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HERMITE AND SIMPSON-LIKE TYPE INEQUALITIES FOR FUNCTIONS WHOSE SECOND DERIVATIVES IN ABSOLUTE VALUES AT CERTAIN POWERS ARE s-CONVEX

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Abstract: In this article a general integral identity for twice differentiable mappings is derived. By using this result, the author establish some new Hermite-Hadamard-like type and Simpson-like type inequalities for twice differentiable s-convex mappings in the second sense.

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1. Introduction

The following inequality, named Simpson's inequality, is one of the best known results.

Theorem 1.1. *Let $f : [a,b] \to R$ be a four times continuously differentiable function on (a,b) and $\| f^{(4)} \|_\infty = \sup_{x\in(a,b)} | f^{(4)}(x) | < \infty$. Then the following inequality holds:*

$$\Big|\frac{1}{6}\big\{f(a) + 4f\big(\frac{a+b}{2}\big) + f(b)\big\} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \leq \frac{1}{2880} \| f^{(4)} \|_\infty (b-a)^4.$$

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For recent refinements, counterparts, generalizations and new Simpson's type inequalities, you may see [1, 7, 16, 17, 18].

S.S. Dragomir et al. [4] proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth:

Theorem 1.2. *Suppose that $f : [a,b] \to R$ is an absolutely continuous function on $[a,b]$ whose derivative belong to $L_p[a,b]$. Then the following inequality holds:*

$$\Big|\frac{1}{6}\{f(a)+4f\big(\frac{a+b}{2}\big)+f(b)\}-\frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\le \frac{1}{6}\Big\{\frac{2^{q+1}+1}{3(q+1)}\Big\}^{\frac{1}{q}}(b-a)^{\frac{1}{q}} \parallel f'' \parallel_q$$

where $q \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In sequel let I^0 be the interior of an interval I in R.

M.Z. Sarikaya et al. [18] and J. Park [13] established the following theorems:

Theorem 1.3. *Let $f : I \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L_1[a,b]$, where $a,b \in I$ with $a<b$. If $\mid f'' \mid^q$ is a convex mapping on $[a,b]$ for $q \ge 1$, then the following inequality holds:*

$$\Big|\frac{1}{6}\{f(a)+4f\big(\frac{a+b}{2}\big)+f(b)\}-\frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\le \frac{(b-a)^2}{162}\Big[\Big\{\frac{59\mid f''(a)\mid^q+133\mid f''(b)\mid^q}{192}\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{133\mid f''(a)\mid^q+59\mid f''(b)\mid^q}{192}\Big\}^{\frac{1}{q}}\Big].$$

Theorem 1.4. *Let $I \subset R$ be an open interval, $a,b \in I$ with $a<b$ and $f : I \to R$ is a twice differentiable mapping such that f'' is integrable. If $\mid f'' \mid^q$ is a convex mapping on $[a,b]$ and $q \ge 1$, then the following inequalities hold:*

(a) for $0 \le r \le \frac{1}{2}$:

$$\Big|(r-1)f\big(\frac{a+b}{2}\big)-r\big\{\frac{f(a)+f(b)}{2}\big\}+\frac{1}{b-a}\int_a^b f(x)dx\Big|$$

$$\le \frac{(b-a)^2}{48}(8r^3-3r+1)^{1-\frac{1}{q}}\big(\frac{1}{2}\big)^{\frac{3}{q}}$$

$$\times \left\{ \begin{array}{c} \left(\mu_1 \mid f''(a) \mid^q + \nu_1 \mid f''(b) \mid^q\right)^{\frac{1}{q}} \\ + \left(\mu_2 \mid f''(a) \mid^q + \nu_2 \mid f''(b) \mid^q\right)^{\frac{1}{q}} \end{array} \right\},$$

where

$$\mu_1 = 2^5 r^4 - 8r + 3, \qquad \mu_2 = 2^5(1-r^2)(1-r)^2 + 48r - 27,$$
$$\nu_1 = 2^5(2-r)r^3 - 16r + 5, \quad \nu_2 = 2^5 r^4 - 8r + 3.$$

(b) for $\frac{1}{2} \leq r \leq 1$ *:*

$$\Big|(r-1)f\big(\frac{a+b}{2}\big) - r\big\{\frac{f(a)+f(b)}{2}\big\} + \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{(b-a)^2}{48}(3r-1)^{1-\frac{1}{q}}(\frac{1}{2})^{\frac{3}{q}}$$
$$\times \left\{ \begin{array}{c} \left((8r-3) \mid f''(a) \mid^q + (16r-5) \mid f''(b) \mid^q\right)^{\frac{1}{q}} \\ + \left((16r-5) \mid f''(a) \mid^q + (8r-3) \mid f''(b) \mid^q\right)^{\frac{1}{q}} \end{array} \right\}.$$

An s-convex function was introduced in Breckner's paper [2]:

Definition 1.1. For some fixed $s \in (0,1]$, a mapping $f : I \subset [0,\infty) \to R$ is said to be s-convex in the second sense on I if the inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds, for all $x, y \in I^0$ and $t \in [0,1]$.

