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# A COMMON FIXED POINT THEOREM FOR FOUR MAPS IN CONE METRIC SPACES

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**Abstract:** In this paper,we prove existence of coincidence points and a common fixed point theorem for four maps in cone metric spaces. Our result generalizes and extends some recent results.

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**Key Words:** cone metric space, common fixed point, coincidence point

## 1. Introduction and Preliminaries

In 2007 Huang and Zhang [7] have generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractive conditions. Subsequently,Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [7], [8] and the references mentioned therein). Recently, Guangxing Song et al [6] have obtained coincidence points and common fixed point theorems for two mappings in cone metric spaces.In this paper we generalize and extend the Theorem 2.1 of Guangxing Song et al [6] for four self-mappings.

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In all that follows, $E$ is a real Banach space. For the mappings $f, g : X \longrightarrow X$, let $C(f, g)$ denotes set of coincidence points of $f$, $g$ that is $C(f, g) := \{z \in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties [7].

**Definition 1.1.** Let $E$ be a real Banach space and $P$ be a subset of $E$. The set $P$ is called a cone if and only if:

$(a)$ $P$ is closed, nonempty and $P \neq \{0\}$;

$(b)$ $a, b \in R$,$a, b \geq 0$,$x, y \in P \Longrightarrow ax + by \in P$;

$(c)$ $x \in P$ and $-x \in P \Longrightarrow x = 0$.

**Definition 1.2.** Let $P$ be a cone in a Banach space $E$ define partial ordering $\leq$ with respect to $p$ by $x \leq y$ if and only if $y - x \in P$.We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set $P$. This Cone $P$ is called an order cone.

**Definition 1.3.** Let $E$ be a Banach Space and $P \subset E$ be an order cone.The order cone $P$ is called normal if there exists $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \quad \text{implies} \quad \| x \| \leq K \| y \| .$$

The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

**Definition 1.4.** Let $X$ be a nonempty set of $E$. Suppose that the map $d : X \times X \longrightarrow E$ satisfies:

$(d1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d3)$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, d)$ is called a cone metric space. It is obvious that the cone metric spaces generalize metric spaces.

**Definition 1.5.** Let $(X, d)$ be a cone metric space.We say that $\{x_n\}$ is:

$(i)$ a Cauchy sequence if for every $c$ in $E$ with $0 \ll c$, there is $N$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

$(ii)$ a convergent sequence if for any $0 \ll c$, there is an $N$ such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed $x$ in $X$. We denote this $x_n \longrightarrow x$

$(n \longrightarrow \infty)$. A cone metric space $X$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

**Definition 1.6.** Let$f, g : X \longrightarrow X$.Then the pair $(f, g)$is said to be(IT)-Commuting at $z \in X$ if $f(g(z))$=$g(f(z))$ with $f(z)$=$g(z)$.

## 2. Common Fixed Point Theorem

In this section we obtain existence of coincidence points and a common fixed point Theorem for four self-mappings defined on a cone metric space.

The following theorem extends and improves Theorem 2.1 in [6].

**Theorem 2.1.** *Let $(X, d)$ be a cone metric space,and $P$ a normal cone with normal constant $K$.let $a_i \geq 0$ (i = 1,2,3,4,5) be constants with $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$. Suppose that the mappings $S, T, f$ and $g$ are four self -maps on $X$ such that $S(X) = T(X)$ and $T(X) \subset f(X), S(X) \subset g(X)$ and satisfy the condition*

$$d(Sx, Ty) \leq a_1 d(fx, gy) + a_2 d(Sx, fx) + a_3 d(Ty, gy) + a_4 d(fx, Ty) + a_5 d(Sx, gy)) \quad (1)$$

*for all $x, y \in X$. If $S(X)$ or $T(X)$ is a complete subspace of $X$. Then the maps $(S, f)$and $(T, g)$ have a coincidence point in $X$. Moreover if $(S, f)$ and $(T, g)$ are (IT)-Commuting at p, then $S$, $T$, $f$ and $g$ have a unique common fixed point.*

*Proof.* Suppose $x_0$ is an arbitrary point of $X$, and define the sequence $\{y_n\}$ in $X$ such that

$$y_{2n} = Sx_{2n} = gx_{2n+1} \quad \text{and} \quad y_{2n+1} = Tx_{2n+1} = fx_{2n+2},$$

for all $n$= 0,1,2,. . . .By the equation (1),we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) =& d(Sx_{2n}, Tx_{2n+1}) \\ \leq& a_1 d(fx_{2n}, gx_{2n+1}) + a_2 d(Sx_{2n}, fx_{2n}) + a_3 d(Tx_{2n+1}, gx_{2n+1}) \\ &+ a_4 d(fx_{2n}, Tx_{2n+1}) + a_5 d(Sx_{2n}, gx_{2n+1}) \\ \leq& a_1 d(y_{2n-1}, y_{2n}) + a_2 d(y_{2n}, y_{2n-1}) + a_3 d(y_{2n+1}, y_{2n}) \\ &+ a_4 d(fx_{2n}, y_{2n+1}) + a_5 d(y_{2n}, y_{2n}) \\ \leq& a_1 d(y_{2n-1,2n}) + a_2 d(y_{2n}, y_{2n-1}) + a_3 d(y_{2n+1}, y_{2n}) \\ &+ a_4[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + a_5 d(y_{2n}, y_{2n}) \end{aligned}$$

$$\leq (a_1 + a_2 + a_4) d(y_{2n-1}, y_{2n}) + (a_3 + a_4) d(y_{2n}, y_{2n+1}).$$

