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MATHEMATICS OF FANS

Yutaka Nishiyama

Department of Business Information
Faculty of Information Management
Osaka University of Economics
2, Osumi Higashiyodogawa Osaka, 533-8533, JAPAN

Abstract: This paper explains why some fans appear to rotate backwards because of a stroboscopic effect. The phenomenon is often seen with fans in movies. After estimating fan rotation by mathematical formulae, many applications of stroboscopes are shown.

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Key Words: stroboscopic effect, virtual rotation, alternating current, movie frames

1. Some Fans Appear to Rotate Backwards

Fans are no longer seen so often, due to the popularity of air conditioners. Even so, I receive questions about fans from time to time, like “Some fans appear to rotate backwards. Why so?”

About 30 years ago, I wrote an essay entitled “Mathematics behind the fans” that suggested that if we can understand the reason for the apparent backward rotation of fans, then we could use that fundamental principle to estimate the speed of a fan’s rotation (see Nishiyama, 1986). Let me explain

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the reason for that by supplementing what I wrote then to account for more modern environments.

The number of fan blades is generally either 3 or 4. Let's assume that it is 3, and that when the fan is powered on its motor begins to rotate in a clockwise direction. As the speed of rotation increases, however, one can see rotations that differ from the actual rotation. These could be clockwise rotations slower than the actual speed, a stationary appearance, or counterclockwise rotations in the opposite direction to the motor's rotation. These three states are possible. They are referred to as virtual rotations, which can be forwards, stationary, or backwards (see Figure 1).

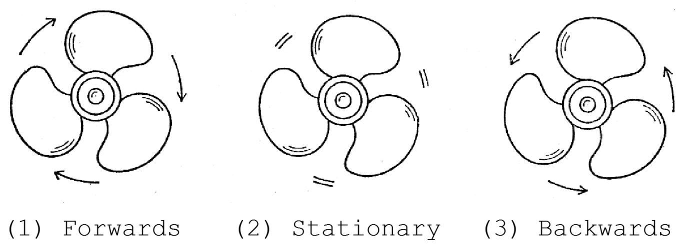

Figure 1: Three virtual rotations

All of these can be explained by the term "stroboscopic effect". Strobe light is a shortening of "stroboscopic light", a flashing lamp used for photography. The rotating visual phenomenon is possible when fans are observed under intermittent fluorescent or candescent lights. Such virtual rotations cannot be observed if the fans are taken outdoors and are observed under the continuous light of the sun, although I may be the only one who has conducted such an odd experiment.

Fluorescent and candescent lights are powered by an alternating current (AC). A direct current (DC) is constant and DC lights are also constant. Alternating current (AC) on the other hand, is not constant. AC has a wave like a Sinusoid with either 50 or 60 cycles per second in Japan. Thus, under such intermittent AC power, a light is also intermittent. Humans are known to recognize only about 10 movie frames per second. This is why we do not sense that the light is blinking intermittently.

So what happens when we observe the fans under intermittent lights? Let's illustrate the answer using Figure 2:

