when $k \in [a, \infty)$.

2. Preliminaries

In this section, we present some basic definitions and results already obtained which is useful for further discussion.

Definition 2.1. Let $u(k),\ k \in [0, \infty)$ be a real valued function and $\ell \in (0, \infty)$. Then, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ on $u(k)$ is defined as

$$\Delta_{\alpha(\ell)} u(k) = u(k+\ell) - \alpha u(k). \tag{2.1}$$

When $\alpha = 1$, the generalized α-difference operator $\Delta_{\alpha(\ell)}$ becomes the generalized difference operator Δ_ℓ [5]. When $\alpha = 1$ and $\ell = 1$, then $\Delta_{\alpha(\ell)}$ is the usual difference operator Δ.

Definition 2.2. (see [5]) Let $u(k),\ k \in [0,\infty)$ be a real valued function and $\ell \in (0,\infty)$. Then, the inverse operator Δ_ℓ^{-1} is defined as,

$$\text{if } \Delta_\ell v(k) \ = \ u(k), \text{ then } \ v(k) \ = \ \Delta_\ell^{-1} u(k) + c_j, \tag{2.2}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j),\ j = k - \left[\frac{k}{\ell}\right]\ell$.

Definition 2.3. The inverse of the generalized α-difference operator denoted by $\Delta_{\alpha(\ell)}^{-1}$ is defined as follows. If $\Delta_{\alpha(\ell)} v(k) \ = \ u(k)$,

$$\text{then } \Delta_{\alpha(\ell)}^{-1} u(k) \ = \ v(k) - \alpha^{\left[\frac{k}{\ell}\right]} c_j. \tag{2.3}$$

The following is an extension of Lemma 4.2 in [5]

Lemma 2.4. *[5] If the real valued function $u(k)$ is defined for all*

$$k \in [a,\infty), \text{ then } \Delta_\ell^{-1} u(k) = \sum_{r=1}^{\left[\frac{k-a}{\ell}\right]} u(k - r\ell) + c_j, \tag{2.4}$$

where c_j is a constant for all $k \in \mathbb{N}_\ell(j)$, $j = k - a - \left[\frac{k-a}{\ell}\right]\ell$.

Theorem 2.5. *If $\Delta_\ell v(k) = u(k)$ for $k \in [k_2,\infty)$ and $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, then* $v(k) - v(k_2 + j) = \displaystyle\sum_{r=0}^{\left[\frac{k-k_2-j-\ell}{\ell}\right]} u(k_2 + j + r\ell)$.

Proof. The proof follows by Definition 2.2, Lemma 2.4 and $c_j = v(k_2 + j)$. □

Definition 2.6. The solution $u(k)$ of (0.1) or (0.2) is called oscillatory by $\ell-$steps if for any $k_1 \in [a,\infty)$ there exists a $k_2 \in [k_1,\infty)$ such that $u(k_2)u(k_2 + \ell) \le 0$. The difference equation itself is called oscillatory by $\ell-$steps if all its solutions are oscillatory by $\ell-$steps. If the solution $u(k)$ is not oscillatory by $\ell-$steps then it is said to be nonoscillatory by $\ell-$steps (i.e. $u(k)u(k+\ell) > 0$ for all $k \in [k_1,\infty)$).

Lemma 2.7. *The relation between Δ_ℓ and $\Delta_{\alpha(\ell)}$ is given by*

$$\alpha^{\left\lceil\frac{k+\ell}{\ell}\right\rceil} \Delta_\ell\Big(\frac{u(k)}{\alpha^{\left\lceil\frac{k}{\ell}\right\rceil}}\Big) = \Delta_{\alpha(\ell)} u(k).$$

3. Main Results

In this section we present the oscillatory behaviour of solutions of (0.1) and (0.2). First we discuss the case when $\delta = 1$.

Theorem 3.1. *Let $\alpha \geq 1$ and let there exists an index $1 \leq i \leq m$ such that $|F_i(u)|$ is bounded away from zero if $|u|$ is bounded away from zero and for some $k_2 \in [0, \infty)$,*

$$\sum_{r=0}^{\infty} \alpha^{-\left\lceil \frac{k_2+j+r\ell}{\ell} \right\rceil} f_i(k_2 + j + r\ell) = \infty \text{ for all } 0 \leq j < \ell. \tag{3.1}$$

Then, every solution $u(k)$ of (0.1) is either oscillatory or $u(k) = o(\alpha^{\lceil \frac{k}{\ell} \rceil})$.

Proof. Let $u(k)$ be a nonoscillatory solution of (0.1) and suppose that $u(k) > 0$ eventually. Then, there exists a $k_1 \in [0, \infty)$ such that $u(k) > 0$ and $u(g_i(k)) > 0,\ 1 \leq i \leq m$ for all $k \in [k_1, \infty)$.

Therefore, we have $\Delta_{\alpha(\ell)} u(k) = \alpha^{\lceil \frac{k+\ell}{\ell} \rceil} \Delta_\ell (u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil}) \leq 0$ for all $k \in [k_1, \infty)$. Hence, $\alpha^{-\lceil \frac{k}{\ell} \rceil} u(k)$ is nonincreasing for all $k \in [k_1, \infty)$, thus

$$\lim_{k \to \infty} \alpha^{-\lceil \frac{k}{\ell} \rceil} u(k) = \mu \geq 0$$

exists. We shall show that $\mu = 0$. Suppose $\mu > 0$, then there exists $k \in [k_2, \infty)$, and from the given hypotheses there exists a positive constant c such that $F_i(u(g_i(k))) \geq c$ for all $k \in [k_2, \infty)$.

