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STAIRWAY LIGHT SWITCHES

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Abstract: This article explains stairway light switches which can be turned on or off from different places. This idea is based on the binary system of Boolean algebra. Mathematics is hidden everywhere in daily life and we may enjoy it if we study the mathematics relating to everyday things.

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1. An Issue at Close Hand

Mathematics clearly serves many purposes, even though many people do not realize it. A good example in daily life involves the use of stairway light switches. Figure 1 shows a model of a stairway light switch of my own design. Before climbing the stairs at night, you first turn on the landing light using the switch in the hallway. Then, after reaching the top, you turn the light off using the switch on the upstairs landing. Everyone must have used such switches to turn the same light on or off from different places. This is an excellent application of mathematics, using a device conceived from the principle of binary numbers.

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Figure 1: Stairway switches on two different floors

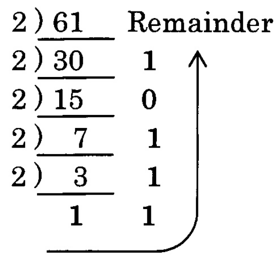
There are very many examples of mathematics being applied all around us. While we can't pay attention to every application, the switch on the stairs is certainly a useful example. And as mathematics wasn't invented to make students suffer in exams, I like to use such simple examples to help us understand that mathematics actually makes our lives easier.

So, what is it about this switch on the stairs? Remote controlled devices are very popular, so is it remote controlled? No, it's not that. There are also sensors that automatically activate when it becomes dark, but it's not that either. Are the switches on the different floors perhaps connected, like the "bedside light switches" known to my generation that allow us when we tire of reading comic books and want to go to sleep to turn off the beside lamp using a cord, without getting out of bed? This type of switch has been very successful. But that's also not it. The answer in fact concerns the *relationship* between the electric wiring and the switch, and the problem begs for the kind of elegant solution a mathematics enthusiast would produce.

The switch on the first floor has two states. The switch on the second floor also has two states. These two states are lit (on) and unlit (off), and may be thought of as being related like binary numbers. So, let's begin by thinking about binary arithmetic problems.

2. Binary and Hexadecimal

Problem 1. Express the decimal number 61 in binary and hexadecimal.



Divide 61 by 2 and obtain the quotient and remainder. 61 divided by 2 yields a quotient of 30 and a remainder of 1. Dividing 30 by 2 yields a quotient of 15 and a remainder of 0. These operations are repeated until a quotient of 1 is reached that cannot be further divided, and the steps of the calculation are recorded as shown in the diagram above. Next, the numbers 11101 written in bold are read off, beginning with the last quotient of 1 and proceeding upwards as indicated by the arrow. 111101 is thus the binary expression.

Considering hexadecimal numbers in terms of binary, they take 4 digit blocks (4 bits). This is because $2^4 = 16$. The conversion to hexadecimal is therefore made simple by splitting 111101 into 4 digit blocks starting from the lowest position. Splitting it into 11 and 1101, and using the fact that 11 is 3, and 1101 is D, the hexadecimal expression can be found, which is 3D.

The origin of the term binary (or base 2) lies in the fact that there are two symbols, '0' and '1', used to represent numbers. These symbols are known as numerals. Being called base 2, we might expect the symbol '2' to be included, but counting '0' and using '1' means that there are already two symbols, so '2' is not included in base 2.

Decimal (or base 10) uses the ten numerals from '0' to '9', which express the values 0 to 9. In the case of hexadecimal (or base 16), there are sixteen symbols used to express numbers. The numerals used to represent 10 and above are the alphabetical characters 'A' to 'F'. The decimal value of 10 is represented by 'A', 11 by 'B', 12 by 'C', 13 by 'D', 14 by 'E', and 15 by 'F'. The decimal value of 16 is then written 10 ('one zero') in hexadecimal.

Numbers such as 11 ('eleven') in decimal and 11 in binary are easy to confuse, so in the binary case it's common to pronounce 1 as 'one', 0 as 'zero', and 11 as 'one one'. A subscript of 2, 10 or 16 may also be written on the right hand side to clarify whether a number is written in binary, decimal or hexadecimal, respectively. The answer to Problem 1 is written as follows.

$$61_{10} = 111101_2 = 3D_{16}$$

After obtaining the answer, it's also important to check back to see if it is correct. The way to convert the obtained binary number back into decimal is as follows.

$$111101_2$$
$$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 32 + 16 + 8 + 4 + 0 + 1$$
$$= 61_{10}$$

Problem 2. Calculate the sum, difference, product and quotient of the binary numbers $A = 1011$ and $B = 111$.