Let's denote the three blades as A, B, and C. Let us further assume that

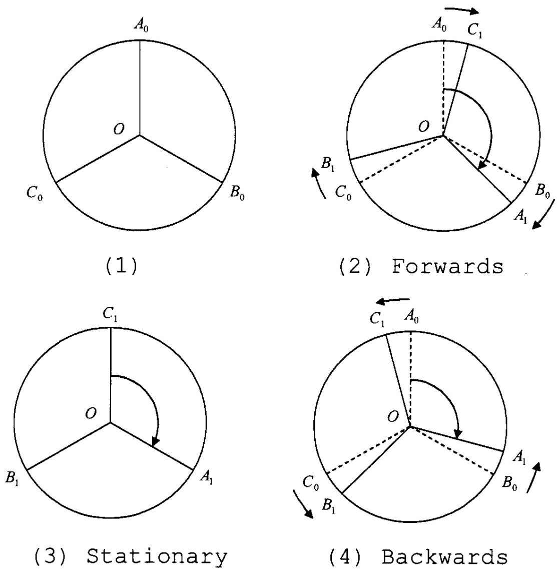

Figure 2: Actual and virtual rotations

the initial blade positions are OA_0, OB_0, and OC_0, respectively (Figure 2(1)). Assume also that, when the light comes on after one AC cycle, the blades have advanced by about 120 degrees to the new positions OA_1, OB_1, and OC_1, respectively. At this time, our eyes do not realize that blades have moved from OA_0 to OA_1, OB_0 to OB_1, and OC_0 to OC_1. Instead, we erroneously regard the blade at the position closest to the original blade's previous position as the original blade. In other words, we feel that OA_0 has moved to OC_1, OB_0 has moved to OA_1, and OC_0 has moved to OB_1.

Therefore, during one cycle in which the light blinks, the fans move forward virtually if the fans advance by slightly more then 120 degrees. The fans are virtually stationary if the fans advance exactly by 120 degrees, and the fans move backwards virtually if the fans advance by less than 120 degrees (Figure 2(2) - (4)).

2. Alternating Current and Rectification

What then is the frequency of the blinking fluorescent and candescent lights? As you may know, the frequency of alternating current in Japan is 60 Hz in

Western Japan and 50 Hz in Eastern Japan.

Why does a small country like Japan have two different frequencies? During the Meiji Restoration, there were lingering power conflicts among political sections, each of which purchased power generators from foreign countries with close ties. Tokyo Dento (Tokyo Electric Power today) adopted a 50 Hz AC power generator from Allgemeine of Germany in 1895, while Osaka Dento adopted a 60 Hz AC power generator from GE of USA in 1897. Thus, 50 Hz was adopted as the standard for Eastern Japan, while 60 Hz became the standard for Western Japan. Although there have been several opportunities to unify the two frequencies, these two have remained separate until today.

My initial understanding was that 50 Hz or 60 Hz AC causes lights to blink 50 or 60 times per second, but some of my readers wrote to me to inform me that because of a process called AC rectification, the lights actually blink twice as fast, *i.e.*, 100 or 120 times.

This is illustrated in Figure 3. Let's assume we are using 50 Hz AC power. Unlike DC, AC is represented by a Sine wave. The wave has positive and negative regions, of which only the positive area is significant for the current. This section constitutes a half-wave rectification. In this state, the blinking occurs at 50 Hz. Some books explain that this half-wave rectification is sufficient for lights not requiring much illumination, like a tail lamp of a bicycle. This half-rectification was possibly used immediately after the war. When two rectifiers are used, converting the current direction of the negative region, a full-wave rectification is created where the light blinks at 100 Hz. Most of the lights today use this full-wave rectification.

In this manner, the lights blink at 100 Hz or 120 Hz. Thus, it is difficult to see virtual rotations if the fans are observed under electric lamps as explained above. Let me explain how the virtual rotations can in fact be observed. In order to do so, I must first explain the fan's specifications.

I once wrote a letter to an electric appliance company asking for the rotational speeds of their fans for "research purposes". I believe the rotational speeds have not changed so much since then. One fan for 60 Hz Western Japan had three settings. They were a light breeze at 720 revolutions/minute, a cool breeze at 1040 revolutions/minute, and a strong breeze at 1390 revolutions/minute.

