International Journal of Pure and Applied Mathematics
Volume 78 No. 4 2012, 491-498
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

PARAMETRIC OPTIMIZATION APPROACH TO UTILITY MAXIMIZATION PROBLEM

J. Enkhbayar[1], R. Enkhbat[2 §]
National University of Mongolia
P.O. Box 46/635, Ulaanbaatar, 210646, MONGOLIA

Abstract: We consider the classical utility maximization problem which arises in consumer theory of economics. The utility maximization problem has been always considered in the literature [1,4,7] as concave maximization problem. So far less attention paid to parametric utility maximization problems. We consider parametric utility maximization problems with one parameter which is treated as time variable. We propose a new method and algorithm for solving the above problem.

AMS Subject Classification: 46N10
Key Words: nonconvex, paramertic optimization problems, utility maximization problem

1. Introduction

The theory of mathematical programming has been applied to a wide variety of problems in economics. It has been used to characterize the solution of fundamental problems in virtually all areas of economics. Microeconomic problems are typically formulated as those of economic agents (consumers and firms) attempting to maximize an objective function subject to certain constraints. Aim of this paper is to classify the optimization problems of consumer theory (utility maximization problem) parametric optimization problems depending on mar-

Received: February 1, 2012

[§]Correspondence author

ket environment. Also, we provide with appropriate methods and algorithms for solving. For instance, the profit maximization problem is the main part of the general equilibrium theory and solution constitutes the market supplies of goods.

2. Parametric Utility Maximization (PUM)

Observations on economic variables such as utility, price and income made over time lead to parametric. In fact, the general equilibrium theory was developed for economic processes operating per a unit time. Consider one of the consumer theory problem is utility maximization problem in parameter (time) of t or its equivalent minimization problem:

$$f(x,t) = -U(x,t) \to min, \quad t \in [t_A, t_B] \tag{2.1}$$

$$\sum_{i=1}^{n} p_i x_i = I \tag{2.2}$$

$$x_i \geq 0, i = 1, 2, ..., n \tag{2.3}$$

where consumer's utility function $U : R^n_+ \to R$ is strictly concave and twice differentiable with respect to x at each $t \in [t_A, t_B]$, $I(t)$ is consumer's income at moment t, $p_i(t)$ is price of goods i per unit. Function $p_i : R_+ \to R_+$, $i = 1, 2, ..., n$ are assumed to be continuous. t is time as parameter, $t \in [t_A, t_B]$. x_i, $i = 1, 2, ..., n$ is quantity of good i a consumer purchases at moment t. Parametric utility maximization problem is a convex programming problem and has a unique solution at each $t \in [t_A, t_B]$, $I(t)$.

This problem is a hard parametric optimization problem.

Lemma. *Assume that the utility function $U(x,t)$ is a differentiable with respect to t and satisfies the Lipschitz condition with constant M for each $x \in D$ i.e.*

$$|U(x,\hat{t}) - U(x,t)| \leq M|\hat{t} - t|, \ \forall t \in [t_A, t_B]$$

Then for a given $\varepsilon > 0$, there exists discretization

$$t_A = t_0 < t_1 < ... < t_{i-1} < t_i < t_{i+1} < ... < t_N = t_B$$

such that

$$|U(x^*(t), t) - U(x^*(t_i), t_i)| < \varepsilon, \ \forall \in [t_A, t_B] \text{ and certain } t_i$$

where

$$U(x^*(t), t) = \max_{x \in D} U(x, t), \ t \in [t_A, t_B]$$
$$D = \{x \in R^n | \sum_{i=1}^n p_i x_i = m\}$$

Proof. We discretize t_A, t_B in the following way

$$t_A = t_0, t_i = t_0 + i\frac{t_A - t_B}{N}, \ i = 1, 2, ..., N.$$

Clearly, for any $t \in [t_A, t_B]$, there exists $j \in \{1, 2, ..., N\}$ such that $t \in [t_j, t_{j+1}]$. Consequently,

$$|t - t_j| < \frac{t_A - t_B}{N}, \ i = 1, 2, ..., N.$$

Due to condition, there exists $M > 0$ such that

$$|U(x(\hat{t}), \hat{t}) - U(x(t), t)| < M|\hat{t} - t|, \ \forall t, \ \hat{t} \in [t_A, t_B].$$

Define $\varepsilon > 0$ as follows:

$$\varepsilon = M\frac{t_A - t_B}{N}.$$

Now take any $t \in [t_A, t_B]$ and compute

$$|U(x^*(t), t) - U(x^*(t_j), t_j)| \leq M|t - t_j| \leq M\frac{t_A - t_B}{N} = \varepsilon$$

which proves the lemma.