On the other hand from (0.1) and Lemma 2.7, we have

$$\Delta_\ell (u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil}) + \alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f_i(k) F_i(u(g_i(k))) \leq 0,\ k \in [k_1, \infty) \tag{3.2}$$

and hence by Theorem 2.5

$$\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} \leq \frac{u(k_2 + j)}{\alpha^{\lceil \frac{k_2+j}{\ell} \rceil}} - \frac{c}{\alpha} \sum_{r=0}^{\frac{k-j-k_2-\ell}{\ell}} \alpha^{-\left\lceil \frac{k_2+j+r\ell}{\ell} \right\rceil} f_i(k_2 + j + r\ell),\ k \in [k_2, \infty),$$

where $j = k - k_2 - \left[\frac{k-k_2}{\ell} \right] \ell$. But, in view of (3.1), this leads to a contradiction to our assumption that $u(k) > 0$ eventually. The case $u(k) < 0$ eventually can be treated similarly. $\square$

Example 3.2. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}u(k)+2\alpha^{\lceil\frac{2k+2\ell}{\ell}\rceil}(-1)^{\lceil\frac{k}{\ell}\rceil}=0$$

and for

$$f_1(k)=\alpha^{\lceil\frac{2k+2\ell}{\ell}\rceil},\quad F_1(u)=\frac{u(k)}{\alpha^{\lceil\frac{2k+\ell}{\ell}\rceil}}$$

all the conditions of Theorem 3.1 hold and hence every solution is oscillatory. Infact $u(k)=(-1)^{\lceil\frac{k}{\ell}\rceil}\alpha^{\lceil\frac{2k+\ell}{\ell}\rceil}$ is one such solution.

Theorem 3.3. *If $0<\alpha<1$ and conditions of Theorem 3.1 hold then every solution of (0.1) is oscillatory or $u(k)=o(\frac{1}{k})$.*

Proof. Let $u(k)>0$ be a nonoscillatory solution of (0.1). Then from (0.1) we obtain $u(k+\ell)\le\alpha u(k)\le u(k)$ for sufficiently large k. Then $u(k)$ is eventually decreasing and the series $\sum\limits_{r=0}^{\infty}u(r\ell+j)$ converges for all $0\le j<\ell$. Hence, it follows from D'Alembert criterion, that $\lim\limits_{k\to\infty}u(k)=0$. □

Example 3.4. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}u(k)+(k(\alpha+1)+\alpha\ell)\alpha^{\lceil\frac{2k+2\ell}{\ell}\rceil}(-1)^{\lceil\frac{k}{\ell}\rceil}=0$$

and for

$$f_1(k)=k(\alpha+1)\alpha^{\lceil\frac{2k+3\ell}{\ell}\rceil},\quad f_2(k)=\ell\alpha^{\lceil\frac{2k+3\ell}{\ell}\rceil},F_1(u)=\frac{u(k)}{k\alpha^{\lceil\frac{2k+\ell}{\ell}\rceil}}$$

and the conditions of Theorem 3.3 hold and every solution is oscillatory. One solution is $u(k)=k(-1)^{\lceil\frac{k}{\ell}\rceil}\alpha^{\lceil\frac{2k+\ell}{\ell}\rceil}$.

Theorem 3.5. *If $\alpha\ge1$ and there exists an index $1\le i\le m$ such that $F_i(u)$ is nonincreasing on $\mathbb{R}\backslash\{0\}$ and*

$$\sum_{r=0}^{\infty}\alpha^{-\lceil\frac{k_2+j+r\ell}{\ell}\rceil}f_i(k_2+j+r\ell)F_i(c\alpha^{\lceil\frac{g(k_2+j+r\ell)}{\ell}\rceil})=\pm\infty,k_2\in[0,\infty),\qquad(3.3)$$

for any $c\neq0$, then every solution of (0.1) is oscillatory.

Proof. Proceeding as in the proof of Theorem 3.1 we obtain the inequality (3.2) and we observe that the sequence $\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}}$ is eventually nonincreasing, say for $k\ge k_1\in[0,\infty)$. Obviously, $k_2\in[k_1,\infty)$ such that $g_i(k)\ge k_1$ for $k\ge k_2$ and so there exist a positive constant c such that $u(g_i(k)) \leq c\alpha^{\left\lceil \frac{g_i(k)}{\ell} \right\rceil}$ for all $k \geq k_2$. Hence by our assumption, we have $F_i(u(g_i(k))) \geq F_i(c\alpha^{\left\lceil \frac{g_i(k)}{\ell} \right\rceil})$, for all $k \geq k_2$. Now from (3.2) we obtain $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}\Big) + \alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f_i(k)F_i(c\alpha^{\left\lceil \frac{g_i(k)}{\ell} \right\rceil}) \leq 0,\ k \geq k_2$. Therefore, from Theorem 2.5 and taking $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, we find

$$\frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}} + \alpha^{-1} \sum_{r=0}^{\frac{k-j-k_2}{\ell}} \alpha^{-\left\lceil \frac{k_2+j+r\ell}{\ell} \right\rceil} f_i(k_2+j+r\ell)F_i(c\alpha^{\left\lceil \frac{g_i(k_2+j+r\ell)}{\ell} \right\rceil}) \leq \frac{u(k_2+j)}{\alpha^{\left\lceil \frac{k_2+j}{\ell} \right\rceil}},$$

which implies for all $k \geq k_2$

$$\sum_{r=0}^{\frac{k-j-k_2}{\ell}} \alpha^{-\left\lceil \frac{k_2+j+r\ell}{\ell} \right\rceil} f_i(k_2+j+r\ell)F_i(c\alpha^{\left\lceil \frac{g_i(k_2+j+r\ell)}{\ell} \right\rceil}) \leq c_1\ (c_1 > 0).$$

By the positivity of $u(k)$. But this contradicts our assumption, and thus our assertion is true. The case $u(k) < 0$ eventually can be treated similarly. □

Example 3.6. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)} u(k) + 2\alpha(-\alpha)^{\lceil \frac{k}{\ell} \rceil} = 0$$

and for $f_1(k) = \alpha(-\alpha)^{2\lceil \frac{k}{\ell} \rceil}, F_1(k) = \frac{1}{(-\alpha)^{\lceil \frac{k}{\ell} \rceil}}$ all the conditions of Theorem 3.5 hold and hence every solution is oscillatory. Infact $u(k) = (-\alpha)^{\lceil \frac{k}{\ell} \rceil}$ is one such solution.