Let us denote the class of s-convex mappings in the second sense on I by $K_s^2(I)$.

We recall that the notion of s-convex mappings generalized that of convex mappings [2]. A number of elementary and further properties of $K_s^2(I)$ and connections with s-convexity in the second sense were discussed in [1, 5, 9, 10, 11, 12, 13].

Theorem 1.5. *Suppose that* $f : [0,\infty) \to [0,\infty)$ *is an* s*-convex function in the second sense, where* $s \in (0,1)$*, and let* $a, b \in [0,\infty)$ *with* $a < b$*. If* $f \in L[a,b]$*, then the following inequalities hold:*

$$2^{s-1} f\big(\frac{a+b}{2}\big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

In recent years many authors have studied error estimations for Hermite-Hadamard's inequality and Simpson's inequality on the class of s-convex mappings in the second sense and (α, m)-convex mappings; for refinements, counterparts, generalizations and new Simpson's type inequalities, see [3, 5, 7, 8, 11, 12, 13, 14, 15, 18, 19].

The main aim of this article is to establish new generalization of Hermite-Hadamard-type and Simpson-type inequalities for functions whose absolute values of derivatives are s-convex. To begin with, the author will derive a general integral identity for twice differentiable mappings. By using this integral equality, the author establish some new inequalities of the Simpson-like and the Hermite-Hadamard-like type for these functions.

2. Main results

In order to prove our main theorem, we need the following lemma:

Lemma 1. *Let $f : I \subset R \to R$ be an absolutely continuous function on I^0 such that f'' is integrable, where $a, b \in I^0$ with $a < b$. Then for $r \geq 2$ the following equality holds:*

$$\begin{aligned}
&H_a^b(f)(r) \\
&\equiv^{\text{let}} \left(\frac{1}{2} - \frac{1}{r}\right)\left\{\frac{f(a) + f(b)}{2}\right\} + \left(\frac{1}{2} + \frac{1}{r}\right) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \\
&= \frac{(b-a)^2}{16} \\
&\quad \times \int_0^1 (1-t)\left(t - \frac{2}{r}\right)\left\{f''\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f''\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\right\}dt.
\end{aligned}$$

Lemma 2. *Let $f : I \subset R \to R$ be an absolutely continuous function on I^0 such that f'' is integrable, where $a, b \in I^0$ with $a < b$. Then for $r \geq 2$ the following equality holds:*

$$\begin{aligned}
&R_a^b(f)(r) \\
&\equiv^{\text{let}} \left(\frac{1}{2} + \frac{1}{r}\right)\left\{\frac{f(a) + f(b)}{2}\right\} + \left(\frac{1}{2} - \frac{1}{r}\right) f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx
\end{aligned}$$

$$= \frac{(b-a)^2}{16}$$

$$\times \int_0^1 (1-t)\big(t+\frac{2}{r}\big)\Big\{f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)+f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\Big\}dt.$$

A simple proof of the equalities can be done by performing an integration by parts in the integrals from the right side and changing the variables.

Remark 1. Note that

$$(a)\Big|H_a^b(f)(2)\Big|=\Big|f\big(\frac{a+b}{2}\big)-\frac{1}{b-a}\int_a^b f(x)dx\Big|,$$

$$(b)\Big|H_a^b(f)(6)\Big| = \Big|\frac{1}{6}\{f(a)+4f\big(\frac{a+b}{2}\big)+f(b)\}-\frac{1}{b-a}\int_a^b f(x)dx\Big|.$$

$$(c)\Big|R_a^b(f)(2)\Big|=\Big|\{\frac{f(a)+f(b)}{2}\}-\frac{1}{b-a}\int_a^b f(x)dx\Big|.$$

