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SOME COMMON FIXED POINT THEOREMS FOR A CLASS OF A-CONTRACTIONS ON 2-METRIC SPACE

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Abstract: In this article we proved fixed point theorems for a class of contraction maps called A-contractions which include the contraction studied by many authors in the literature. We prove common fixed point theorems in 2-metric space using four self mappings satisfying weak compatibility and A-contractions.

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Key Words: 2-metric space, A-contractions, common fixed point, weakly compatible

1. Introduction and Preliminaries

The concept of 2-metric space is a natural generalization of the classical one of metric space. It has been investigated, initially by Gahler and has been developed extensively by Gahler and many other mathematicians([4]-[5]). M. Akram [1] defined A-contractions on metric space and proved some common fixed point theorems. G. Akinbo [2] generalizes the result using concept of weakly compatible mapping. Mantu Saha [3] also proved fixed point theorem on A-contraction in the setting of 2-metric space. Many other authors R. Bian-

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chini [7], M.S. Khan [8] also studied contraction type mappings. This article represents an appreciable generalization of the results of G. Akinbo [2].

Definition 1. Let X be a non-empty set. A real valued function d on $X \times X \times X$ is said to be a 2-metric on X if

1. given distinct elements x, y of X, there exists an element z of X such that $d(x, y, z) \neq 0$.
2. $d(x, y, z) = 0$ when at least two of x, y, z are equal.
3. $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$

for all x, y, z, w in X. When d is a 2-metric on X, then ordered pair (X, d) is called a 2-metric space.

Definition 2. A sequence $\{x_n\}$ in 2-metric space is said to be cauchy sequence if for each $a \in X$, $\lim d(x_n, x_m, a) = 0$ as $n, m \to \infty$.

Definition 3. A sequence $\{x_n\}$ in 2-metric space X is convergent to an element $x \in X$ if for each $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$

Definition 4. A complete 2-metric space is one in which every cauchy sequence in X converges to an element of X.

Definition 5. A 2-metric space X is said to be complete, if every cauchy sequence in X is convergent to an element of X.

On the other hand, Akram [1] defined A-contractions as follows:
Let a non-empty set A consisting of all functions $\alpha : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying

1. α is continuous on the set $\mathbb{R}_+^3$ of all triplet of non-negative reals (with respect to the Euclidean metric on $\mathbb{R}^3$.
2. $a \leq kb$ for some $k \in [0, 1]$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in \mathbb{R}_+$.

Definition 6. A self map T on a 2-metric space X is said to be A-contractions if for each $u \in X$,

$$d(Tx, Ty, u) \leq \alpha\left(d(x, y, u), d(x, Tx, u), d(y, Ty, u)\right)$$

holds for all $x, y \in X$ and $\alpha \in A$.

2. Main Result

Theorem 7. *Let F, G, S and T be self maps of 2-metric space X satisfying*

$$SX \subseteq FX, \qquad TX \subseteq GX \tag{1}$$

and for all $x, y, u \in X$, we have,

$$d(Sx, Ty, u) \leq \alpha\left[d(Gx, Fy, u), d(Gx, Sx, u), d(Fy, Ty, u\right] \tag{2}$$

Where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then the set $C(T, F)$ and $C(S, G)$ are non-empty, where $C(T, F)$ denotes the set of coincidence points of T and F.

Suppose further that (T, F) and (S, G) are weakly compatible, then F, G, S, T have a unique common fixed point.

Proof. Here $SX \subseteq FX$, $TX \subseteq GX$, then for any point $x_0 \in X$, we can find $x_1, x_2, x_3, \dots$ all in X, such that $Sx_0 = Fx_1$, $Tx_1 = Gx_2$, $Sx_2 = Fx_3$, $\dots\ \dots\ \dots$ Therefore by induction, we can define a sequence $\{y_n\}$ in X as,

$$y_n = \begin{cases} Sx_n = Fx_{n+1}, & \text{when } n \text{ is even} \\ Tx_n = Gx_{n+1}, & \text{when } n \text{ is odd} \end{cases}$$

assuming $n \in \mathbb{N}$ is even, then

$$\begin{aligned} d\left(y_n, y_{n+1}, u\right) &= d\left(Sx_n, Tx_{n+1}, u\right) \\ &\leq \alpha\left[d(Gx_n, Fx_{n+1}, u), d(Gx_n, Sx_n, u),\right. \\ &\quad \left. d(Fx_{n+1}, Tx_{n+1}, u)\right] \\ &= \alpha\left[d(y_{n-1}, y_n, u), d(y_{n-1}, y_n, u), d(y_n, y_{n+1}, u)\right] \\ \implies \qquad d(y_n, y_{n+1}, u) &\leq kd(y_{n-1}, y_n, u) \end{aligned}$$