Theorem 3.7. *If $0 < \alpha < 1$ and for some index $1 \leq i \leq m$, (3.1)*

$$\textit{hold and } \lim_{u\to 0^+} \inf F_i(u) = a > 0,\ \lim_{u\to 0^-} \sup F_i(u) = b < 0, \tag{3.4}$$

then, every solution of (0.1) is oscillatory.

Proof. Suppose there exist a nonoscillatory solution $u(k)$ and assume that $u(k)$ is eventually positive. Then by Theorem 3.3 we obtain $\lim_{k\to\infty} u(k) = 0$. Hence $u(g_i(k)) \to 0$ as $k \to \infty$. From our assumption $F_i(u(g_i(k))) \geq \frac{a}{2}$ for all sufficiently large k, for $k \geq k_1 \in [0, \infty)$. Furthermore from (0.1) we obtain the inequality (3.2). Which implies $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}\Big) + \frac{a}{2\alpha}\alpha^{-\lceil \frac{k}{\ell} \rceil} f_i(k) \leq 0,\ k \geq k_1$.

This implies by Theorem 2.5

$$\frac{u(k+\ell)}{\alpha^{\lceil\frac{k+\ell}{\ell}\rceil}}+\frac{a}{2\alpha}\sum_{r=0}^{\frac{k-k_1-j}{\ell}}\alpha^{-\lceil\frac{k_1+j+r\ell}{\ell}\rceil}f_i(k_1+j+r\ell)\le\frac{u(k_1+j)}{\alpha^{\lceil\frac{k_1+j}{\ell}\rceil}},$$

which yields $\sum_{r=0}^{\frac{k-k_1-j}{\ell}}\alpha^{-\lceil\frac{k_1+j+r\ell}{\ell}\rceil}f_i(k_1+j+r\ell)\le c_2$ $(c_2>0)$ for $k\ge k_1$, since $u(k)>0$ for $k\ge k_1$, this contradicts our assumption. A similar argument hold in the case of an eventually negative solution, which completes the proof. □

Example 3.8. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}u(k)+u(k)(\alpha^2+1)+\ell\alpha^2\frac{u(k)}{k+\ell}=0$$

and for

$$f_1(k)=(\alpha^2+1)\alpha^{\lceil\frac{2k+3\ell}{\ell}\rceil},$$

$$f_2(k)=\ell\alpha^{\lceil\frac{2k+4\ell}{\ell}\rceil},F_1(k)=\frac{u(k)}{\alpha^{\lceil\frac{2k+3\ell}{\ell}\rceil}},F_2(k)=\frac{u(k)}{(k+\ell)\alpha^{\lceil\frac{2k+2\ell}{\ell}\rceil}}$$

and the conditions of Theorem 3.7 hold and every solution is oscillatory. $u(k)=(k+\ell)(-1)^{\lceil\frac{k}{\ell}\rceil}\alpha^{\lceil\frac{2k+\ell}{\ell}\rceil}$ is one such solution.

Theorem 3.9. *Let there exists an index* $1\le i\le m$ *and a positive constant* L *such that* $|F_i(u)|\ge L\,|u|$ *for* $u\in\mathbb{R}$, $\mathbb{N}_{(i)}=\{k\in[0,\infty):g_i(k)\le k\}$ *is an infinite set,* $Lf_i(k)\alpha^{\lceil\frac{g_i(k)-k-\ell}{\ell}\rceil}\ge 1$ *for all* $k\in\mathbb{N}_{(i)}$. *Then, the difference equation (0.1) is oscillatory.*

Proof. Let $u(k)$ be as in Theorem 3.1 so that $u(k)/\alpha^{\lceil\frac{k}{\ell}\rceil}$ is nonincreasing for all $k\in[k_1,\infty)$. Thus, for all $k\in\mathbb{N}_{(i)}\cap[k_1,\infty)$ it follows that

$$u(g_i(k))\ge u(k)\alpha^{\lceil\frac{g_i(k)-k}{\ell}\rceil}.$$

On the other hand, we have

$$\begin{aligned}u(k+\ell)\le&\alpha u(k)-f_i(k)F_i(u(g_i(k))),\ k\in[k_1,\infty)\\ \le&\alpha u(k)-Lf_i(k)u(g_i(k)),\ k\in[k_1,\infty)\\ \le&\alpha u(k)(1-Lf_i(k)\alpha^{\lceil\frac{g_i(k)-k-\ell}{\ell}\rceil})\le 0,\quad k\in\mathbb{N}_{(i)}\cap[k_1,\infty).\end{aligned}$$

But, this contradicts our assumption that $u(k)>0$ eventually. A similar contradiction hold for $u(k)<0$ eventually. □

Example 3.10. For the generalized $\alpha-$difference equation $\Delta_{\alpha(\ell)}u(k) + 2\alpha u(k) = 0$ and for $g(k) = k, f_1(k) = \alpha, F_1(k) = u(k)$ all the conditions of Theorem 3.9 hold and hence every solution is oscillatory. $u(k) = (-\alpha)^{\lceil \frac{k}{\ell} \rceil}$ is one such solution.