The above lemma allows us to find $\varepsilon-$ approximate solution of problem 2.1 by solving a finite number of nonlinear optimization problems.

3. General Parametric Utility Maximization

Consider one of the consumer theory problem is general parametric utility maximization problem depend on time of t or its equivalent minimization problem:

$$f(x, t) = -U(x, t) \to min, \tag{3.1}$$

$$\sum_{i=1}^n p_i(t) x_i = I(t) \quad t \in [t_A, t_B] \tag{3.2}$$

$$x_i \geq 0, i = 1, 2, ..., n \tag{3.3}$$

where consumer's utility function $U : R^n_+ \to R$ is strictly concave and twice differentiable with respect to x at each $t \in [t_A, t_B]$, $I(t)$ is consumer's income at moment t, $p_i(t)$ is price of goods i per unit. Function $p_i : R_+ \to R_+$, $i = 1, 2, ..., n$ are assumed to be continuous. t is time as parameter, $t \in [t_A, t_B]$. x_i, $i = 1, 2, ..., n$ is quantity of good i a consumer purchases at moment t. Parametric utility maximization problem is a convex programming problem and has a unique solution at each $t \in [t_A, t_B]$, $I(t)$.

The Karush-Kuhn-Tucker (KKT) condition for the above problem is as follows:

$$\begin{cases} \dfrac{\partial f(x,t)}{\partial x_i} + \lambda(t)p_i(t) - \mu(t) = 0 \quad i = 1, 2, ..., n \\ \sum_{i=1}^{n} p_i(t)x_i = m(t) \\ \mu_j(t)x_j = 0, j = 1, ..., n \\ \mu_j \geq 0, x_j \geq 0 \end{cases} \tag{3.4}$$

where $\mu = \{\mu_1, ...\mu_n\}$ This condition can be used for solving (PUMP) directly.

We assume that [5]:

a1. There exists a continuous function $x : [t_A, t_B] \to R^n_+$ such that $x(t)$ is a global minimizer for Discret Utility Maximization Problem.

a2. $x(t_A)$ is known.

Let $\tilde{J} \subset J = 1, ..., n$ for further purpose consider the auxiliary parametric optimization problem:

$$f(x,t) \to min, \quad t \in [t_A, t_B] \tag{3.5}$$

subject to:

$$\sum_{i=1}^{n} p_i(t)x_i = I(t) \tag{3.6}$$

$$x_j = 0, j \in \tilde{J} \tag{3.7}$$

Let $v^0 = (x^0(t), \mu^0(t), \lambda^0(t))$ satisfy the KKT conditions for problem (3.5)-(3.7) with $\tilde{J} = J_0$,

$$\begin{cases} \dfrac{\partial f(x^0(t),t)}{\partial x_i} + \lambda^0(t)p_i(t) - \mu^0(t) = 0 \quad i = 1, 2, ..., n \\ \sum_{i=1}^{n} p_i(t)x_i^0 = m(t) \\ x_j(t) = 0, j \in J \end{cases} \tag{3.8}$$

Convergence of the algorithm is given by the following theorem.

Assume that the assumption (a1)-(a2) hold. Then, for all $\varepsilon, \varepsilon_t, \varepsilon_v, \varepsilon_{\dot{v}}$ and Δt_{min}, Δt_{max} sufficiently small, algorithm PUM [5] generates a discretization $t_A = t_0 < t_1 < ... < t_i < t_{(i+1)} < t_N = t_B$, with corresponding points $\widetilde{x}^i, \widetilde{\lambda}^i, \widetilde{\mu}^i$ such that $\|(\widetilde{x}^i, \widetilde{\lambda}^i, \widetilde{\mu}^i) - (x(t_i), \lambda(t_i), \mu(t_i))\| < \varepsilon$ $i = 1, 2, ..., N$

Algorithm PUM

Step 1. Given $x^0, J_0, \lambda^0, \mu^0, \varepsilon, \varepsilon_t, \varepsilon_v, \varepsilon_{\dot{v}}, \Delta t_{min}, \Delta t_{max}, t_0 := t_A, k := 1$

Step 2. Determine a step size $\Delta t_k \in [\Delta t_{min}, \Delta t_{max}]$

Step 3. Find an approximate KKT point $v^k = (x^k, \lambda^k, \mu^k)$ solving system (2.6) for $t = t_k$ with

$$\|v^k - v(t_k)\| \leq \varepsilon_v, v(t_k) = (x(t_k), \lambda(t_k), \mu(t_k)).$$

Step 4. If

$$\begin{cases} x_j^r(t_k) > \varepsilon_v, j \in J_{k-1} \\ \mu_j^r(t_k) > \varepsilon_v, j \in J_{k-1} \end{cases}$$

then $J_k := J_{k-1}$ and go to step 6.