On the other hand, assuming $n \in \mathbb{N}$ is odd

$$\begin{aligned} d(y_n, y_{n+1}, u) &= d(Tx_n, Sx_{n+1}, u) \\ &\leq \alpha\left[d(Gx_{n+1}, Fx_n, u), d(Gx_{n+1}, Sx_{n+1}, u),\right. \\ &\quad \left. d(Fx_n, Tx_n, u)\right] \\ &= \alpha\left[d(y_n, y_{n-1}, u), d(y_n, y_{n+1}, u), d(y_{n-1}, y_n, u)\right] \end{aligned}$$

this means,

$$d(y_n, y_{n+1}, u) \leq kd(y_{n-1}, y_n, u)$$

Thus whether n is even or odd, we have,

$$d(y_n, y_{n+1}, u) \leq kd(y_{n-1}, y_n, u)$$

for some $k \in [0,1]$

Inductively,

$$\begin{aligned} d(y_n, y_{n+1}, u) &\leq kd(y_{n-1}, y_n, u) \leq k^2 d(y_{n-2}, y_{n-1}, u) \\ &\leq \ldots \ldots \ldots \leq k^n d(y_0, y_1, u) \end{aligned}$$

That is

$$d(y_n, y_{n+1}, u) \leq k^n d(y_0, y_1, u) \tag{3}$$

for some $k \in [0,1]$.

Next,

$$\begin{aligned} d(y_n, y_{n+2}, u) &\leq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, u) + d(y_{n+1}, y_{n+2}, u) \\ &\leq d(y_n, y_{n+2}, y_{n+1}) + \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u) \end{aligned} \tag{4}$$

Now assuming $n \in N$ is odd:

$$\begin{aligned} d(y_n, y_{n+2}, y_{n+1}) &= d(y_{n+1}, y_{n+2}, y_n) \\ &= d(Sx_{n+1}, Tx_{n+2}, y_n) \\ &\leq \alpha\,[d(Gx_{n+1}, Fx_{n+2}, y_n), d(Gx_{n+1}, Sx_{n+1}, y_n), \\ &\quad d(Fx_{n+2}, Tx_{n+2}, y_n)] \\ &= \alpha d(y_n, y_{n+1}, y_n), d(y_n, y_{n+1}, y_n), \\ &\quad d(y_{n+1}, y_{n+2}, y_n)] \\ \Longrightarrow \qquad d(y_n, y_{n+2}, y_{n+1}) &\leq kd(y_n, y_{n+1}, y_n) \end{aligned}$$

for some $k \in [0,1]$ since $\alpha \in A$. So it follows that

$$d(y_n, y_{n+2}, y_{n+1}) = 0 \tag{5}$$

In the same way for $n \in N$, n is even:

$$d(y_n, y_{n+2}, y_{n+1}) = 0$$

Hence for $n \in N$, we have,

$$d(y_n, y_{n+2}, y_{n+1}) = 0 \tag{6}$$

So from (4), (5) and (6), we get,

$$d(y_n, y_{n+2}, u) \leq \sum_{r=0}^{1} d(y_{n+r}, y_{n+r+1}, u) \tag{7}$$

proceeding in the same manner, we get from any integer $p > 0$

$$d(y_n, y_{n+p}, u) \leq \sum_{r=0}^{p-1} d(y_{n+r}, y_{n+r+1}, u)$$

So by (3), we have for any integer $p > 0$

$$d(y_n, y_{n+p}, u) \leq \frac{k^n}{1-k} d(y_0, y_1, u) \to 0$$

as $n \to \infty$ since $k \in [0, 1]$ Hence $\{y_n\}$ is a cauchy sequence in X. Observe that $\{y_n\}$ is contained in $FX \cup GX$ which is complete, there exist a point $p \in FX \cup GX$ such that

$$\lim_{n\to\infty} y_n = p$$

without loss of generality let $p \in GX$. It means we can find a point $q \in X$ such that $p = Gq$. Putting $x = q$ and $y = x_m$, m odd in (2)

$$\begin{aligned} & d(Sq, Ty, u) \leq \alpha\left[d(Gq, Fx_m, u), d(Gq, Sq, u), d(Fx_m, Tx_m, u)\right] \\ i.e \quad & d(Sq, y_m, u) \leq \alpha\left[d(p, y_{m-1}, u), d(p, Sq, u), d(y_{m-1}, y_m, u)\right] \end{aligned}$$

letting $m \to \infty$, recalling that α is continuous on $\mathbb{R}^3_+$, we obtain,

$$\begin{aligned} & d(Sq, p, u) \leq \alpha\left[d(p, p, u), d(p, Sq, u), d(p, p, u)\right] \\ i.e. \quad & d(Sq, p, u) \leq \alpha\left[0, d(p, Sq, u), 0\right] \\ \Longrightarrow \quad & d(Sq, p, u) \leq k \cdot 0 = 0 \end{aligned}$$

Consequently $Sq = p$. From $SX \subseteq FX$, we know that there exists a point $v \in X$ such that

$$Fv = Sq = p = Gq$$

Choosing $x = q,\ y = v$ in(2) gives

$$d(p, Tv, u) \leq \alpha[0, 0, d(p, Tv, u)]$$

so that

$$d(p, Tv, u) \leq k \cdot 0 = 0$$

hence,

$$Fv = Tv = p = Sq = Gq$$

This proves the first part of theorem.