The four arithmetic operations in binary are as follows.

$$\begin{array}{r} 1011 \\ +111 \\ \hline 10010 \end{array} \qquad \begin{array}{r} 1011 \\ -111 \\ \hline 100 \end{array} \qquad \begin{array}{r} 1011 \\ \times 111 \\ \hline 1011 \\ 1011 \\ 1011 \\ \hline 1001101 \end{array} \qquad \begin{array}{r} 1 \\ 111\overline{)1011} \\ 111 \\ \hline 100 \end{array}$$

Addition Subtraction Multiplication Division

In decimal, these are the four arithmetical calculations for the numbers $A = 11$ and $B = 7$. Check each one!

3. Boolean Algebra and Truth Tables

Problem 3. Think about the mechanism of the stairway switch that can turn the light on or off on two different floors. "How does this mechanism involve maths?"

Normal switches are known as single state switches where in the "on" position the light would be on, and in the "off" position the light would be off. However, the on and off positions of the stairway switch cannot be distinguished. It is the *relationship* between the switches on the first and second floors that determines whether the light is on or off.

Let us denote the switch on the first floor as A, its two states as 0 and 1, and likewise denote the switch on the second floor as B and its two states as 0 and 1. When the light is on its state is taken to be 1, and when it is off its state is 0. The switches A and B and the light each have only two states, so this is clearly a binary world.

The relationship between switches A and B and the light is summarized in Table 1. This kind of table is known as a truth table. There are two cases for switch A, and two for switch B, so in total there are $2 \times 2 = 4$ cases.

	(1)	(2)	(3)	(4)
A	0	1	1	0
B	0	0	1	1
The light	1	0	1	0

Table 1. Truth table (switches in two different places)

This may also be expressed as a Venn diagram, as shown in Figure 2. The cases when the light is on are shown in the diagonal region. Venn diagrams are a special case of Euler diagrams. The region shown in this Venn diagram represents the cases when switches A and B are both 0, or when A and B are both 1, and in these cases the light is on. In the truth table these are cases (1) and (3).

Writing this as a logical formula yields the following.

$$shining = AB + \overline{A}\,\overline{B}$$

AB is the product of A and B ,+ indicates a sum, $\bar{A}$ is the negation of A, and $\bar{B}$ the negation of B. A circuit of this type is known as an equivalence circuit. An equivalence circuit is the logical negation of exclusive or (Exclusive OR, XOR) as shown below.

Computers are constructed using binary as a basis. Electronic circuits are constructed from the following six fundamental circuits: logical and (AND), logical or (OR), negation (NOT), negative logical and (NAND), negative logical or (NOR), and exclusive logical or (XOR).

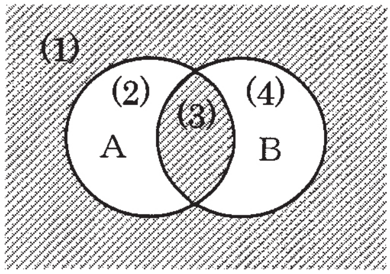


Figure 2: Venn diagram (switches in two different places)

There are four basic arithmetic operations (addition, subtraction, multiplication and division). Multiplications are repeated additions, and divisions are repeated subtractions. All of the four arithmetical operations can therefore be expressed using the fundamental operation of addition in the one digit case.