Theorem 3.11. *Let $\alpha \geq 1$ and let there exists an index $1 \leq i \leq m$ such that $F_i(u)$ is nondecreasing on $\mathbb{R}\backslash\{0\}$, and*

$$\int_0^{\beta} \frac{dt}{F_i(t)} < \infty \text{ and } \int_0^{-\beta} \frac{dt}{F_i(t)} < \infty \text{ for every } \beta > 0, \tag{3.5}$$

$$\sum_{r=0}^{\infty} \chi_{\mathbb{N}_{(i)}}(j + r\ell)\alpha^{-\lceil \frac{j+r\ell}{\ell} \rceil} f_i(j + r\ell) = \infty, \text{ for all } 0 \leq j < \ell \tag{3.6}$$

where $\chi_{\mathbb{N}_{(i)}}(j + r)$ is the characteristic function of the set $\mathbb{N}_{(i)}$ defined in Theorem 3.9. Then, the difference equation (0.1) is oscillatory.

Proof. Let $u(k)$ be as in Theorem 3.1 so that $u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil}$ is nonincreasing for all $k \in [k_1, \infty)$, and the inequality (3.2) hold. Let $k_2 \in [k_1, \infty)$ be so large that $g_i(k) \geq k_1$ for all $k \in [k_2, \infty)$. Hence, for all $k \in \mathbb{N}_{(i)} \cap [k_2, \infty)$, we have $u(g_i(k)) \geq u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil}$, and consequently $F_i(u(g_i(k))) \geq F_i(u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil})$. Then, from (3.2) it follows that

$$-\frac{\Delta_\ell(u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil})}{F_i(u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil})} \geq \alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f_i(k), \quad k \in \mathbb{N}_{(i)} \cap [k_2, \infty).$$

However, for

$$\frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}} \leq t \leq \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}, \ [F_i(t)]^{-1} \geq [F_i(u(k)/\alpha^{\lceil \frac{k}{\ell} \rceil})]^{-1},$$

the above inequality implies

$$\alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f_i(k) \leq \int_{\psi_{\alpha(\ell)}(k+\ell)}^{\psi_{\alpha(\ell)}(k)} \frac{dt}{F_i(t)}, \quad k \in \mathbb{N}_{(i)} \cap [k_2, \infty),$$

where $\psi_{\alpha(\ell)}(k) = \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}$. Thus, summing over k leads to the inequality

$$\sum_{r=0}^{\frac{k-j-k_2}{\ell}} \chi_{\mathbb{N}_{(i)}}(k_2 + j + r\ell)\alpha^{-\lceil \frac{(k_2+j+r\ell)+\ell}{\ell} \rceil} f_i(k_2 + j + r\ell)$$

$$\leq \sum_{r=0}^{\frac{k-j-k_2}{\ell}} \chi_{\mathbb{N}_{(i)}}(k_2+j+r\ell) \int_{\psi_{\alpha(\ell)}(k_2+j+r\ell+\ell)}^{\psi_{\alpha(\ell)}(k_2+j+r\ell)} \frac{dt}{F_i(t)}$$
$$\leq \int_{\psi_{\alpha(\ell)}(k+\ell)}^{\psi_{\alpha(\ell)}(k_2+j)} \frac{dt}{F_i(t)}, \;\; k \in [k_2, \infty).$$

But, from (3.5) and (3.6) this leads to a contradiction. A similar argument hold for $u(k) < 0$ eventually. □

Example 3.12. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)} u(k) + \left(u(k)(\alpha + 8\alpha^2) - \frac{2\ell u(k+\ell)}{(2k+3\ell)} \right) = 0$$

and for

$$f_1(k) = (\alpha + 8\alpha^2)\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}, \quad f_2(k) = 2\ell\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil},$$
$$F_1(k) = \frac{u(k)}{\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}}, \quad F_2(k) = -\frac{u(k+\ell)}{(2k+3\ell)\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}}$$

all the conditions of Theorem 3.11 hold and every solution is oscillatory. Infact $u(k) = (-\alpha)^{\lceil \frac{2k+\ell}{\ell} \rceil} 2^{\frac{3k}{\ell}}(2k+\ell)$ is one such solution for some k.

Theorem 3.13. *Let $j = k - \left[\frac{k}{\ell}\right]\ell$ for $k \in [k_1, \infty)$. If $0 < \alpha \leq 1$ and there exists an index $1 \leq i \leq m$ such that $F_i(u)$ is nonincreasing on $\mathbb{R}\backslash\{0\}$,*

$$\lim_{k\to\infty} \sum_{r=0}^{\frac{k-j}{\ell}} \alpha^{\lceil \frac{k-j-r\ell}{\ell} \rceil} f_i(k-j-r\ell) = \infty, \tag{3.7}$$

then every bounded solution of (0.1) is oscillatory.

Proof. Arguing as in the proof of Theorem 3.15 we obtain the inequality (3.9). Since $u(k)$ is bounded there exist a positive constant M such that $F_i(u(g_i(k))) \geq M$ for all $k \geq k_1$. Now from (3.9) we have

$$\Delta_\ell \Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} \Big) \geq M\alpha^{-\lceil \frac{k+\ell}{\ell} \rceil} f_i(k), \quad k \geq k_1,$$

from where and by (3.7), Theorem 2.5 and $j = k - k_1 - \left[\frac{k-k_1}{\ell}\right]\ell$ it follows that

$$u(k+\ell) \geq \alpha^{\lceil \frac{k}{\ell} \rceil} u(k_1+j)$$

$$+M\sum_{r=0}^{\frac{k-j-k_1}{\ell}}\alpha^{\left\lceil\frac{k-j-k_1-r\ell}{\ell}\right\rceil}f_i(k_1+j+r\ell)\to\infty \text{ as } k\to\infty.$$

Thus we have a contradiction and the theorem follows. □

Example 3.14. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)}u(k)+\frac{(1+2\alpha^2)u(k)}{2\alpha}=0$$

and for

$$F_1(u)=(1+2\alpha^2)u(k),\quad f_1(k)=\frac{1}{2\alpha},$$

all the conditions of Theorem 3.13 hold and hence every solution is oscillatory. Infact $u(k)=\frac{1}{(-2\alpha)^{\lceil\frac{k}{\ell}\rceil}}$ is one such solution.