Now suppose (F, T) and (S, G) are weakly compatible pair, then F and T commute at v and G and S commute at q so that

$$Fp = F(Fv) = FTv = TFv = Tp$$

and

$$Sp = SSq = SGq = GSq = Gp \tag{8}$$

now with $x = p$, $y = v$,(2) and (7) yields

$$d(Sp, p, u) \leq \alpha\,[d(Sp, p, u), 0, 0]$$

$$\implies \quad d(Sp, p, u) \leq k \cdot 0 = 0$$

Therefore $p = Sp = Gp$.

In the similar way, letting $x = y = p$ (2) and (8) yields

$$p = Tp = Fp$$

Thus

$$Sp = Gp = p = Tp = Fp$$

Finally we show p is unique in X. Suppose p' is another common fixed point of the four maps then from (2),

$$x = p', \quad y = p$$

$$\implies \quad d(Sp', Tp, u) \leq \alpha\left[d(Gp', Fp, u), d(Gp', Sp', u), d(Fp, Tp, u)\right]$$

$$\implies \quad d(p', p, u) \leq \alpha\left[d(p', p, u), 0, 0\right]$$

$$\implies \quad d(p', p, u) \leq k \cdot 0 = 0$$

hence $p' = p$ and this completes the proof. □

Taking $F = G$ in the above theorem, we obtain the corollary.

Corollary 8. *Let F, S and T be self maps of 2-metric space X satisfying $SX \cup TX \subseteq FX$ and for all $x, y, u \in X$.*

$$d(Sx, Ty, u) \leq \alpha\,[d(Fx, Fy, u), d(Fx, Sx, u), d(Fy, Ty, u)]$$

where $\alpha \in A$. Suppose FX is a complete subspace of X then F, S and T have a coincidence point.

Suppose further that F commutes with both S and T at this coincidence point, then F, S and T have a unique common fixed point.

Choosing F to be identify map of X, in corollary 8 the following result follows immediately.

Corollary 9. *Let S and T be self maps of a complete 2-metric space X satisfying*

$$d(Sx, Ty, u) \leq \alpha\,[d(x, y, u), d(x, Sx, u), d(y, Ty, u)]$$

for all $x, y, u \in X$ where $\alpha \in A$. Then S and T have a unique common fixed point.

Theorem 10. *Let F, G, S, T be self maps of a 2-metric space X and let $\{S_n\}_{n=1}^{\infty}$ and $\{T_n\}_{n=1}^{\infty}$ be sequences on S and T satisfying,*

$$S_nX \subseteq FX,\ T_nX \subseteq GX, \quad n = 1, 2, \ldots \tag{9}$$

and for all $x, y, u \in X$

$$d(S_ix, T_jy, u) \leq \alpha\,[d(Gx, Fy, u), d(Gx, S_ix, u), d(F_y, T_jy, u)] \tag{10}$$

where $\alpha \in A$. Suppose $FX \cup GX$ is a complete subspace of X, then for each $n \in \mathbb{N}$,

1. *The sets $C(F, T_n)$ and $C(G, S_n)$ are non-empty. Further if T_n commutes with F and S_n commutes with G at their coincidence points, then*

2. *F, G, S_n and T_n have a unique common fixed point.*

Proof. For any arbitrary $x_0 \in X$ and $n = 0, 1, 2, \ldots$ following a similar argument as in theorem (1). We can define a sequence $\{y'_n\}$ in X as

$$y'_n = \begin{cases} S_nx_n = Fx_{n+1}, & \text{when } n \text{ is even} \\ T_nx_n = Gx_{n+1}, & \text{when } n \text{ is odd} \end{cases}$$

Now for each $i = 1, 3, 5, \ldots$ and $j = 2, 4, 6, \ldots$ from (10), we have

$$d(y'_i, y'_{i+1}, u) \leq kd(y'_{i-1}, y'_i, u)$$

and $$d(y'_j, y'_{j+1}, u) \leq kd(y'_{j-1}, y'_j, u)$$

i.e. $$d(y'_n, y'_{n+1}, u) \leq kd(y'_{n-1}, y'_n, u) \qquad n = 1, 2, 3, \ldots$$

By induction (as in the proof of theorem 1), we have,
$[d(y'_n, y'_{n+1}, u) \leq k^n d(y'_0, y'_1, u)$ for some $k \in [0, 1]$. Consequently $\{y'_n\}$ is cauchy in $FX \cup GX$, a complete subspace of X.

The rest of the proof is similar to the corresponding part of the proof of Theorem 7. ∎

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