Step 5. find $\bar{t}$ solving system:

$$\begin{cases} x_j^r(t_k) \geq \varepsilon_v, j \in J \setminus J_{k-1} \\ \mu_j^r(t_k) >\geq \varepsilon_v, j \in J_{k-1} \end{cases}$$

Step 6. Solve system (2.5) approximately,i.e,

$$\|\tilde{t} - \bar{t}\| \leq \varepsilon_t \ \|\tilde{v}^k - v(\tilde{t})\| \leq \varepsilon_v \ \|\tilde{\dot{v}}^k - \dot{v}(\tilde{t})\| \leq \varepsilon_v, \ \tilde{v}^k = (\tilde{x}^k), \tilde{\lambda}^k, \tilde{\mu}^k)$$

Step 7. For index sets:

$$\tilde{J} = J_{k-1} \bigcup \{j \in J_{k-1} : |\tilde{\dot{x}}_j^k \varepsilon_t - \tilde{x}_j^k \geq -\varepsilon_{t-2} \varepsilon_v\}$$
$$\tilde{J}^+ = \{j \in J_{k-1} : \tilde{\mu}_j^k \geq |\tilde{\dot{\mu}}_j^k| \varepsilon_t + \varepsilon_v + \varepsilon_{\tilde{t}}$$
$$\tilde{J}^0 = \tilde{J} \setminus \tilde{J}^+.$$

Step 8. If $|\tilde{J}^0| = 1$ then construct the index set J_k as:

$$J_k = \{j : \tilde{x}_j^k = 0\}$$

Otherwise, go to next step.

Step 9. Solve problem (2.1)-(2.4) for $t = \tilde{t}$ and

$$J_k = \{j : x_j(\tilde{t}) = 0\}$$

Step 10. Set $k := k + 1$ and go to step 2.

Remark 3.1. Finding an approximate KKT point in step 3 is based on the method in [5].

Remark 3.2. A search for $\tilde{t}$ in step 5 is carried out by bisection strategy in [5].

Remark 3.3. Choice of parameters $\varepsilon_t, \varepsilon_v, \varepsilon_{\dot{v}}$ and $\Delta t_{min}, \Delta t_{max}$ is done according to [5].

4. Numerical Example

Consider the utility maximization problem:

$$UM : u(x) = \prod_{i=1}^{n} (x_i - c_i)_i^{\alpha} \to max, \quad t \in [t_A, t_B], \tag{4.1}$$

$$\sum_{i=1}^{n} p_i(t)x_i = I(t), \tag{4.2}$$

$$x_i \geq 0, i = 1, 2, ..., n. \tag{4.3}$$

where consumer's utility function $u(x)$ is strictly concave and twice differentiable with respect to x at each $t \in [t_A, t_B]$, $x_i \quad i = 1, 2, ..., n$ is a quantity of good i and $c_i \quad i = 1, 2, ..., n$ is minima required quantity of good i, $I(t)$ is consumer's income at moment t, $p_i(t)$ is price of goods i per unit. Function $p_i : R_+ \to R_+$, $i = 1, 2, ..., n$ are assumed to be continuous. t is time as parameter, $t \in [t_A, t_B]$. x_i, $i = 1, 2, ..., n$ is quantity of good i a consumer purchases at moment t.

Problem (4.1) called Stone-Geary's utility function (SG function) in microeconomic. We estimated parameters SG function for 5 goods used time series data of statistics of Mongolia and solved following cases.