The following results are related to the case $\delta=-1$.

Theorem 3.15. *Let $0<\alpha\le 1$ and let there exists an index $1\le i\le m$ such that $F_i(u)$ is nondecreasing on $\mathbb{R}\backslash\{0\}$. If $j=k-k_2-\left[\frac{k-k_2}{\ell}\right]\ell$ for $k_2\in[0,\infty)$ and for any $c\neq 0$*

$$\lim_{k\to\infty}\sum_{r=0}^{\frac{k-k_2-j}{\ell}}\alpha^{\left\lceil\frac{k-k_2-j-r\ell}{\ell}\right\rceil}f_i(k_2+j+r\ell)F_i(c\alpha^{\left\lceil\frac{g_i(k_2+j+r\ell)}{\ell}\right\rceil})=\pm\infty, \tag{3.8}$$

then, every solution $u(k)$ of (0.1) is oscillatory or $|u(k)|\to\infty$ as $k\to\infty$.

Proof. Let $u(k)$ be a nonoscillatory solution of (0.1), and suppose that $u(k)>0$ eventually. Then, there exists a $k_1\in[0,\infty)$ such that $u(k)>0$ and $u(g_i(k))>0$ for $1\le i\le m$ for all $k\in[k_1,\infty)$. Therefore, we have $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}}\Big)\ge 0$ for all $k\in[k_1,\infty)$. Hence, $\alpha^{-\lceil\frac{k}{\ell}\rceil}u(k)$ is nondecreasing for all $k\in[k_1,\infty)$, which yields $u(k)\ge\alpha^{\left\lceil\frac{k-k_1}{\ell}\right\rceil}u(k_1)$, $k\in[k_1,\infty)$. Obviously, we can choose $k_2\in[k_1,\infty)$ such that $g_i(k)\ge k_1$ for all $k\in[k_2,\infty)$, and so $u(g_i(k))\ge c\alpha^{\left\lceil\frac{g_i(k)}{\ell}\right\rceil}$, $k\in[k_2,\infty)$, where $c=\frac{u(k_1)}{\alpha^{\lceil\frac{k_1}{\ell}\rceil}}$. On the other hand from (0.1), we obtain

$$\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil\frac{k}{\ell}\rceil}}\Big)\ge\alpha^{-\lceil\frac{k+\ell}{\ell}\rceil}f_i(k)F_i(u(g_i(k))),\ k\in[k_1,\infty) \tag{3.9}$$

and hence $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell}\rceil}}\Big) \geq \alpha^{-\lceil \frac{k+\ell}{\ell}\rceil} f_i(k)F_i(c\alpha^{g_i(k)})$, $k \in [k_2, \infty)$ from which and by Theorem 2.5 it follows that for $k \in [k_2, \infty)$,

$$u(k) \geq \alpha^{\left\lceil \frac{k-j-k_2}{\ell}\right\rceil} u(k_2 + j) + \sum_{r=0}^{\frac{k-j-k_2-\ell}{\ell}} \alpha^{\left\lceil \frac{k-j-k_2-r\ell-\ell}{\ell}\right\rceil} f_i(k_2 + j + r\ell)F_i(c\alpha^{\left\lceil \frac{g_i(k_2+j+r\ell)}{\ell}\right\rceil}).$$

The result now follows from (3.8). The case $u(k) < 0$ can be treated analogously. ☐

Example 3.16. For the generalized $\alpha-$difference equation $\Delta_{\alpha(\ell)}u(k) - 2u(k + \ell) = 0$ and for $F_1(u) = u(2(k + \ell))$, $f_1(k) = \frac{1}{(-\alpha)^{\lceil \frac{k+\ell}{\ell}\rceil}}$ all the conditions of Theorem 3.15 hold and hence every solution is oscillatory. One such solution is $u(k) = (-\alpha)^{\lceil \frac{k}{\ell}\rceil}$.

Theorem 3.17. *Let $\overline{\mathbb{N}}_{(i)} = \{k \in [0, \infty) : g_i(k) \geq k + \ell\}$. Suppose that $\alpha \geq 1$ and there exists an index $1 \leq i \leq m$ such that $F_i(s)$ is nondecreasing on $\mathbb{R}\backslash\{0\}$ and*

$$\int_b^\infty \frac{ds}{F_i(s)} < \infty \text{ and } \int_{-b}^{-\infty} \frac{ds}{F_i(s)} < \infty \text{ for any } b > 0, \tag{3.10}$$

$$\sum_{r=0}^{\infty} \chi_{\overline{\mathbb{N}}_{(i)}}(k + r\ell)\alpha^{-\lceil \frac{k+r\ell}{\ell}\rceil} f_i(k + r\ell) = \infty, \tag{3.11}$$

where $\chi_{\overline{\mathbb{N}}_{(i)}}(k)$ is the characteristic function of the set $\overline{\mathbb{N}}_{(i)}$. Then all the solutions of (0.1) are oscillatory.

