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OPERATORS INDUCED BY STACKS ON A TOPOLOGICAL SPACE

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Abstract: The purpose of this paper is to introduce the notions of two operators φ_{S} and Ψ_{S} induced by a given stack $\mathcal{S}$ and a topology τ, and to investigate some properties of them. In particular, we study the collection $\tau\mathcal{S}$ of sets induced by the operator Ψ_{S}.

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1. Introduction

The concept of grill on a topological space was introduced by Choquet [2]: A *grill* $\mathcal{G}$ on X is a nonempty subset $\mathcal{G}$ of the power set $P(X)$ of X satisfying the following conditions:

(i) $\emptyset \notin \mathcal{G}$.

(ii) $A \subseteq B$, $A \in \mathcal{G}$ implies $B \in \mathcal{G}$.

(iii) $A, B \in \mathcal{G}$ implies $A \cup B \in \mathcal{G}$.

A nonempty family $\mathcal{S} \subseteq P(X)$ is called *a stack* [3,6] if it satisfies the above

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conditions (i) and (ii).

In 2007, Roy and Mukherjee [5] introduced an operator defined by a grill on a given topological space and showed that it satisfies Kuratowski's closure axioms [4]. They also investigated an associated topology induced by a grill on a given topological space. In 2008, Al-Omari and Noiri [1] introduced and investigated a new generalized open set $\widetilde{\Psi}_{\mathsf{G}}$ induced by the operation defined by a grill $\mathcal{G}$ on a given topological space.

In this paper, we are going to introduce another operator with the help of the stack on a given topology. It is obviously a generalized notion of the operator defined by a grill on a given topological space in [5]. First, we introduce the notion of operator φ_{S} defined by a given stack $\mathcal{S}$ and a topology τ, and investigate some basic properties. Second, we consider another operator Ψ_{S} defined by the operator φ_{S}, and study its properties and the collection $\tau\mathcal{S}$ of all subsets induced by the operator Ψ_{S}. In particular, we show that the collection $\tau\mathcal{S}$ satisfies the following: (i) $\emptyset, X \in \tau\mathcal{S}$; (ii) the union of any subfamily of $\tau\mathcal{S}$ is also in it.

2. Operator $\varphi_{\mathcal{S}}$

In this section, we introduce the notion of operator φ_{S} defined by a given stack $\mathcal{S}$ and a topology τ, and investigate some basic properties.

For a topological space (X, τ) and $x \in X$, we will denote $\tau(x)$ the collection of all open sets containing x.

Definition 2.1. Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. we define a mapping $\varphi_{\mathsf{S}} : P(X) \to P(X)$ as the following:

$$\varphi_{\mathsf{S}}(A) = \{x \in X : A \cap U \in \mathcal{S} \text{ for all } U \in \tau(x)\}.$$

Lemma 2.2. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A \subseteq X$,*
$x \notin \varphi_{\mathsf{S}}(A)$ iff there exists $U \in \tau(x)$ such that $U \cap A \notin \mathcal{S}$.

Theorem 2.3. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X.*
(i) $\varphi_{\mathsf{S}}(\emptyset) = \emptyset$.
(ii) $A \subseteq B \subseteq X$ implies $\varphi_{\mathsf{S}}(A) \subseteq \varphi_{\mathsf{S}}(B)$.
(iii) $\varphi_{\mathsf{S}}(A) \subseteq cl(A)$.
(iv) $\varphi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A)) \subseteq \varphi_{\mathsf{S}}(A)$.
(v) $\varphi_{\mathsf{S}}(A)$ is closed.

Proof. (i) Obvious.

(ii) It is easily obtained from the notion of stack.

(iii) Suppose that $x \notin cl(A)$; then there exists $U \in \tau(x)$ such that $U \cap A = \emptyset$. It implies $U \cap A \notin \mathcal{S}$ and so $x \notin \varphi_{\mathsf{S}}(A)$ by Lemma 2.2.

(iv) Let $x \in \varphi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A))$. Then for every $U \in \tau(x)$, $\varphi_{\mathsf{S}}(A) \cap U \in \mathcal{S}$. Since $\varphi_{\mathsf{S}}(A) \cap U \neq \emptyset$, there exists an element $z \in \varphi_{\mathsf{S}}(A) \cap U$. So $z \in \varphi_{\mathsf{S}}(A)$ and U is also an open set containing z. From the definition of operation φ_{S}, we have $A \cap U \in \mathcal{S}$, and so $x \in \varphi_{\mathsf{S}}(A)$.