a) If $p_i(t)$ is price of goods i and $I(t)$ is consumer's budget depending on t linearly we can formulate following:

$$\begin{aligned} ln[u(x)] = y = -(&ln(x(1) - 9.57493) * 0.16121 + ln(x(2) - 2.7174) * 0.16517 \\ &+ ln(x(3) - 8.494) * 0.326 + ln(x(4) - 11.79) * 0.1412 + \\ &ln(x(5) - 0.8779) * 0.205)) \to min, \qquad t \in [2010, 2014] \end{aligned} \tag{4.4}$$

$$\sum_{i=1}^{5} p_i(t)x_i = I(t), \tag{4.5}$$

$$\begin{cases} p_1(t) = 26.21 * t - 46170 \\ p_2(t) = 48.11 * t - 95767 \\ p_3(t) = 220.3 * t - 43934 \\ p_4(t) = 26.4 * t - 52361 \\ p_5(t) = 419.9 * t - 83665 \\ I(t) = 1341.4 * t - 2.6727e + 006 \end{cases}$$

$$x_i \geq 0, i = 1, 2, ..., n. \tag{4.6}$$

b). If $p_i(t)$ is price of goods i and $I(t)$ is consumer's budget depending on t quadratic or cubic polynomial we can formulate following:

$$\begin{aligned} &ln[u(x)] = y = -(ln(x(1) - 9.57493) * 0.16121 + ln(x(2) - 2.7174) * 0.16517 \\ &+ln(x(3) - 8.494) * 0.326 + ln(x(4) - 11.79) * 0.1412+ \\ &ln(x(5) - 0.8779) * 0.205)) \rightarrow min, \qquad t \in [2011, 2015] \end{aligned} \tag{4.7}$$

$$\sum_{i=1}^{5} p_i(t)x_i = I(t), \tag{4.8}$$

$$\begin{cases} p_1(t) = c_1t^2 + c_2t + c_3 \quad (c_1 = 5.2132,\ c_2 = -20849,\ c_3 = 2.0846e + 007) \\ p_2(t) = c_1t^2 + c_2t + c_3 \quad (c_1 = 10.849,\ c_2 = -43383,\ c_3 = 4.3371e + 007) \\ p_3(t) = c_1t^2 + c_2t + c_3 \quad (c_1 = 20.672,\ c_2 = -82558,\ c_3 = 8.243e + 007) \\ p_4(t) = 0.91792t^3 - 5513.6t^2 + (1.104e + 007)t - 7.3678e + 009 \\ p_5(t) = 3.2037t^3 + -19226t^2 + (3.8459e + 007)t + -2.5644e + 010 \\ I(t) = 10.019t^3 - 60068t^2 + (1.2005e + 008)t + -7.9975e + 010 \end{cases}$$

$$x_i \geq 0, i = 1, 2, ..., n. \tag{4.9}$$

The numerical results using *Matlab* are given in the following tables, respectively.

time(t)	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	f_{max}
2010	0.267	0.64	0.034	10.7636	0.0001	1.006
2011	0.0845	0.7381	0.0380	12.8164	0.0000	1.000
2012	0.1875	0.5654	0.0438	8.7568	0.0018	1.9985
2013	0.2817	0.7042	0.0420	12.8187	0.0000	1.0017
2014	0.2817	0.7042	0.0420	12.8187	0.0001	1.1495

Table 1: Example 2(a)

time(t)	x_1^*	x_2^*	x_3^*	x_4^*	x_5^*	f_{max}
2011	3.9798	1.7282	1.2753	4.6522	0.9159	1.6695
2012	5.1683	0.1901	2.5762	4.9632	1.8201	1.7825
2013	2.1996	0.6598	5.2251	4.5652	0.6817	1.0715
2014	2.0621	1.0025	5.2830	4.7399	1.1805	1.1495
2015	2.0621	1.0025	5.2830	4.7399	1.1805	1.1495

Table 2: Example 2(b)

Acknowledgments

This work was supported by ARC and KFAC.

References

[1] David Romer, *Advanced Macroeconomics*, University of California, Berkeley (1996).

[2] R. Enkhbat, *Quasiconvex Programming and its Applications*, Lambert Publisher,Germany (2009).

[3] R. Enkhbat, J. Guddat, *Parametric Utility Problem*, Mongolian Mathematician Journal, Ulaanbaatar (2006).

[4] Gregory H. Mankiw, *Macro Economics*, Harvard University, Worth Publisher (2000).

[5] J. Guddat, Vasquet F. Guerra, H.Th. Jongen, *Parametric Optimization: Singularities, Parthtollowing and Jumps*, Jonhn Wiley and Sons, New York (1990).

[6] R. Horst, M. Pardalos Panos, N. Thoat, *Introduction to Global Optimization*, Kluwer Academic Publishers (1995).

[7] Paul Modden, *Concavity and Optimization in Microeconomics*, Oxford University Press (1986).

[8] O.V. Vasiliev, *Optimization Methods*, World Federation Publisher (1995).