Theorem 2.1. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on I^0 such that f'' is integrable, where $a, b \in I^0$ with $a < b$. If $| f'' | \in K_s^2[a,b]$, for some fixed $s \in (0,1]$, then, for $r \geq 2$ the following inequality holds:*

$$(a)\Big|H_a^b(f)(r)\Big|$$
$$\leq \frac{(b-a)^2}{16}\Big\{2\big(\mu_{11}(r,s)+\mu_{21}(r,s)\big)$$
$$-\big(\mu_{12}(r,s)+\mu_{22}(r,s)+\mu_{31}(r,s)\big)\Big\}\Big\{\ | f''(a) | + | f''(b) |\ \Big\},$$

where

$$\mu_{11}(r,s)=\frac{(r+2)^{s+2}\{r(s+5)-2(s+1)\}}{2^{s+4}r^{s+3}(s+1)(s+2)(s+3)},$$

$$\mu_{12}(r,s)=\frac{2(s+3)^2+r(s+5)}{2^{s+4}r(s+1)(s+2)(s+3)},$$

$$\mu_{21}(r,s)=\frac{(r-2)^2}{2^{s+4}r^{s+3}(s+2)(s+3)},$$

$$\mu_{22}(r,s)=\frac{r-6-2s}{2^{s+4}r(s+2)(s+3)},$$

$$\mu_{31}(r,s)=\frac{2(s+3)-r(s-1)}{4r(s+1)(s+2)(s+3)}.$$

$$(b)\Big|R_a^b(f)(r)\Big|$$

$$\leq \Big\{\frac{8(s+3)+4r(s-1)}{r(s+1)(s+2)(s+3)} + \frac{(r-2)}{2^{s-1}r(s+1)(s+2)}\}\Big\}$$
$$\times \Big\{ \mid f''(a) \mid + \mid f''(b) \mid \Big\}.$$

Proof. (a) From Lemma 1 and by definition of $I1_a^b(f)(r,s)$, we get:

$$\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big|$$
$$\leq \int_0^1 \big|(1-t)(t-\frac{2}{r})\big|\big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|dt$$
$$+\int_0^1 \big|(1-t)(t-\frac{2}{r})\big|\big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|dt. \tag{1}$$

Since $\mid f'' \mid$ is s-convex, we have

$$\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big|$$
$$\leq \int_0^1 \big|(1-t)(t-\frac{2}{r})\big|\big\{\big(\frac{1+t}{2}\big)^s \mid f''(a) \mid + \big(\frac{1-t}{2}\big)^s \mid f''(b) \mid \big\}dt$$
$$+\int_0^1 \big|(1-t)(t-\frac{2}{r})\big|\big\{\big(\frac{1+t}{2}\big)^s \mid f''(b) \mid + \big(\frac{1-t}{2}\big)^s \mid f''(a) \mid \big\}dt$$
$$= \Big\{\int_0^{\frac{2}{r}} (1-t)\big(\frac{2}{r}-t\big)\big(\frac{1+t}{2}\big)^s dt + \int_{\frac{2}{r}}^1 (1-t)\big(t-\frac{2}{r}\big)\big(\frac{1+t}{2}\big)^s dt$$
$$+\int_0^{\frac{2}{r}} (1-t)\big(\frac{2}{r}-t\big)\big(\frac{1-t}{2}\big)^s dt + \int_{\frac{2}{r}}^1 (1-t)\big(t-\frac{2}{r}\big)\big(\frac{1-t}{2}\big)^s dt\Big\}$$
$$\times \Big\{ \mid f''(a) \mid + \mid f''(b) \mid \Big\}. \tag{2}$$

By a simple calculations, we get

$$\int_0^{\frac{2}{r}} (1-t)\big(\frac{2}{r}-t\big)\big(\frac{1+t}{2}\big)^s dt = \mu_{11}(r,s) - \mu_{12}(r,s),$$
$$\int_{\frac{2}{r}}^1 (1-t)\big(t-\frac{2}{r}\big)\big(\frac{1+t}{2}\big)^s dt = \mu_{21}(r,s) - \mu_{22}(r,s),$$

$$\int_0^{\frac{2}{r}} (1-t)\big(\frac{2}{r}-t\big)\big(\frac{1-t}{2}\big)^s dt = \mu_{11}(r,s) - \mu_{31}(r,s),$$

$$\int_{\frac{2}{r}}^{1} (1-t)\big(t-\frac{2}{r}\big)\big(\frac{1-t}{2}\big)^{s} dt = \mu_{21}(r,s). \tag{3}$$