(v) If $x \in cl(\varphi_{\mathsf{S}}(A))$, then for each $U \in \tau(x)$, $U \cap \varphi_{\mathsf{S}}(A) \neq \emptyset$ and $U \cap A \in \varphi_{\mathsf{S}}$. So $x \in \varphi_{\mathsf{S}}(A)$. □

The reverse inclusion of (vi) in Theorem 3.3 may not be hold as shown in the following example.

Example 2.4. Let $X = \{a, b, c, d, e\}$, a topology $\tau = \{\emptyset, \{a\}, \{b, c, d\}, \{a, b, c, d\}, X\}$ and a stack $\mathcal{S} = \{\{a, b, d\}, \{a, b, d, e\}, \{a, b, c, d\}, X\}$. For a set $A = \{a, b, d\}$, $\varphi_{\mathsf{S}}(A) = \{e\}$ and $\varphi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A)) = \emptyset$. So $\varphi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A)) \neq \varphi_{\mathsf{S}}(A)$.

Remark 2.5. In the above example, we know $\varphi_{\mathsf{S}}(X) = \{e\}$. So, it is not true $\varphi_{\mathsf{S}}(X) = X$ in general. Furthermore, for a set $A = \{a, b, d\}$, we have $\varphi_{\mathsf{S}}(A) = \{e\} \neq A$. Consequently, we know that there is no any inclusion relation between $\varphi_{\mathsf{S}}(A)$ and A.

Theorem 2.6. *If $U \cap A \notin \mathcal{S}$ for some $U \in \tau(x)$, then $U \cap \varphi_{\mathsf{S}}(A) \notin \mathcal{S}$ and in particular, $U \cap \varphi_{\mathsf{S}}(A) = \emptyset$.*

Proof. Let $U \in \tau(x)$ with $U \cap A \notin \mathcal{S}$. Assume that $U \cap \varphi_{\mathsf{S}}(A) \in \mathcal{S}$; then from $U \cap \varphi_{\mathsf{S}}(A) \in \mathcal{S}$, it follows $U \cap \varphi_{\mathsf{S}}(A) \neq \emptyset$ and there exists an element $y \in U \cap \varphi_{\mathsf{S}}(A)$. Since U is an open set containing y and $y \in \varphi_{\mathsf{S}}(A)$, from the definition of φ_{S}, $U \cap A \in \mathcal{S}$ and it is a contradiction. Consequently, $U \cap \varphi_{\mathsf{S}}(A) \notin \mathcal{S}$.

For the proof of the last statement, assume that $U \cap \varphi_{\mathsf{S}}(A) \neq \emptyset$ for some $U \in \tau(x)$. Then there exists an element $y \in U \cap \varphi_{\mathsf{S}}(A)$ and so $U \cap A \in \mathcal{S}$. So the proof of last statement is completed. □

Theorem 2.7. *For $A \subseteq X$, $\varphi_{\mathsf{S}}(A \cup \varphi_{\mathsf{S}}(A)) = \varphi_{\mathsf{S}}(A)$.*

Proof. From (ii) of Theorem 2.3, it is obviously $\varphi_{\mathsf{S}}(A) \subseteq \varphi_{\mathsf{S}}(A \cup \varphi_{\mathsf{S}}(A))$.

For the other inclusion, let $x \notin \varphi_{\mathsf{S}}(A)$; then there exists an open set U containing x such that $U \cap A \notin \mathcal{S}$. From Theorem 2.6, $U \cap \varphi_{\mathsf{S}}(A) = \emptyset$ and $U \cap (A \cup \varphi_{\mathsf{S}}(A)) = (U \cap A) \cup (U \cap \varphi_{\mathsf{S}}(A)) = U \cap A \notin \mathcal{S}$. So $x \notin \varphi_{\mathsf{S}}(A \cup \varphi_{\mathsf{S}}(A))$. □

Lemma 2.8. *Let (X,τ) be a topological space and $\mathcal{S}$ be a stack on X. If $U \in \tau$, then $U \cap \varphi_{\mathsf{S}}(A) = U \cap \varphi_{\mathsf{S}}(U \cap A)$.*

Proof. From (ii) of Theorem 2.3, it is obviously obtained that $U \cap \varphi_{\mathsf{S}}(U \cap A) \subseteq U \cap \varphi_{\mathsf{S}}(A)$.