Although the motor revolutions are often expressed per minute, the electric light frequency is usually expressed per second. Let me therefore convert these numbers to a scale of seconds. The light breeze is 12 revolutions/second, the cool breeze is 17.3 revolutions/second, and the strong breeze is 23.2 revolutions/second. The average comes to roughly 20 revolutions/second. This

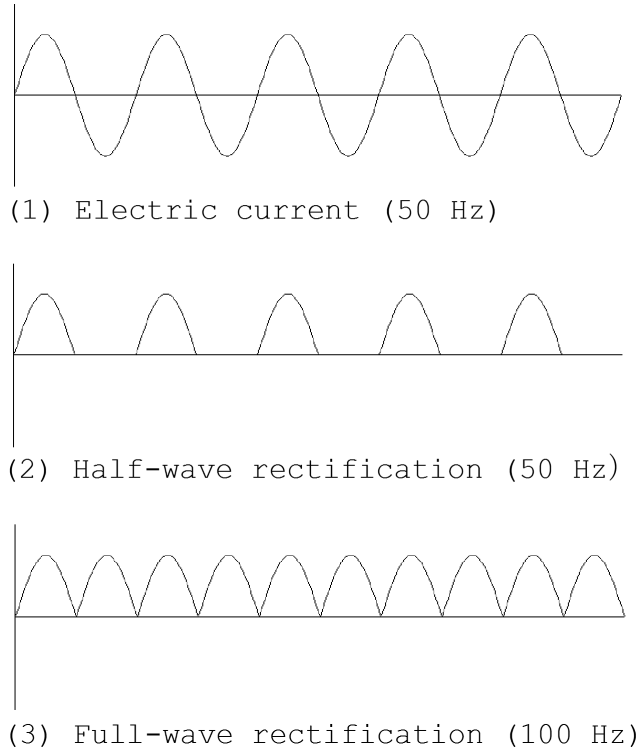

Figure 3: Rectification

means that if the fan has three blades, under an electric light at 60 Hz, it rotates around 120 degrees every 1/60th of a second. In other words, before and after the fan goes into the strong breeze speed, we should be able to observe forward, stationary and backward virtual rotations.

	50Hz	60Hz
Light breeze	800(13.3)	720(12.0)
Cool breeze	1050(17.5)	1040(17.3)
Strong breeze	1280(21.3)	1390(23.2)

Unit: Rotations/Minute, numbers in ( ) show rotations/second.

Table. Fan rotation speeds

However, it is difficult to observe these phenomena because the electric light frequency is either 100 Hz or 120 Hz due to rectification as explained above.

What should we do then? Give up the fluorescent light and candescent light in favor of TVs. Braun tube TV scanning lines transmit 30 image frames per second. 30 Hz is a smaller frequency than that of the fluorescent lamps. When you wave your hands in front of a TV display, your hands seem to flicker. This is because of the intermittent images displayed 30 times per second by the TV.

TVs using Braun tubes use 525 scanning lines which run from the top left corner to the bottom right corner 30 times per second. However, since this TV image flickers and is not good for human eyes, an interlace scanning method which scans odd number lines at 60 Hz and even number lines at 60 Hz has been used more lately. Furthermore, since TVs have moved from old Braun tubes to LCD and plasma displays, it might not be possible to see such phenomena these days. So, it is true that there are fewer opportunities to observe fans rotating backwards these days.

3. Fans in the Movies

However, I recently received the following question. Apparently some fans are seen rotating in a backwards direction, not under an electric light but in the daylight. This suggests that the stroboscopic effect is not the reason for the backwards rotation.

We can imagine two scenarios here. Were these fans rotating in a backwards direction observed under actual sun light, or were they seen on TV screens or in movies? In the latter scenario, since the movie films are not displayed continuously but are made up from a certain number of still images shown each second, a stroboscopic effect would result. Movies show 24 frames per second, while TVs show 30 frames per second.

There is a history behind this frequency of 24 frames per second in the movies. The projection speed of the movies was obtained empirically based on afterimages, which are a human visual characteristic. Initially, it was 10 frames per second. But this did not produce smooth images, and it was increased to 16 frames per second. The reason for the low number of frames was to minimize the consumption of film itself, which was very costly in those days. Furthermore, the film was flammable and higher speeds with more frames per second also increased the risk of fires. With the advent of talkies, when sound was recorded on the edge of the film, 16 frames per second produced jerky sounds. The speed was thus eventually increased to 24 frames per second, which is the world standard today.