By (1),(2) and (3), we get the desired result.
(b) By the similar way as Part (a), using Lemma 2 this is proved:

$$\begin{aligned}
&\frac{16}{(b-a)^2}\Big|R_a^b(f)(r)\Big| \\
&\leq \int_0^1 (1-t)\big(t+\frac{2}{r}\big)\Big\{\big(\frac{1+t}{2}\big)^s \mid f''(a)\mid + \big(\frac{1-t}{2}\big)^s \mid f''(b)\mid \Big\}dt \\
&\quad + \int_0^1 (1-t)\big(t+\frac{2}{r}\big)\Big\{\big(\frac{1-t}{2}\big)^s \mid f''(a)\mid + \big(\frac{1+t}{2}\big)^s \mid f''(b)\mid \Big\}dt \\
&= \Big\{\frac{8(s+3)+4r(s-1)}{r(s+1)(s+2)(s+3)} + \frac{(r-2)2^{1-s}}{r(s+1)(s+2)}\Big\}\Big\{\mid f''(a)\mid + \mid f''(b)\mid\Big\}.
\end{aligned}$$

Corollary 2.2. *In Theorem 2.1,*
(a) if we choose $r=2$ and $s=1$ in Part(a) , then we get:

$$\Big|f\big(\frac{a+b}{2}\big) - \frac{1}{b-a}\int_a^b f(x)dx\Big| \leq \frac{(b-a)^2}{48}\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\}.$$

(b) if we choose $r=4$ and $s=1$ in Part(a), then we get:

$$\begin{aligned}
&\Big|\frac{1}{8}\{f(a)+6f\big(\frac{a+b}{2}\big)+f(b)\} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \\
&\leq \frac{(b-a)^2}{128}\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\}.
\end{aligned}$$

(c) if we choose $r=6$ and $s=1$ in Part(a), then we get:

$$\begin{aligned}
&\Big|\frac{1}{6}\{f(a)+4f\big(\frac{a+b}{2}\big)+f(b)\} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \\
&\leq \frac{(b-a)^2}{2\cdot 3^4}\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\}.
\end{aligned}$$

(d) if we choose $r=2$ and $s=1$ in Part(b), then we get:

$$\Big|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \leq \frac{(b-a)^2}{24}\Big\{\big|f''(a)\big| + \big|f''(b)\big|\Big\}.$$

Let us consider the following special functions:
(1) The Gamma function:

$$\Gamma(a)=\int_0^\infty e^{-t}t^{a-1}dt, a>0.$$

(2) The incomplete Beta function:

$$\beta(x,a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt, a>0.$$

(3) The hypergeometric function:

$${}_2F_1(a,b;c;x)=\Sigma_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}, 0<x<1$$

where the Pochhammer symbol $(z)_n$ of z is defined by

$$(z)_n=\begin{cases}1 & \text{if } n=0\\ z(z+1)\cdots(z+n-1) & \text{if } n>0,\end{cases}$$

where c is not $0,-1,-2,\cdots$.

Theorem 2.3. *Let $f: I\subset[0,\infty)\to R$ be a twice differentiable function on I^0 such that f'' is integrable, where $a,b\in I^0$ with $a<b$. If $\mid f''\mid^q$ is s-convex on $[a,b]$, for some fixed $s\in(0,1]$ and $q>1$, then, for $r\geq 2$ the following inequality holds:*

$$(a)\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big|$$
$$\leq\Big\{\Big(\frac{2^{p+1}}{r^{p+1}(p+1)}\Big){}_2F_1(1,-p;p+2;\frac{2}{r})+\Big(\frac{r-2}{r}\Big)^{2p+1}\beta(1,p+1,p+1)\Big\}^{\frac{1}{p}}$$
$$\times\Big\{\Big(\frac{\mid f''(a)\mid^q+\mid f''(\frac{a+b}{2})\mid^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{\mid f''(\frac{a+b}{2})\mid^q+\mid f''(b)\mid^q}{s+1}\Big)^{\frac{1}{q}}\Big\}.$$

$$(b)\frac{16}{(b-a)^2}\Big|R_a^b(f)(r)\Big|$$
$$\leq\Big(\frac{2}{r}\Big)\Big(\frac{{}_2F_1(1,-p;p+2;-\frac{r}{2})}{p+1}\Big)^{\frac{1}{p}}$$
$$\times\Big\{\Big(\frac{\mid f''(a)\mid^q+\mid f''(\frac{a+b}{2})\mid^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{\mid f''(\frac{a+b}{2})\mid^q+\mid f''(b)\mid^q}{s+1}\Big)^{\frac{1}{q}}\Big\}.$$

Proof. (a) From Lemma 1 and by the Hölder's integral inequality, we have

$$
\begin{aligned}
&\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big| \\
&\leq \Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|^q dt\Big\}^{\frac{1}{q}} \\
&+\Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q dt\Big\}^{\frac{1}{q}}. \qquad (4)
\end{aligned}
$$