Proof. As in the proof of Theorem 3.15, an eventually positive solution $u(k)$ of (0.1) satisfies the inequality (3.9) for all $k \geq k_1$ and $\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell}\rceil}}$ is nondecreasing for $k \geq k_1$. Choose $k_2 \in [k_1, \infty)$ so large that $g_i(k) \geq k_1$ for all $k \in [k_2, \infty)$ and note that for $k \in \overline{\mathbb{N}}_{(i)}$, $g_i(k) \geq k + \ell$. So, for all $k \in \overline{\mathbb{N}}_{(i)} \cap [k_2, \infty)$ we have $u(g_i(k)) \geq \frac{u(k + \ell)}{\alpha^{\lceil \frac{k+\ell}{\ell}\rceil}}$ and by our assumption,

$$F_i(u(g_i(k))) \geq F_i\Big(\frac{u(k + \ell)}{\alpha^{\lceil \frac{k+\ell}{\ell}\rceil}}\Big).$$

Therefore (3.9) implies

$$\frac{\Delta_\ell\left(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell}\rceil}}\right)}{F_i\left(\frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell}\rceil}}\right)} \geq \alpha^{-\lceil \frac{k+\ell}{\ell}\rceil} f_i(k) \text{ for all } k \in \overline{\mathbb{N}}_{(i)} \cap [k_2, \infty).$$

Next, by monotonicity of $F_i(s)$, we get

$$\alpha^{-\lceil \frac{k+\ell}{\ell}\rceil} f_i(k) \leq \int_{\psi_{\alpha(\ell)}(k)}^{\psi_{\alpha(\ell)}(k+\ell)} \frac{ds}{F_i(s)}, \text{ for all } k \in \overline{\mathbb{N}}_{(i)} \cap [k_2, \infty),$$

from which by taking $j = k - k_2 - \left[\frac{k-k_2}{\ell}\right]\ell$, it follows that

$$\sum_{r=0}^{\frac{k-j-k_2}{\ell}} \chi_{\overline{\mathbb{N}}_{(i)}}(k_2+j+r\ell)\alpha^{\left\lceil \frac{-k_2-j-\ell-r\ell}{\ell}\right\rceil} f_i(k_2+j+r\ell)$$
$$\leq \int_{\psi_{\alpha(\ell)}(k_2)}^{\psi_{\alpha(\ell)}(k+\ell)} \frac{ds}{F_i(s)} \quad \text{for all} \quad k \geq k_2.$$

Then, by (3.10), there is a constant $c > 0$ such that

$$\sum_{r=0}^{\frac{k-j-k_2}{\ell}} \chi_{\overline{\mathbb{N}}_{(i)}}(k_2+j+r\ell)\alpha^{\left\lceil \frac{-k_2-j-r\ell}{\ell}\right\rceil} f_i(k_2+j+r\ell) \leq c \text{ for all } k \geq k_2,$$

which contradicts our assumption (3.11). This completes the proof. □

Example 3.18. For the generalized $\alpha-$difference equation

$$\Delta_{\alpha(\ell)} u(k) - \frac{u(k+\ell)}{(2k+5\ell)}\Big(\frac{3}{2}(2k+3\ell)+2\ell\Big) = 0$$

and for

$$f_1(k) = \frac{3}{2}(2k+3\ell)\alpha^{\lceil \frac{2k+2\ell}{\ell}\rceil}, \quad f_2(k) = 2\ell\alpha^{\lceil \frac{2k+2\ell}{\ell}\rceil},$$

$$F_1(k) = \frac{u(k+\ell)}{(2k+5\ell)\alpha^{\lceil \frac{2k+2\ell}{\ell}\rceil}}$$

all the conditions of Theorem 3.17 hold and hence every solution is oscillatory. Infact $u(k) = -(-\alpha)^{\lceil \frac{k+3\ell}{\ell}\rceil} 2^{\frac{k+3\ell}{\ell}}(2k+3\ell)$ is one such solution.

Following results are due to the oscillatory and nonoscillatory behaviours of Equation (0.2).

Theorem 3.19. *Let* $\alpha > 0$ *and* $S = \{[0,\infty) \times \{(u,v) \in \mathbb{R}^2 : v + (\beta - \alpha)u = 0\}\}$. *Further, let*

(i) $f(k,u,v) = 0$ *if* $(k,u,v) \in S$.

(ii)$f(k,u,v)[v + (\beta - \alpha)u]$
$+\alpha[v + (\beta - \alpha)u]^2 > 0$ *if* $(k,u,v) \in [0,\infty) \times \mathbb{R}^2 \backslash S$.
Then, the generalized difference equation (0.2) is nonoscillatory.

Proof. We observe that $(k, u(k), \Delta_{\beta(\ell)} u(k)) \in S$ is equivalent to

$$u(k+\ell) - \alpha u(k) = 0.$$

Therefore, if the solution $u(k)$ of (0.2) is such that for a fixed $k_1 \in [0,\infty)$, $(k_1, u(k_1), \Delta_{\beta(\ell)} u(k_1)) \in S$, then from the hypothesis (i), it follows that

$$\Delta^2_{\alpha(\ell)} u(k_1) = 0.$$

However, since

$$\Delta^2_{\alpha(\ell)} u(k_1) = \Delta_{\alpha(\ell)} u(k_1+\ell) - \alpha \Delta_{\alpha(\ell)} u(k_1) = u(k_1+2\ell) - \alpha u(k_1+\ell) = 0$$