For the other inclusion, let $x \in U \cap \varphi_{\mathsf{S}}(A)$ and V any open set containing x. Then $x \in U \cap V$ and $x \in \varphi_{\mathsf{S}}(A)$. So $(U \cap V) \cap A = (U \cap A) \cap V \in \mathcal{S}$. This implies $x \in \varphi_{\mathsf{S}}(U \cap A)$ and hence, $x \in U \cap \varphi_{\mathsf{S}}(U \cap A)$. □

Theorem 2.9. *Let (X,τ) be a topological space and $\mathcal{S}$ be a stack on X. If $\tau - \{\emptyset\} \subseteq \mathcal{S}$ and $U \in \tau$, then $U \subseteq \varphi_{\mathsf{S}}(U)$.*

Proof. By hypothesis, $\varphi_{\mathsf{S}}(X) = X$. From Lemma 2.8, it follows $U = U \cap \varphi_{\mathsf{S}}(X) = U \cap \varphi_{\mathsf{S}}(U \cap X) = U \cap \varphi_{\mathsf{S}}(U) \subseteq \varphi_{\mathsf{S}}(U)$. So it implies $U \subseteq \varphi_{\mathsf{S}}(U)$. □

Theorem 2.10. *Let (X,τ) be a topological space and $\mathcal{S}$ be a stack on X. If $\tau - \{\emptyset\} \subseteq \mathcal{S}$, then $cl(U) = \varphi_{\mathsf{S}}(U)$ for $U \in \tau$.*

Proof. Let $x \notin cl(U)$. Then there exists an open set G containing x such that $U \cap G = \emptyset$. By the notion of stack, $U \cap G \notin \mathcal{S}$. It implies $x \notin \varphi_{\mathsf{S}}(U)$ and $\varphi_{\mathsf{S}}(U) \subseteq cl(U)$. Finally, by Theorem 2.3 (v) and Theorem 2.9, we have $cl(U) = \varphi_{\mathsf{S}}(U)$. □

3. Operator $\Psi_{\mathcal{S}}$

In this section, we consider another operator Ψ_{S} defined by the operator φ_{S}, and study its properties and the collection $\tau\mathcal{S}$ of sets defined by the operator Ψ_{S}.

Definition 3.1. Let (X,τ) be a topological space and $\mathcal{S}$ be a stack on X. We define an operator $\Psi_{\mathsf{S}} : P(X) \to P(X)$ as the following: For $A \subseteq X$,

$$\Psi_{\mathsf{S}}(A) = A \cup \varphi_{\mathsf{S}}(A).$$

Theorem 3.2. *Let (X,τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A \subseteq X$,*

(i) $\Psi_{\mathsf{S}}(\emptyset) = \emptyset$.

(ii) $A \subseteq \Psi_{\mathsf{S}}(A)$; moreover $\Psi_{\mathsf{S}}(X) = X$.

(iii) $A \subseteq B \subseteq X$ implies $\Psi_{\mathsf{S}}(A) \subseteq \Psi_{\mathsf{S}}(B)$.

(iv) $\Psi_{\mathsf{S}}(\Psi_{\mathsf{S}}(A)) = \Psi_{\mathsf{S}}(A)$.

(v) For $A, B \subseteq X$, $\Psi_{\mathsf{S}}(A \cap B) \subseteq \Psi_{\mathsf{S}}(A) \cap \Psi_{\mathsf{S}}(B)$.

Proof. (i) Since $\varphi_{\mathsf{S}}(\emptyset) = \emptyset$, $\Psi_{\mathsf{S}}(\emptyset) = \emptyset \cup \varphi_{\mathsf{S}}(\emptyset) = \emptyset$.

(ii) Obvious.

(iii) For $A \subseteq B \subseteq X$, it is obtained by (i) of Theorem 2.3.