Since $\mid f'' \mid^q$ is s-convex on $[a, b]$, we get

$$
\int_0^1 \big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|^q dt \leq \frac{\mid f''(a)\mid^q + \mid f''\big(\frac{a+b}{2}\big)\mid^q}{s+1} \qquad (5)
$$

and

$$
\int_0^1 \big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q dt \leq \frac{\mid f''\big(\frac{a+b}{2}\big)\mid^q + \mid f''(b)\mid^q}{s+1}. \qquad (6)
$$

By (4),(5) and (6), we have

$$
\begin{aligned}
&\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big| \\
&\leq \Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\frac{\mid f''(a)\mid^q + \mid f''(\frac{a+b}{2})\mid^q}{s+1}\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\frac{\mid f''(\frac{a+b}{2})\mid^q + \mid f''(b)\mid^q}{s+1}\Big\}^{\frac{1}{q}} \\
&= \Big\{\int_0^{\frac{2}{r}} (1-t)^p\big(\frac{2}{r}-t\big)^p dt + \int_{\frac{2}{r}}^1 (1-t)^p\big(t-\frac{2}{r}\big)^p dt\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{\mid f''(a)\mid^q + \mid f''(\frac{a+b}{2})\mid^q}{s+1}\Big)^{\frac{1}{q}} + \Big(\frac{\mid f''(\frac{a+b}{2})\mid^q + \mid f''(b)\mid^q}{s+1}\Big)^{\frac{1}{q}}\Big\} \\
&\leq \Big\{\big(\frac{2^{p+1}}{r^{p+1}(p+1)}\big)\cdot {}_2F_1\big(1,-p;p+2;\frac{2}{r}\big) \\
&\qquad + \big(\frac{r-2}{r}\big)^{2p+1}\beta(1,p+1,p+1)\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{\mid f''(a)\mid^q + \mid f''(\frac{a+b}{2})\mid^q}{s+1}\Big)^{\frac{1}{q}} + \Big(\frac{\mid f''(\frac{a+b}{2})\mid^q + \mid f''(b)\mid^q}{s+1}\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}
$$

(b) By the similar way as Part (a), using Lemma 2 this is proved:

$$\begin{aligned}
&\frac{16}{(b-a)^2}\left|R_a^b(f)(r)\right| \\
&\le \Big\{\int_0^1 \left|(1-t)\big(t+\frac{2}{r}\big)\right|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \left|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\right|^q dt\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\int_0^1 \left|(1-t)\big(t+\frac{2}{r}\big)\right|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \left|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\right|^q dt\Big\}^{\frac{1}{q}} \\
&\le \Big\{\int_0^1 (1-t)^p\big(t+\frac{2}{r}\big)^p dt\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{|f''(a)|^q+|f''(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\frac{a+b}{2})|^q+|f''(b)|^q}{s+1}\Big)^{\frac{1}{q}}\Big\} \\
&=\Big\{\frac{2^p\cdot{}_2F_1(1,-p;p+2;-\frac{r}{2})}{r^p(p+1)}\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{|f''(a)|^q+|f''(\frac{a+b}{2})|^q}{s+1}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\frac{a+b}{2})|^q+|f''(b)|^q}{s+1}\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

Corollary 2.4. *In Theorem 2.3,*
(a) if we choose $r=2$ and $s=1$ in Part(a), then we get:

$$\begin{aligned}
&\left|f\big(\frac{a+b}{2}\big)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{16}\big(\frac{1}{2p+1}\big)^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{|f''(a)|^q+|f''(\frac{a+b}{2})|^q}{2}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\frac{a+b}{2})|^q+|f''(b)|^q}{2}\Big)^{\frac{1}{q}}\Big\}.
\end{aligned}$$

(b) if we choose $r=4$ and $s=1$ in Part(a), then we get:

$$\begin{aligned}
&\left|\frac{1}{8}\{f(a)+6f\big(\frac{a+b}{2}\big)+f(b)\}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&\le \frac{(b-a)^2}{16}\Big\{\big(\frac{1}{2}\big)^{p+1}\frac{{}_2F_1(1,-p,p+2;\frac{1}{2})}{p+1} \\
&\qquad+\big(\frac{1}{2}\big)^{2p+1}\beta(1,p+1,p+1)\Big\}^{\frac{1}{p}} \\
&\quad\times\Big\{\Big(\frac{|f''(a)|^q+|f''(\frac{a+b}{2})|^q}{2}\Big)^{\frac{1}{q}}+\Big(\frac{|f''(\frac{a+b}{2})|^q+|f''(b)|^q}{2}\Big)^{\frac{1}{q}}\Big\}
\end{aligned}$$

$$= \left(\frac{b-a}{16}\right)^2 \left(\frac{1}{2}\right)^{\frac{2}{p}} (-1)$$
$$\times \left\{ \frac{1}{2}\beta(9, \frac{1}{2}, p+1) - \left(\frac{1}{2} - (-1)^p\right)\beta(1, \frac{1}{2}, p+1) \right\}^{\frac{1}{p}}$$
$$\times \left\{ \left(\frac{| f''(a) |^q + | f''(\frac{a+b}{2}) |^q}{2} \right)^{\frac{1}{q}} + \left(\frac{| f''(\frac{a+b}{2}) |^q + | f''(b) |^q}{2} \right)^{\frac{1}{q}} \right\}.$$