When thinking about such single digit addition, two locations for the result are necessary, the ‘sum’ and the ‘carry’. For the sum, the following four patterns must be satisfied; the sum of 0 and 0 is 0, for 0 and 1 it is 1, for 1 and 0 it is 1, and for 1 and 1 it is 0. Likewise, the carry is 0 for 0 and 0, 0 for 1 and 0, 0 for 0 and 1, and for 1 and 1, it is 1.

$$\begin{array}{r} 0 \\ +\,0 \\ \hline 0 \end{array} \qquad \begin{array}{r} 0 \\ +\,1 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\,0 \\ \hline 1 \end{array} \qquad \begin{array}{r} 1 \\ +\;\;1 \\ \hline 10 \end{array}$$

The addition circuit that provides these both simultaneously only requires an exclusive logical or (XOR) circuit for the ‘sum’ location, and a logical and (AND) circuit for the ‘carry’ location. Make a Venn diagram for each to confirm this. Now, I mentioned that the exclusive logical or and the stairway switch circuits are related by negation, but let’s confirm this using logic equations. The exclusive logical or of A and B is written as $A \oplus B$. The fact that exclusive logical or is the negation of the equivalence circuit may be confirmed using equations by means of De Morgan’s law, and is also made clear by Venn diagrams.

$$\overline{AB + \bar{A}\bar{B}} = \overline{AB} \times \overline{\bar{A}\bar{B}} = (\bar{A} + \bar{B})(\bar{\bar{A}} + \bar{\bar{B}})$$
$$= (\bar{A} + \bar{B})(A + B) = \bar{A}A + \bar{A}B + \bar{B}A + \bar{B}B$$

$$= A\bar{B} + \bar{A}B$$

By replacing binary calculations with logical set operations, the above equations can be reformulated using Boolean algebra. Boolean algebra is the basis of computing today. George Boole (1854), who formulated the system, made a great contribution without yet experiencing the emergence of computers. Some people say that mathematics anticipates developments 100 years in the future, but perhaps this is a romantic notion that only mathematicians can appreciate.

Now, back to the stairway switch. Maybe you have realized that a normal single state switch is no good for this problem. In fact the solution uses something called a 3-way switch (Figure 3). Since the light is only on when switches A and B are both 0 or both 1, *i.e.*, when they both have the same value, there is not just one wire between the switches on the two floors, but two. The final circuit diagram is as shown in Figure 4.

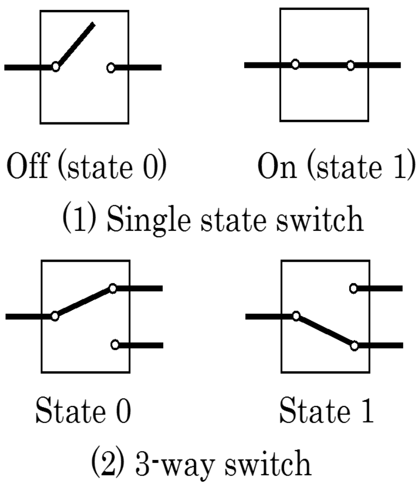


Figure 3: Single state switch and 3-way switch

4. An Infinite Story House

Problem 4. Consider the design of a light switch system that can turn the same light on or off in three different places.

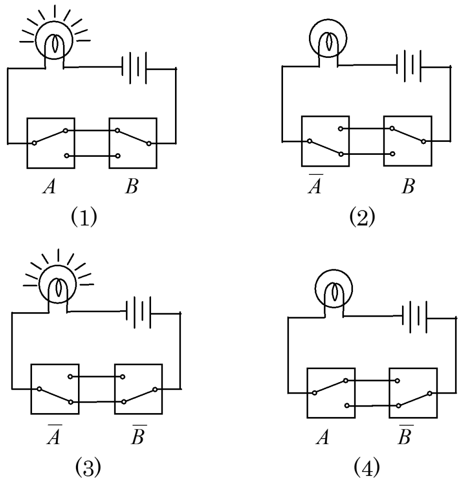


Figure 4: Switches in two places (set of four)

Now, the switch system in our house that can be used in the hall and on the upstairs landing to turn on and off the same light is extremely useful. This type of design can be used not only for switches in two locations, but also in three locations, or five locations, and such devices are in fact available in practice. Let's think about the three-location switch system. Figure 5 shows a switch system I designed for a three story house.