(iv) From Theorem 2.7, it follows

$$\begin{aligned}
\Psi_{\mathsf{S}}(\Psi_{\mathsf{S}}(A)) &= \Psi_{\mathsf{S}}(A \cup \varphi_{\mathsf{S}}(A)) \\
&= (A \cup \varphi_{\mathsf{S}}(A)) \cup \varphi_{\mathsf{S}}(A \cup \varphi_{\mathsf{S}}(A)) \\
&= A \cup \varphi_{\mathsf{S}}(A) \cup \varphi_{\mathsf{S}}(A) \\
&= A \cup \varphi_{\mathsf{S}}(A) \\
&= \Psi_{\mathsf{S}}(A)
\end{aligned}$$

(v) Obvious. □

The equality in (v) of Theorem 3.3 is not true in general.

Example 3.3. In Example 2.4, consider $A = \{a, b, d\}$ and $B = \{d, e\}$. Note that $\varphi_{\mathsf{S}}(A) = \{e\}$, $\varphi_{\mathsf{S}}(B) = \emptyset$ and $\varphi_{\mathsf{S}}(A \cap B) = \varphi_{\mathsf{S}}(\{d\}) = \emptyset$. So $\Psi_{\mathsf{S}}(A) = \{a, b, d, e\}$ and $\Psi_{\mathsf{S}}(B) = B$. From these facts, $\Psi_{\mathsf{S}}(A \cap B) = \{d\}$; $\Psi_{\mathsf{S}}(A) \cap \Psi_{\mathsf{S}}(B) = \{d, e\}$. So $\Psi_{\mathsf{S}}(A \cap B) \neq \Psi_{\mathsf{S}}(A) \cap \Psi_{\mathsf{S}}(B)$.

Definition 3.4. Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. we define a mapping $\Psi_{\mathsf{S}} : P(X) \to P(X)$ as the following:

$$\tau\mathcal{S} = \{U \subseteq X : \Psi_{\mathsf{S}}(X - U) = X - U\}.$$

Theorem 3.5. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. Then*

(i) $\emptyset, X \in \tau\mathcal{S}$.

(ii) If $U_\alpha \in \tau\mathcal{S}$ for $\alpha \in J$, then $\cup U_\alpha \in \tau\mathcal{S}$.

Proof. (i) Since $\Psi_{\mathsf{S}}(X) = X$ and $\Psi_{\mathsf{S}}(\emptyset) = \emptyset$, both $\emptyset$ and X are in $\tau\mathcal{S}$.

(ii) Let $U_\alpha \in \tau\mathcal{S}$ for $\alpha \in J$. Then from $\varphi_{\mathsf{S}}(X - \cup U_\alpha) \subseteq \varphi_{\mathsf{S}}(X - U_\alpha)$ and $U_\alpha \in \tau\mathcal{S}$, we have $\varphi_{\mathsf{S}}(X - \cup U_\alpha) \subseteq \varphi_{\mathsf{S}}(X - U_\alpha) \subseteq X - U_\alpha$ and $\varphi_{\mathsf{S}}(X - \cup U_\alpha) \subseteq \cap(X - U_\alpha) = X - \cup U_\alpha$. So $\Psi_{\mathsf{S}}(X - \cup U_\alpha) = (X - \cup U_\alpha) \cup \varphi_{\mathsf{S}}(X - \cup U_\alpha) = X - \cup U_\alpha$, and hence $\cup U_\alpha \in \tau\mathcal{S}$. □

For two elements of $\tau\mathcal{S}$, the intersection may not be an element of $\tau\mathcal{S}$ as shown in the next example.

Example 3.6. In Example 2.4, consider $U_1 = \{d, e\}$ and $U_2 = \{a, e\}$. Note that $\varphi_{\mathsf{S}}(X - U_1) = \varphi_{\mathsf{S}}(\{a, b, c\}) = \emptyset$ and $\varphi_{\mathsf{S}}(X - U_2) = \varphi_{\mathsf{S}}(\{b, c, d\}) = \emptyset$. It implies $\Psi_{\mathsf{S}}(X - U_1) = X - U_1$ and $\Psi_{\mathsf{S}}(X - U_2) = X - U_2$, i.e. $U_1, U_2 \in \tau\mathcal{S}$. But for $U_1 \cap U_2 = \{e\}$, $\varphi_{\mathsf{S}}(X - (U_1 \cap U_2)) = \varphi_{\mathsf{S}}(\{a, b, c, d\}) = \{e\}$ and $\Psi_{\mathsf{S}}(X - (U_1 \cap U_2)) = X$. Since $\Psi_{\mathsf{S}}(X - (U_1 \cap U_2)) \neq X - (U_1 \cap U_2)$, we have $U_1 \cap U_2 \notin \tau\mathcal{S}$.

Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. Then the elements of τ_{S} are said to be τ_{S}-*open*. If the complement of a subset of X is τ_{S}-open, then the subset is said to be τ_{S}-*closed*.

Now we define the operators $i_{\tau\mathsf{S}}, c_{\tau\mathsf{S}} : P(X) \to P(X)$ as the following: For $A \subseteq X$,

$$i_{\tau\mathsf{S}}(A) = \cup\{U \subseteq X : U \subseteq A, U \in \tau\mathcal{S}\};$$

$$c_{\tau\mathsf{S}}(A) = \cap\{F \subseteq X : A \subseteq F, X - F \in \tau\mathcal{S}\} :$$

The following two theorems are obviously obtained from the definitions of operations $i_{\tau\mathsf{S}}, c_{\tau\mathsf{S}}$:

Theorem 3.7. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A, B \subseteq X$*

(i) $i_{\tau\mathsf{S}}(\emptyset) = \emptyset$.

(ii) $i_{\tau\mathsf{S}}(A) \subseteq A$.

(iii) If $A \subseteq B$, then $i_{\tau\mathsf{S}}(A) \subseteq i_{\tau\mathsf{S}}(B)$.

(iv) $i_{\tau\mathsf{S}}(i_{\tau\mathsf{S}}(A)) = i_{\tau\mathsf{S}}(A)$.

Theorem 3.8. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A, B \subseteq X$*

(i) $c_{\tau\mathsf{S}}(X) = X$.

(ii) $A \subseteq c_{\tau\mathsf{S}}(A)$.

(iii) If $A \subseteq B$, then $c_{\tau\mathsf{S}}(A) \subseteq c_{\tau\mathsf{S}}(B)$.

(iv) $c_{\tau\mathsf{S}}(c_{\tau\mathsf{S}}(A)) = c_{\tau\mathsf{S}}(A)$.

Theorem 3.9. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A \subseteq X$, A is $\tau\mathcal{S}$-closed iff $\Psi_{\mathsf{S}}(A) = A$.*

Proof. A is $\tau\mathcal{S}$-closed iff $X - A$ is $\tau\mathcal{S}$-open iff $\Psi_{\mathsf{S}}(X - (X - A)) = X - (X - A)$. So we have the statement. □

Theorem 3.10. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For $A \subseteq X$,*

(i) $x \in i_{\tau\mathsf{S}}(A)$ iff there exists a $\tau\mathcal{S}$-open set U containing x such that $U \subseteq A$.

(ii) $x \in c_{\tau\mathsf{S}}(A)$ iff for each $\tau\mathcal{S}$-open set V containing x, $A \cap V \neq \emptyset$.

Proof. Obvious. □

Theorem 3.11. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. If $x \in i_{\tau\mathsf{S}}(A)$, then there exists some $W \in \tau(x)$ satisfying $A^c \cap W \notin \mathcal{S}$.*

Proof. For $x \in i_{\tau\mathsf{S}}(A)$, by Theorem 3.10, there exists a $\tau\mathcal{S}$-open set U containing x such that $U \subseteq A$. From $X - A \subseteq X - U = \Psi_{\mathsf{S}}(X - U)$, we have $x \notin \varphi_{\mathsf{S}}(X - U)$, and so there exists an open set W containing x such that $(X - U) \cap W \notin \mathcal{S}$. Since $\mathcal{S}$ is a stack and $A^c \subseteq U^c$, we have $A^c \cap W \notin \mathcal{S}$. □

Theorem 3.12. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X.*
(i) $c_{\tau\mathsf{S}}(A) = \Psi_{\mathsf{S}}(A)$ for $A \subseteq X$.
(ii) If $A \notin \mathcal{S}$, then $X - A \in \tau\mathcal{S}$ for $A \subseteq X$.
(iii) $\varphi_{\mathsf{S}}(A)$ is $\tau\mathcal{S}$-closed for $A \subseteq X$.