Which implies that

$$d(y_{2n}, y_{2n+1}) \leq \frac{(a_1 + a_2 + a_4)}{1 - (a_3 + a_4)} d(y_{2n-1}, y_{2n}),$$

$$d(y_{2n}, y_{2n+1}) \leq \delta d(y_{2n-1}, y_{2n}),$$

where $\delta = \frac{(a_1+a_2+a_4)}{1-(a_3+a_4)} < 1$. Since $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$. Similarly, it can be shown that $d(y_{2n+1}, y_{2n+2}) \leq \delta d(y_{2n}, y_{2n+1})$.

Therefore, for all $n$

$$d(y_{n+1}, y_{n+2}) \leq \delta d(y_n, y_{n+1}) \leq \cdots \leq \delta^{n+1} d(y_0, y_1).$$

Now, for any $m > n$,

$$\begin{aligned} d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{m-1}, y_m). \\ &\leq [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}] d(y_1, y_0) \\ &\leq \frac{\delta^n}{1-\delta} d(y_1, y_0). \end{aligned}$$

From definition (1.3), we have

$$\| d(y_n, y_m) \| \leq \frac{\delta^n}{1-\delta} K \| d(y_1, y_0) \| .$$

Which implies that $d(y_n, y_m) \to 0 \ as \ \ n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence.Since $\{y_n\}$ is a Cauchy sequence in $T(X)$ which is complete there exists $z \in T(X)$ such that $y_n \longrightarrow z$. Since $T(X) \subset f(X)$, then there exists a point $u \in X$ such that $z = fu$. Let us prove that $z = Su$. Then by the triangle inequality and (1), we have

$$\begin{aligned} d(Su, z) \leq & d(Su, Tx_{2n-1}) + d(Tx_{2n-1}, z) \\ \leq & a_1 d(fu, gx_{2n_1}) + a_2 d(Su, fu) + a_3 d(Tx_{2n-1}, gx_{2n-1}) \\ & + a_4 d(fu, Tx_{2n-1}) + a_5 d(Su, gx_{2n-1}) + d(Tx_{2n-1}, z). \end{aligned}$$

By (1.3), we have

$$\begin{aligned} \| d(Su, z) \| \leq & K(a_1 \| d(fu, gx_{2n-1}) \| + a_2 \| d(Su, fu) \| \\ & + a_3 \| d(Tx_{2n-1}, gx_{2n-1}) \| \\ & + a_4 \| d(fu, Tx_{2n-1}) \| + a_5 \| d(Su, gx_{2n-1}) \| \end{aligned}$$

$$+ \parallel d(Tx_{2n-1}, z) \parallel).$$

Letting $n \longrightarrow \infty$, yields

$$\begin{aligned} d(Su, z) \leq & a_1 d(z, z) + a_2 d(Su, z) + a_3 d(z, z) \\ & + a_4 d(z, z) + a_5 d(Su, z) + d(z, z) \\ \leq & (a_2 + a_3) d(Su, z), \end{aligned}$$

which is a contradiction, since $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$. Implies $Su = z$.

Therefore,

$$z = Su = fu, \; u \text{ is a coincidence point of } \{S, f\}. \tag{2}$$

Since, $S(X) \subseteq g(X)$ there exists a point $v \in X$ such that $z = gv$.We shall show that $Tv = z$.Then by (1), we have

$$\begin{aligned} d(z, Tv) = & d(Su, Tv) \\ \leq & a_1 d(fu, gv) + a_2 d(Su, fu) + a_3 d(Tv, gv) \\ & + a_4 d(fu, Tv) + a_5 d(Su, gv) \\ \leq & a_1 d(z, z) + a_2 d(z, z) + a_3 d(Tv, z) \\ & + a_4 d(z, Tv) + a_5 d(z, z) \\ \leq & (a_3 + a_4) d(z, Tv), \end{aligned}$$

which is a contradiction, since $a_1 + a_2 + a_3 + 2a_4 + a_5 < 1$. Implies $z = Tv$.