(c) if we choose $r = 6$ and $s = 1$ in Part(a), then we get:

$$\left| \frac{1}{6}\{ f(a) + 4f(\frac{a+b}{2}) + f(b)\} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{16}$$
$$\times \left\{ \left(\frac{1}{3}\right)^{p+1} \frac{{}_2F_1(1, -p, p+2; \frac{1}{3})}{p+1} + \left(\frac{2}{3}\right)^{2p+1} \beta(1, p+1, p+1) \right\}^{\frac{1}{p}}$$
$$\times \left[\left\{ \frac{| f''(a) |^q + | f''(\frac{a+b}{2}) |^q}{2} \right\}^{\frac{1}{q}} + \left\{ \frac{| f''(\frac{a+b}{2}) |^q + | f''(b) |^q}{2} \right\}^{\frac{1}{q}} \right].$$

(d) if we choose $r = 2$ and $s = 1$ in Part(b), then we get:

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{16} \left\{ \frac{{}_2F_1(1, -p; p+2; -1)}{(p+1)} \right\}^{\frac{1}{p}}$$
$$\times \left\{ \left(\frac{| f''(a) |^q + | f''(\frac{a+b}{2}) |^q}{2} \right)^{\frac{1}{q}} + \left(\frac{| f''(\frac{a+b}{2}) |^q + | f''(b) |^q}{2} \right)^{\frac{1}{q}} \right\}.$$

Theorem 2.5. *Let $f : I \subset [0, \infty) \to R$ be a twice differentiable function on I^0 such that f'' is integrable, where $a, b \in I^0$ with $a < b$. If $| f'' |^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then, for $r \geq 2$ the following inequality holds:*

$$(a) \frac{16}{(b-a)^2} \left| H_a^b(f)(r) \right|$$
$$\leq \left[\left\{ \frac{2^{p+1}}{r^{p+1}(p+1)} \right\} {}_2F_1(1, -p; p+2; \frac{2}{r}) \right.$$
$$\left. + \left\{ \frac{r-2}{r} \right\}^{2p+1} \beta(1, p+1, p+1) \right]^{\frac{1}{p}}$$

$$\times \Big[\Big\{\frac{(2^{s+1}-1) \mid f''(a) \mid^q + \mid f''(b) \mid^q}{2^s(s+1)}\Big\}^{\frac{1}{q}} + \Big\{\frac{\mid f''(a) \mid^q + (2^{s+1}-1) \mid f''(b) \mid^q}{2^s(s+1)}\Big\}^{\frac{1}{q}}\Big].$$

$$(b) \frac{16}{(b-a)^2}\Big|R_a^b(f)(r)\Big| \leq \big(\frac{2}{r}\big)\Big\{\frac{{}_2F_1(1,-p;p+2;-\frac{r}{2})}{p+1}\Big\}^{\frac{1}{p}} \times \Big[\Big\{\frac{(2^{s+1}-1) \mid f''(a) \mid^q + \mid f''(b) \mid^q}{(s+1)}\Big\}^{\frac{1}{q}} + \Big\{\frac{\mid f''(a) \mid^q + (2^{s+1}-1) \mid f''(b) \mid^q}{(s+1)}\Big\}^{\frac{1}{q}}\Big].$$

Proof. (a) From Lemma 1 and by the Hölder's integral inequality, we have

$$\begin{aligned}
&\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big| \\
&\leq \Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big) \big|^q\, dt\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\int_0^1 \big|(1-t)\big(t-\frac{2}{r}\big)\big|^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q dt\Big\}^{\frac{1}{q}} \\
&= \Big\{\int_0^{\frac{2}{r}} (1-t)^p\big(\frac{2}{r}-t\big)^p dt + \int_{\frac{2}{r}}^1 (1-t)^p\big(t-\frac{2}{r}\big)^p dt\Big\}^{\frac{1}{p}} \\
&\quad \times \Big[\Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|^q dt\Big\}^{\frac{1}{q}} \\
&\qquad + \Big\{\int_0^1 \big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q dt\Big\}^{\frac{1}{q}}\Big]. \qquad (7)
\end{aligned}$$