Proof. (i) First, by Theorem 3.2 (iv) and Theorem 3.9, $\Psi_{\mathsf{S}}(A)$ is $\tau\mathcal{S}$-closed. From $A \subseteq \Psi_{\mathsf{S}}(A)$, it follows $A \subseteq c_{\tau\mathsf{S}}(A) \subseteq \Psi_{\mathsf{S}}(A)$.
Furthermore, since $A \subseteq c_{\tau\mathsf{S}}(A)$, from Theorem 3.2(iii) and Theorem 3.9, it follows $\Psi_{\mathsf{S}}(A) \subseteq \Psi_{\mathsf{S}}(c_{\tau\mathsf{S}}(A)) = c_{\tau\mathsf{S}}(A)$. Consequently, $c_{\tau\mathsf{S}}(A) = \Psi_{\mathsf{S}}(A)$.

(ii) If $A \notin \mathcal{S}$, then by Lemma 2.2, we know that $\varphi_{\mathsf{S}}(A) = \emptyset$. So $\Psi_{\mathsf{S}}(X - (X - A)) = \Psi_{\mathsf{S}}(A) = A \cup \varphi_{\mathsf{S}}(A) = A = X - (X - A)$. So $X - A \in \tau\mathcal{S}$.

(iii) For $A \subseteq X$, from Theorem 2.3 (iv), $\Psi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A)) = \varphi_{\mathsf{S}}(A) \cup \varphi_{\mathsf{S}}(\varphi_{\mathsf{S}}(A)) = \varphi_{\mathsf{S}}(A)$. By Theorem 3.9, $\varphi_{\mathsf{S}}(A)$ is $\tau\mathcal{S}$-closed □

Theorem 3.13. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. Then for $U \in \tau$ and $A \notin \mathcal{S}$,*

$$U - A \in \tau\mathcal{S}.$$

Proof. First, we show that $\varphi_{\mathsf{S}}(U^c \cup A) \subseteq U^c \cup A$ for $U \in \tau$ and $A \notin \mathcal{S}$. Assume $x \in \varphi_{\mathsf{S}}(U^c \cup A)$; then for every $G \in \tau(x)$, $G \cap (U^c \cup A) = (G \cap U^c) \cup (G \cap A) \in \mathcal{S}$. If there exists an open set $G' \in \tau(x)$ such that $G' \cap U^c = \emptyset$, since $G' \cap A \in \mathcal{S}$ and $\mathcal{S}$ is a stack, we have $A \in \mathcal{S}$. It contradicts to the fact $A \notin \mathcal{S}$. So $G \cap U^c \neq \emptyset$ for every $G \in \tau(x)$ and this implies $x \in cl(U^c) = U^c \subseteq U^c \cup A$. Hence $\varphi_{\mathsf{S}}(U^c \cup A) \subseteq U^c \cup A$ and from the fact, it follows $\Psi_{\mathsf{S}}(X - (U - A)) = X - (U - A) \cup \varphi_{\mathsf{S}}(X - (U - A)) = X - (U - A)$. Hence $U - A \in \tau\mathcal{S}$. □

Theorem 3.14. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. For any $W \in \tau\mathcal{S}$, $W = \cup(U - A)$ for $U \in \tau$ and $A \notin \mathcal{S}$.*

Proof. For any $W \in \tau\mathcal{S}$, let $x \in W$. Then $\Psi_{\mathsf{S}}(X - W) = (X - W) \cup \varphi_{\mathsf{S}}(X - W) = X - W$ and $x \notin X - W$. So $x \notin \varphi_{\mathsf{S}}(X - W)$ and there exists some $U \in \tau(x)$ such that $(X - W) \cap U \notin \mathcal{S}$. Put $A = (X - W) \cap U$. Then $x \notin A$ and $A \notin \mathcal{S}$. Moreover, we have $x \in U - A \subseteq W$. So the proof is completed. □

Theorem 3.15. *Let (X, τ) be a topological space and $\mathcal{S}$ be a stack on X. Then $\tau \subseteq \tau\mathcal{S}$.*

Proof. It is obvious from Theorem 3.14 and $\emptyset \notin \mathcal{S}$. □

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