inductively, we have $u(k_1 + t\ell) - \alpha^t u(k_1) = 0$, t is a positive integer and this solution is of course nonoscillatory. Now let $u(k)$ be a solution of (0.2) such that for any $k \in [0,\infty)$, $(k, u(k), \Delta_{\beta(\ell)} u(k)) \notin S$, and this solution is oscillatory. Then, there exists a $k_2 \in [0,\infty)$ such that $u(k_2) > 0, u(k_2+\ell) \leq 0$ and hence $\Delta_{\alpha(\ell)} u(k_2) < 0$. Setting $k = k_2$ in (0.2) and multiplying the resulting equation by $\Delta_{\alpha(\ell)} u(k_2)$ we obtain

$$\begin{aligned}\Delta_{\alpha(\ell)} u(k_2) \Delta_{\alpha(\ell)} u(k_2+\ell) &= f(k_2, u(k_2), \Delta_{\beta(\ell)} u(k_2)) \\ &\times [\Delta_{\beta(\ell)} u(k_2) + (\beta - \alpha) u(k_2)] + \alpha [\Delta_{\beta(\ell)} u(k_2) + (\beta - \alpha) u(k_2)]^2.\end{aligned}$$

Therefore, from the hypothesis (ii), we find that

$$\Delta_{\alpha(\ell)} u(k_2) \Delta_{\alpha(\ell)} u(k_2+\ell) > 0,$$

and hence

$$\Delta_{\alpha(\ell)} u(k_2+\ell) < 0.$$

Repeating this reasoning we get $\Delta_{\alpha(\ell)} u(k) < 0$ for all $k \in [k_2, \infty)$. This implies that $u(k) < 0$ for all $k \in [k_2 + 2\ell, \infty)$, which contradicts our assumption. The proof for the case $u(k_2) \geq 0, u(k_2+\ell) < 0$ is similar. □

Example 3.20. The generalized $\alpha-$difference equation

(i) $\Delta^2_{\alpha(\ell)}u(k) = 0$

(ii) $\Delta^2_{\alpha(\ell)}u(k) = 2\ell^2\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}$ are clearly nonoscillatory, since $u(k) = \alpha^{\lceil \frac{k}{\ell} \rceil}$ and $u(k) = k^2\alpha^{\lceil \frac{k}{\ell} \rceil}$ are the solutions respectively.

Theorem 3.21. *Let $\alpha > 0$ and $T = \{[0,\infty) \times \{(u,v) \in \mathbb{R}^2 : v + \beta u = 0\}\}$. Further, let*

$$f(k,u,v)(v+\beta u) + \alpha(v+\beta u)[v + (\beta-\alpha)u] > 0$$

if $(k,u,v) \in [0,\infty)\times\mathbb{R}^2 \backslash T$. Then, the difference equation (0.2) is nonoscillatory.

Proof. We observe that $(k, u(k), \Delta_{\beta(\ell)}u(k)) \in T$ is equivalent to $u(k+\ell) = 0$. Since we consider only nontrivial solution, there exists a $k_1 \in [k_2,\infty)$ such that $u(k_1+\ell) = \Delta_{\beta(\ell)}u(k_1) + \beta u(k_1) \neq 0$. Setting $k = k_1$ in (0.2) and multiplying the resulting equation by $u(k_1+\ell)$, we obtain

$$\begin{aligned} u(k_1+\ell)\Delta_{\alpha(\ell)}u(k_1+\ell) =& f(k_1, u(k_1), \Delta_{\beta(\ell)}u(k_1))u(k_1+\ell) \\ &+ \alpha u(k_1+\ell)\Delta_{\alpha(\ell)}u(k_1) \\ =& f(k_1, u(k_1), \Delta_{\beta(\ell)}u(k_1))(\Delta_{\beta(\ell)}u(k_1) + \beta u(k_1)) \\ &+ \alpha(\Delta_{\beta(\ell)}u(k_1) + \beta u(k_1))[\Delta_{\beta(\ell)}u(k_1) + (\beta-\alpha)u(k_1)]. \end{aligned}$$

Hence, from the given hypothesis it follows that $u(k_1+\ell)\Delta_{\alpha(\ell)}u(k_1+\ell) > 0$. If $u(k_1+\ell) > 0$, then $\Delta_{\alpha(\ell)}u(k_1+\ell) > 0$ implies $u(k_1+2\ell) > \alpha u(k_1+\ell) > 0$.

Repeating the above reasoning we obtain $\Delta_{\alpha(\ell)}u(k) > 0$ for all $k \in [k_1 + \ell,\infty)$, and from this $u(k) > \alpha^{\left\lceil \frac{k-k_1-\ell}{\ell} \right\rceil}u(k_1+\ell) > 0$ for all $k \in [k_1+2\ell,\infty)$. This solution is positive and therefore nonoscillatory. A similar proof hold for $u(k_1+\ell) < 0$. □

Example 3.22. The generalized $\alpha-$difference equation

$$\Delta^2_{\alpha(\ell)}u(k) = 2\ell^2\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}$$

is nonoscillatory, since $u(k) = k^2\alpha^{\lceil \frac{k}{\ell} \rceil}$ is clearly nonoscillatory.

Theorem 3.23. *Let $\alpha = \beta = 1$ and*

$$f(k,u,v)(u+v) \geq 0$$

if $(k,u,v) \in [0,\infty) \times \mathbb{R}^2$. Then, the generalized difference equation (0.2) is nonoscillatory.

Proof. Suppose there exists an oscillatory solution $u(k)$ of (0.2). Then, there exists $k_1, k_2 \in [0, \infty)$ such that $u(k_1) \leq 0, u(k_2) \geq 0$, and there exists a $t \in [k_1 + \ell, k_2 - \ell]$ such that $u(t) > 0$ (the case $u(t) < 0$ can be considered similarly) and simultaneously $u(t) > (\geq) u(t + \ell)$, $u(t) \geq (>) u(t - \ell)$.