Since $\mid f'' \mid^q$ is s-convex on $[a,b]$, we get

$$\big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|^q \leq \big(\frac{1+t}{2}\big)^s \mid f''(a) \mid^q + \big(\frac{1-t}{2}\big)^s \mid f''(b) \mid^q \qquad (8)$$

and

$$\big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q \leq \big(\frac{1+t}{2}\big)^s \mid f''(b) \mid^q + \big(\frac{1-t}{2}\big)^s \mid f''(a) \mid^q, \qquad (9)$$

which implies that

$$\int_0^1 | f''(\frac{1+t}{2}a + \frac{1-t}{2}b) |^q \, dt$$
$$\leq \{\frac{2^{s+1}-1}{s+1}\} | f''(a) |^q + \{\frac{1}{s+1}\} | f''(b) |^q \tag{10}$$

and

$$\int_0^1 |f''(\frac{1+t}{2}b + \frac{1-t}{2}a)|^q dt$$
$$\leq \{\frac{1}{s+1}\} | f''(a) |^q + \{\frac{2^{s+1}-1}{s+1}\} | f''(b) |^q . \tag{11}$$

Note that

$$\int_0^{\frac{2}{r}} (1-t)^p (\frac{2}{r} - t)^p dt = \{\frac{2^{p+1}}{r^{p+1}(p+1)}\} {}_2F_1(1, -p; p+2; \frac{2}{r}).$$
$$\int_{\frac{2}{r}}^1 (1-t)^p (t - \frac{2}{r})^p dt = \{\frac{r-2}{r}\}^{2p+1} \beta(1, p+1, p+1). \tag{12}$$

By (7),(10),(11) and (12), the assertion in this theorem is proved.
(b) By the similar way as Part (a), using Lemma 2 this is proved.

Corollary 2.6. *In Theorem 2.5:*

(a) if we choose $r = 2$ and $s = 1$, then we get:

$$\Big| f(\frac{a+b}{2}) - \frac{1}{b-a} \int_a^b f(x)dx \Big|$$
$$\leq \frac{(b-a)^2}{16} (\frac{1}{2p+1})^{\frac{1}{p}}$$
$$\times \Big\{ \Big(\frac{3 | f''(a) |^q + | f''(b) |^q}{4} \Big)^{\frac{1}{q}} + \Big(\frac{| f''(a) |^q + 3 | f''(b) |^q}{4} \Big)^{\frac{1}{q}} \Big\}.$$

(b) if we choose $r = 4$ and $s = 1$, then we get:

$$\Big| \frac{1}{8}\{ f(a) + 6f(\frac{a+b}{2}) + f(b)\} - \frac{1}{b-a} \int_a^b f(x)dx \Big|$$
$$\leq (\frac{b-a}{16})^2 (\frac{1}{2})^{\frac{2}{p}} (-1)$$

$$\times \Big\{\frac{1}{2}\beta(9,\frac{1}{2},p+1) - \big(\frac{1}{2} - (-1)^p\big)\beta(1,\frac{1}{2},p+1)\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\Big(\frac{3\mid f''(a)\mid^q + \mid f''(b)\mid^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{\mid f''(a)\mid^q + 3\mid f''(b)\mid^q}{4}\Big)^{\frac{1}{q}}\Big\}.$$

(c) if we choose $r = 6$ and $s = 1$, then we get:

$$\Big|\frac{1}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\} - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{(b-a)^2}{16}$$
$$\times \Big\{\big(\frac{1}{3}\big)^{p+1}\frac{{}_2F_1(1,-p,p+2;\frac{1}{3})}{p+1} + \big(\frac{2}{3}\big)^{2p+1}\beta(1,p+1,p+1)\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\Big(\frac{3\mid f''(a)\mid^q + \mid f''(b)\mid^q}{4}\Big)^{\frac{1}{q}} + \Big(\frac{\mid f''(a)\mid^q + 3\mid f''(b)\mid^q}{4}\Big)^{\frac{1}{q}}\Big\}.$$

(d) if we choose $r = 2$ and $s = 1$ in Part(b), then we get:

$$\Big|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{(b-a)^2}{16}\Big\{\frac{{}_2F_1(1,-p;p+2;-1)}{p+1}\Big\}^{\frac{1}{p}}$$
$$\times \Big[\Big\{\frac{3\mid f''(a)\mid^q + \mid f''(b)\mid^q}{2}\Big\}^{\frac{1}{q}} + \Big\{\frac{\mid f''(a)\mid^q + 3\mid f''(b)\mid^q}{2}\Big\}^{\frac{1}{q}}\Big].$$