What would happen to the images if fans were recorded with a 24 frame movie or by a 30 frame TV? As I explained above in relation to stroboscopic effects of fluorescent lamps and incandescent lamps, when we observe images at 24 or 30 frames per second, the same phenomenon occurs. Since movies and TVs use smaller frequencies than electric lights, it is easier to observe the virtual rotations. Many of us watched fans in old black-and-white movies, wheels of covered wagons in Westerns, helicopter blades, and automobile wheels in TV commercials -all rotating backwards.

4. Estimating fan Rotations

Let me try to explain this effect using mathematical formulae. Let the strobe light frequency be n_1 (Hz) and the fan frequency be n_2 (Hz). Both of these are periodic functions which can be expressed as Sine waves. Let the time be $t(0 \leq t \leq 1)$. Then we get:

$$y_1(t) = \sin 2\pi n_1 t \qquad (1)$$
$$y_2(t) = \sin 2\pi n_2 t \qquad (2)$$

At this time, the frequency of the observed virtual rotation n_3 (Hz) is related by $n_3 = n_2 - n_1$, which produces the following:

$$y_3(t) = \sin 2\pi(n_2 - n_1)t \qquad (3)$$

As an example, if we observe a blade rotating at 9 rotations/second under a strobe light blinking at 10 rotations/second, we will see a virtual rotation at 1 rotation/second. In numerical values, this would be $n_1 = 10, n_2 = 9, n_3 = 9 - 10 = -1$. This is illustrated in Figure 4.

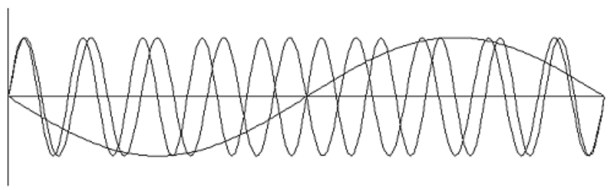

Figure 4: Relationship between a rotating object and a strobe light

Likewise, if a propeller rotating at a frequency of 100 Hz is illuminated under a 100 Hz light, the propeller should be observed as stationary. Furthermore, if 99 Hz is used for the illumination then the virtual rotation should move forwards slowly, in the same direction as the propeller at a speed of 1 Hz. A 101 Hz light would produce a slow virtual rotation moving backwards. Helicopter blades seem to rotate slowly in the movies if a multiple of the number of movie frames per second is almost the same as the number of revolutions of the propellers.

Let's place a fan in front of a TV screen and observe the fan. The TV receiver emits images at 30 Hz, *i.e.*, 30 intermittent images per second. If the blades rotate 1/3, 2/3, or 1 complete rotation after 1/30th second, the fan appears stationary. On a per second basis, such phenomena are observed when the blades are rotating $30/3 = 10$ times, $30 \times 2/3 = 20$ times, and $30 \times 1 = 30$ times per second.

Let the actual rotation speed be n_1 rotations/second, and the virtual rotation speed be n_2 rotations/second. Then, we get the following relationship:

$$n_2 = f(n_1) = n_1 - 10i \quad (10i - 5 \leq n_1 < 10i + 5, \quad i = 0, 1, \cdots, m)$$

As the linear function on the left side of Figure 5 shows, rotations proportional to the rotational speed of the fan should be observed. However, as the saw-tooth function on the right side shows, virtual rotations are observed within a maximum and minimum range of ± 5 rotations/second. $n_2 < 0$ produces backwards rotation, $n_2 = 0$ produces a stationary appearance, and $n_2 > 0$ produces forwards rotation. These are repeated.