Figure 5: Stairway switches on three floors

The three switches are denoted A,B and C, and the relationship between them determines the state of the light. Since each of the switches has the states 0 and 1, there are $2 \times 2 \times 2 = 8$ different cases. For each of these cases, denoting the situation when the light is on by 1, and off by 0, we can draw up a truth table as shown in Table 2.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
A	1	1	1	1	0	0	0	0
B	1	1	0	0	1	1	0	0
C	1	0	1	0	1	0	1	0
The light	0	1	1	0	1	0	0	1

Table 2. Truth table (switches in three places)

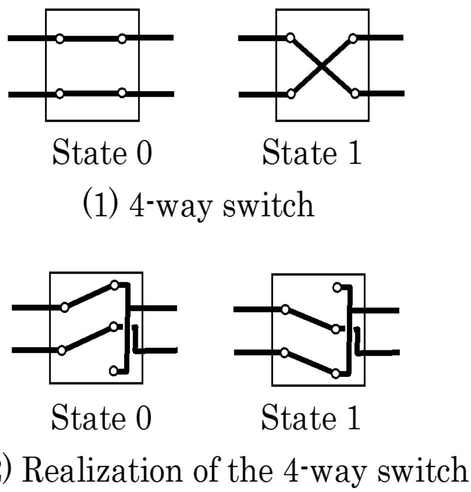


Figure 6: 4-way switch

Expressing this as a logical formula yields the following.

$$Shining = AB\bar{C} + A\bar{B}C + \bar{A}BC + \bar{A}\bar{B}\bar{C}$$

It's necessary to think about the corresponding circuit, but rather than being composed of electric circuits for AND, XOR, *etc.*, it is actually simple to construct.

The cases when the light is on are (2), (3) , (5) and (8). The cases when it is off are (1), (4), (6) and (7). Comparing these reveals that it is on when the sum $A + B + C$ is an even number, and off when it is odd. If instead the light were on when $A + B + C$ was odd, the functionality would be the same.

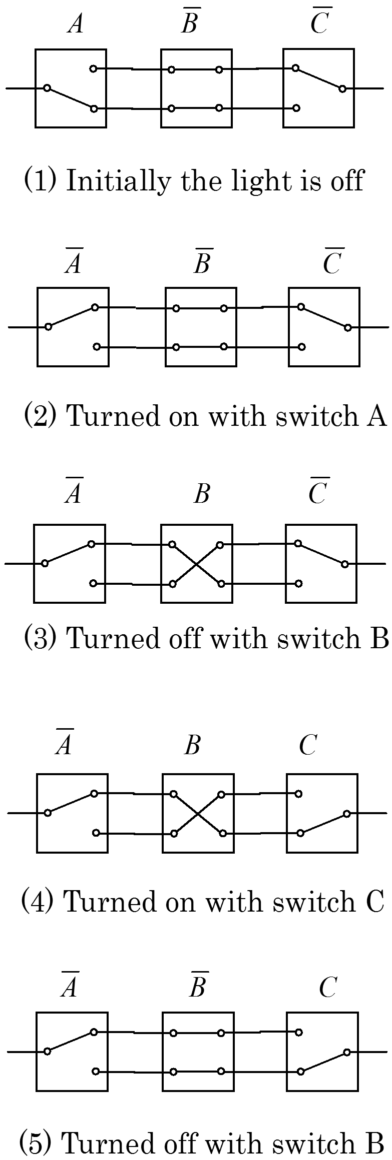


Figure 7: Confirming the three location switch

Now then, time for the actual circuit. Firstly, the switches on the first and third floors, A and C, use the 3-way switch shown above. The switch on the second floor uses one called a 4-way switch. The 4-way switch takes a value of 1 in the parallel condition, and 0 in the cross condition. The functionality of the cross condition is complicated, and its realization is shown in Figure 6(2).

Let's confirm that the three location switch operates correctly by means of Figure 7. Suppose that the light is initially off. Switches A, B and C are in states 1, 0 and 0 (case 1). Next, the light is turned on with switch A. Switches A, B and C are now in states 0, 0 and 0 (case 2). Next, the light is turned off using switch B. Switches A, B and C are now in states 0, 1 and 0 (case 3). Next, switch C is pressed to turn on the light. Switches A, B and C are now in states 0, 1 and 1 (case 4). Then the light is turned off using switch B. Switches A, B and C are now in states 0, 0 and 1 (case 5).

The thinking behind the three location switch can be applied to an unlimited number of locations to build our infinite story house. By using a 3-way switch at the left and right ends, and 4-way switches for all the switches in between, a circuit can be made by which a light can be turned on or off using switches in ten locations or even a hundred locations. The idea behind this kind of circuit is a by-product of computer technology. (see more Nishiyama, 1986).

References

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