Theorem 2.7. *Let $f : I \subset [0,\infty) \to R$ be a twice differentiable function on I^0 such that f'' is integrable, where $a, b \in I^0$ with $a < b$. If $\mid f'' \mid^q$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then, for $r \geq 2$ the following inequality holds:*

$$(a)\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big|$$
$$\leq \Big\{\frac{r^3 - 6r^2 + 24 - 16}{6r^3}\Big\}^{1-\frac{1}{q}}\big(\frac{1}{2}\big)^{\frac{s}{q}}$$
$$\times \Big[\Big\{\big(2\mu_{11} - \mu_{12} - \mu_{31}\big)\mid f''(a)\mid^q + \big(2\mu_{21} - \mu_{22}\big)\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+ \Big\{\big(2\mu_{21} - \mu_{22}\big)\mid f''(a)\mid^q + \big(2\mu_{11} - \mu_{12} - \mu_{31}\big)\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}\Big],$$

where μ_{ij} is defined in Theorem 2.1.

$$(b)\frac{16}{(b-a)^2}\Big|R_a^b(f)(r)\Big|$$
$$\leq \Big(\frac{r+2}{6r}\Big)^{1-\frac{1}{q}}\Big[\Big\{\mu_{41}\big|f''(a)\big|^q+\mu_{42}\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\mu_{42}\big|f''(a)\big|^q+\mu_{41}\big|f''(b)\big|^q\Big\}^{\frac{1}{q}}\Big],$$

where

$$\mu_{41}=\frac{6+r+2s}{2^s r(s+2)(s+3)},$$
$$\mu_{42}=\frac{2^{s+2}\{2+r(s-1)\}+(s+5)r-2(s+3)^2}{2^s r(s+1)(s+2)(s+3)}.$$

Proof. (a) From Lemma 1 and by the power mean integral inequality, we have

$$\frac{16}{(b-a)^2}\big|H_a^b(f)(r)\big|$$
$$\leq \Big\{\int_0^1\big|(1-t)\big(t-\frac{2}{r}\big)\big|dt\Big\}^{1-\frac{1}{q}}$$
$$\times\Big\{\Big(\int_0^1\big|(1-t)\big(t-\frac{2}{r}\big)\big|\big|f''\big(\frac{1+t}{2}a+\frac{1-t}{2}b\big)\big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1\big|(1-t)\big(t-\frac{2}{r}\big)\big|\big|f''\big(\frac{1+t}{2}b+\frac{1-t}{2}a\big)\big|^q dt\Big)^{\frac{1}{q}}\Big\}. \tag{13}$$

By (10), (11) and (13), we have

$$\frac{16}{(b-a)^2}\Big|H_a^b(f)(r)\Big|$$
$$\leq\Big\{\Big(\frac{6r-4}{3r^3}\Big)+\Big(\frac{r^3-6r^2+12r-8}{6r^3}\Big)\Big\}^{1-\frac{1}{q}}$$
$$\times\Big[\Big\{\Big(\int_0^{\frac{2}{r}}(1-t)\big(\frac{2}{r}-t\big)\big(\frac{1+t}{2}\big)^s dt$$
$$+\int_{\frac{2}{r}}^1(1-t)\big(t-\frac{2}{r}\big)\big(\frac{1+t}{2}\big)^s dt\Big)\mid f''(a)\mid^q$$
$$+\Big(\int_0^{\frac{2}{r}}(1-t)\big(\frac{2}{r}-t\big)\big(\frac{1-t}{2}\big)^s dt$$

$$+\int_{\frac{2}{r}}^{1}(1-t)\big(t-\frac{2}{r}\big)\big(\frac{1-t}{2}\big)^s dt\Big)\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}$$

$$+\Big\{\Big(\int_0^{\frac{2}{r}}(1-t)\big(\frac{2}{r}-t\big)\big(\frac{1-t}{2}\big)^s dt$$

$$+\int_{\frac{2}{r}}^{1}(1-t)\big(t-\frac{2}{r}\big)\big(\frac{1-t}{2}\big)^s dt\Big)\mid f''(a)\mid^q$$

$$+\Big(\int_0^{\frac{2}{r}}(1-t)\big(\frac{2}{r}-t\big)\big(\frac{1+t}{2}\big)^s dt$$

$$+\int_{\frac{2}{r}}^{1}(1-t)\big(t-\frac{2}{r}\big)\big(\frac{1+t}{2}\big)^s dt\Big)\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}\Big].$$

By using the equalities (5), the assertion in this theorem is proved.
(b) By the similar way as Part (a), using Lemma 2 this is proved.

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