Therefore,

$$z = Tv = gv, \; v \text{ is a coincidence point of } \{T, g\}. \tag{3}$$

From (2) and (3) it follows

$$Su = fu = Tv = gv (= z). \tag{4}$$

Since, $(S, f)$, $(T, g)$ are $(IT)$-commuting

$$\begin{aligned} d(SSu, Su) = & d(SSu, fu) \\ = & d(SSu, Tv) \\ \leq & a_1 d(fSu, gv) + a_2 d(SSu, fSu) + a_3 d(Tv, gv) \\ & + a_4 d(fSu, Tv) + a_5 d(SSu, gv) \\ = & a_1 d(Sfu, gv) + a_2 d(SSu, Sfu) + a_3 d(Tv, gv) \\ & + a_4 d(Sfu, Tv) + a_5 d(SSu, gv) \end{aligned}$$

$$\begin{aligned}&=a_1d(SSu,Su)+a_2d(SSu,SSu)+a_3d(z,z)\\&\quad+a_4d(SSu,Su)+a_5d(SSu,Su),\end{aligned}$$

$$d(SSu,Su)=(a_1+a_4+a_5)<d(SSu,Su).$$

Contradiction, since $a_1+a_2+a_3+2a_4+a_5<1$. Implies $SSu=Su(=z)$, $Su=SSu=Sfu=fSu$ (since $C(S,f)$ is (IT)-commuting), $SSu=fSu=Su(=z)$.

Therefore

$$Su=z,\ \text{is a common fixed point of } S \text{ and } f. \tag{5}$$

Similarly

$$Tv=TTv=Tgv=gTv \quad\Rightarrow\quad TTv=gTv=Tv(z). \tag{6}$$

Therefore,

$$Tv(=z)\quad\text{is a common fixed point of } T \text{ and } g. \tag{7}$$

In view of (5) and (6) it follows that $S$, $T$, $f$ and $g$ have a common fixed point namely $z$. uniqueness, let $z_1$ be another common fixed point of $S$, $T$, $f$ and $g$. Then

$$\begin{aligned}d(z,z_1)&=d(Sz,Tz_1)\\&\le a_1d(fz,gz_1)+a_2d(Sz,fz)+a_3d(Tz_1,gz_1)\\&\quad+a_4d(fz,Tz_1)+a_5d(Sz,gz_1)\\&\le a_1d(z,z_1)+a_2d(z,z)+a_3d(z_1,z_1)\\&\quad+a_4d(z,z_1)+a_5d(z,z_1)\\&\le(a_1+a_4+a_5)d(z,z)<d(z,z)\end{aligned}$$

(since $a_1+a_2+a_3+2a_4+a_5<1$), which is a contradiction. Hence $z=z_1$ and moreover $S$, $T$, $f$ and $g$ have a unique common fixed point. □

**Corollary 2.2.** *Let $(X,d)$ be a cone metric space, and $P$ a normal cone with normal constant $K$. Let $a_i\ge 0$ $(i=1,2,3,4,5)$ be constants with $a_1+a_2+a_3+2a_4+a_5<1$. Suppose that the mappings $S,T$ and $f$ are three self-maps on $X$ such that $S(X)=T(X)$ and $T(X)\subset f(X), S(X)\subset f(X)$ and satisfy the condition*

$$\begin{aligned}d(Sx,Ty)\le a_1d(fx,fy)+a_2d(Sx,fx)+a_3d(Ty,fy)\\+a_4d(fx,Ty)+a_5d(Sx,fy))\end{aligned} \tag{8}$$

*for all $x, y \in X$. If $S(X)$ or $T(X)$ is a complete subspace of $X$. Then the maps $(S, f)$and $(T, f)$ have a coincidence point in $X$. Moreover if $(S, f)$ and $(T, g)$ are (IT)-Commuting, then $S$, $T$ and $f$ have a unique common fixed point.*

*Proof.* We easy complete the proof if we put $f = g$ in Theorem 2.1. □

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## References

[1] M. Abbas, G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, **341** (2008), 416-420.

[2] M. Abbas, B.E. Rhoades, Fixed and periodic point results in cone metric Spaces, *Appl. Math. Lett.*, **22** (2009), 511-515.

[3] I. Beg, M. Abbas, Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition, *Fixed Point Theory Appl.* (2006), 1-7, Article ID 74503.

[4] V. Berinde, A common fixed point theoremfor compactible quasi contractive self mappings in metric spaces, *Appl. Math. Comput.*, **213** (2009), 348-354.

[5] B. Fisher, Four mappings with a common fixed point, *J. Univ. Kuwait Sci.* (1981), 131-139.

[6] Guang Xing Song, X. Huang, Xiaoyan Sun, Yian Zhao, Guotao Wang, New common fixed point theorems for maps on cone metric spaces, *Applied Mathematics Letters*, **23** (2010), 1033-1037.

[7] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332**, No. 2 (2007), 1468-1476.

[8] S. Rezapour, Halbarani, Some notes on the paper "cone metric spaces and fixed point theorem of contractive mappings", *J. Math. Anal. Appl.*, **345** (2008), 719-724.