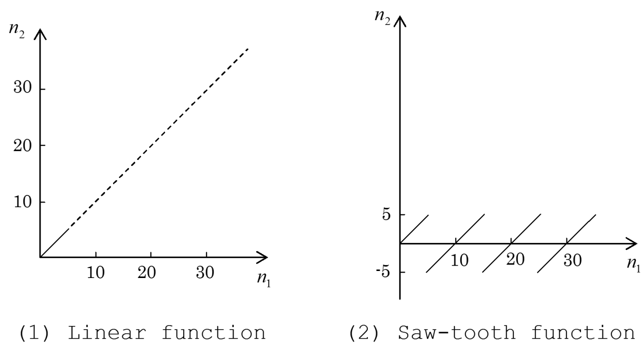

Figure 5: Numbers of actual and virtual rotations.

Further, although the fan we were considering has only three blades, it is also possible to observe the blades in multiples of 3, such as 6 blades or 9 blades, depending on the location of the blades (Figure 6). In fact, it is often possible to observe 12 thin blades. Armed with these observations, we can also estimate the actual rotational speed of the fan from the virtual rotations.

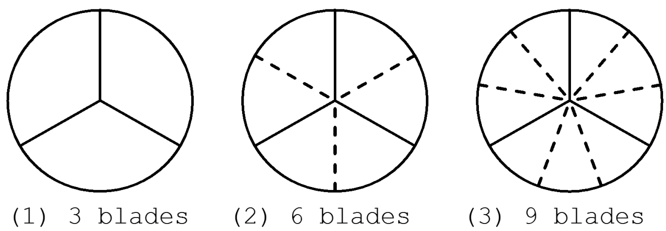

Figure 6: Generation of multiples of the number of blades.

5. Application of Stroboscopes

I have explained above that fans seem to rotate backwards due to stroboscopic effects. Stroboscopes make use of this basic principle. The following are some examples.

Figure 7 is an adjustment sheet for rotational speed, which was very popular during analog players' heyday. The younger generation of today, at a time when the CD players rule the world, may not be familiar with such devices. But in those days, the rotational speeds of players were 33 1/3 rotations/minute and 45 rotations/minute. Different models were available for the 50 Hz and 60 Hz areas. Unfortunately, I was unable to confirm this since I do not know what has happened to my own analog player, but under the old lamps, this pattern would have appeared stationary at the correct rotational speeds.

For industrial uses, there are non-contact speedometers. There were contact and non-contact speedometers, and the non-contact types utilized the stroboscopic effect.

The rotational speed of a revolving object could be measured without contact by attaching a reflective tape to the revolving object and illuminating it under a visible red LED light. According to the specification of one particular product, its accuracy is listed as $\pm 0.02\%$ for 1 - 99999 rotations/minute, which is fairly high. A 3V Xenon flash lamp is supposed to generate a strong intermittent light. Measurements are made by adjusting the flashing rate so that the subject of measurement appears to be stationary and then reading off the value indicating the rate at that moment. The subjects of measurement are revolving objects rotating at high speeds, such as motors.

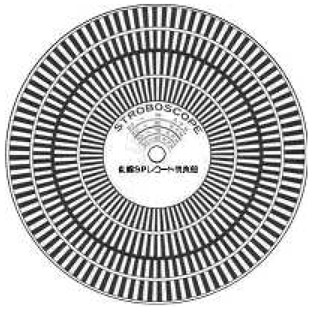

Figure 7: Rotational speed adjustment sheet

Stroboscopes are also used with medical equipment. The vocal cords vibrate between 100 - 300 Hz for voicing. When there are disorders with the vocal cords, precise diagnosis during vocal emission requires accurate frequency measurements. Such examination of diseased sections is possible visually if a CCD camera or other imaging device is attached to an endoscope. There are also devices that observe molecules moving at super-high speeds using stroboscopes with laser lights. Stroboscopic effects are thus used for many purposes, but the basic principle behind all of them is the phenomenon which makes fans look as if they are rotating in a backwards direction.

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