Thus, $\Delta_\ell^2 u(t - \ell) = \Delta_\ell u(t) - \Delta_\ell u(t - \ell) < 0$. But, setting $k = t - \ell$ in (0.2) and multiplying the resulting equation by $u(t)$ we obtain

$$u(t)\Delta_\ell^2 u(t - \ell) = u(t) f(t - \ell, u(t - \ell), \Delta_\ell u(t - \ell))$$

and hence, from the given hypothesis we have $u(t)\Delta_\ell^2 u(t - \ell) > 0$, i.e. $\Delta_\ell^2 u(t - \ell) > 0$. This contradiction completes the proof. □

Example 3.24. For the generalized $\alpha-$difference equation $\Delta_{\alpha(\ell)}^2 u(k) = 6\ell^2 k$ the solutions are nonoscillatory.

Theorem 3.25. *Let $\alpha < 0$ and $S = \{[0, \infty) \times \{(u, v) \in \mathbb{R}^2 : v + (\beta - \alpha) u = 0\}\}$. Further, let*

(i) $f(k, u, v) = 0$ if $(k, u, v) \in S$,

(ii)$f(k, u, v)[v + (\beta - \alpha) u] + \alpha [v + (\beta - \alpha) u]^2 < 0$ if $(k, u, v) \in [0, \infty) \times \mathbb{R}^2 \backslash S$. Then, the difference equation (0.2) is oscillatory.

Proof. Suppose $u(k)$ is a nonoscillatory solution of (0.2) which is positive for all $k \in [a, \infty)$, where $a \in [0, \infty)$. If there is some $k_1 \in [a, \infty)$ so that $(k_1, u(k_1), \Delta_{\beta(\ell)} u(k_1)) \in S$, then as in Theorem 3.19, we find that

$$u(t) = \alpha^{\left\lceil \frac{t\ell - k_1}{\ell} \right\rceil} u(k_1), t \in [k_1, \infty).$$

However, since $\alpha < 0$, this solution is oscillatory.

Thus, for all $k \in [a, \infty), (k, u(k), \Delta_{\beta(\ell)} u(k)) \in [0, \infty) \times \mathbb{R}^2 \backslash S$.

But, then for all $k \in [a, \infty)$,

$$\Delta_{\alpha(\ell)} u(k + \ell) \Delta_{\alpha(\ell)} u(k)$$
$$= \Delta_{\alpha(\ell)} u(k) f(k, u(k), \Delta_{\beta(\ell)} u(k)) + \alpha (\Delta_{\alpha(\ell)} u(k))^2 < 0,$$

i.e.

$$\Delta_{\alpha(\ell)} u(k + \ell) \Delta_{\alpha(\ell)} u(k) = \alpha^{\left\lceil \frac{k+2\ell}{\ell} \right\rceil} \Delta_\ell \Big(\frac{u(k + \ell)}{\alpha^{\left\lceil \frac{k+\ell}{\ell} \right\rceil}} \Big) \alpha^{\left\lceil \frac{k+\ell}{\ell} \right\rceil} \Delta_\ell \Big(\frac{u(k)}{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}} \Big) < 0.$$

Thus, if k is even then $\Delta_\ell \Big(\frac{u(k + \ell)}{\alpha^{\left\lceil \frac{k+\ell}{\ell} \right\rceil}} \Big) \Delta_\ell \Big(\frac{u(k)}{\alpha^{\left\lceil \frac{k}{\ell} \right\rceil}} \Big) > 0.$

If $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}\Big) > 0$, then

$$\frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}} > \frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}} > 0.$$

Therefore, $u(k+\ell) < 0$ and we obtain a contradiction. Hence, it turns out to be that $\Delta_\ell\Big(\frac{u(k)}{\alpha^{\lceil \frac{k}{\ell} \rceil}}\Big) < 0$. Then, $\Delta_\ell\Big(\frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}}\Big) < 0$, i.e.

$$\frac{u(k+2\ell)}{\alpha^{\lceil \frac{k+2\ell}{\ell} \rceil}} < \frac{u(k+\ell)}{\alpha^{\lceil \frac{k+\ell}{\ell} \rceil}} < 0,$$

which implies that $u(k+2\ell) < 0$. This contradiction completes the proof. □

Example 3.26. For the generalized $\alpha-$difference equation

$$\Delta^2_{\alpha(\ell)} u(k) = ((k+2\ell)^2 + 2(k+\ell)^2 + k^2)(-\alpha)^{\lceil \frac{k+2\ell}{\ell} \rceil}$$

the solutions are oscillatory.

Theorem 3.27. *Let $\alpha < 0$ and $T = \{[0,\infty) \times \{(u,v) \in \mathbb{R}^2 : v + \beta u = 0\}\}$. Further, let*

$$f(k,u,v)(v+\beta u) + \alpha(v+\beta u)[v + (\beta-\alpha)u] < 0 \text{ if } (k,u,v) \in [0,\infty) \times \mathbb{R}^2 \backslash T.$$

Then, the difference equation (0.2) is oscillatory.

Proof. Similar reasoning as in the proof of Theorem 3.21 gives us

$$u(k_1+\ell)\Delta_{\alpha(\ell)} u(k_1+\ell) = u(k_1+\ell)u(k_1+2\ell) - \alpha u^2(k_1+\ell) < 0.$$

But this inequality hold only for an oscillatory solution. □

Example 3.28. For the generalized $\alpha-$difference equation

$$\Delta^2_{\alpha(\ell)} u(k) = (4k+8\ell)(-\alpha)^{\lceil \frac{k+2\ell}{\ell} \rceil}, \quad u(k) = (k+\ell)(-\alpha)^{\lceil \frac{k}{\ell} \rceil}$$

which is clearly oscillatory.

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