International Journal of Pure and Applied Mathematics
Volume 78 No. 7 2012, 1005-1009
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

AN AK TIME TO BUILD GROWTH MODEL

Luca Guerrini
Department of Mathematics for Economic and Social Sciences
University of Bologna
5, Viale Filopanti 5, Bologna, 40126, ITALY

Abstract: We investigate how the choice of time delay may change the dynamics of the standard AK Solow-Swan model. In particular, the condition under which an Hopf bifurcations occurs is obtained.

AMS Subject Classification: 34K18, 91B62
Key Words: AK model, time delay, Hopf bifurcation

1. Introduction

Kalecki [16] was the first to formally examine the conjecture that production lags cause cycles in output. Whether lags induce cycles in the Solow-Swan model was investigated by Asea and Zak [1]. However, no analysis was done for the so-called AK Solow-Swan models (Romer [18]; Rebelo [17], Jones and Manuelli [15]). In these models, according to the value of savings rate adjusted by the level of technology relative to the depreciation rate of physical capital, we have that economic growth can be either equal to zero or negative or else positive. The purpose of this paper is to analyze the AK model with lag between investment and production (time-to-build). With this setup, the stability of the model is investigated. In particular, it is proved that a Hopf bifurcation occurs as the delay increases. Unlike the standard AK Solow-Swan model, the per-capita capital stock does not monotonically converge over time toward zero when the value of savings rate adjusted by the level of technology relative is less than the depreciation rate of physical capital. For future research, we aim to investigate the consequence of introducing a time lag in other economic growth

Received: February 26, 2012

models (see, e.g., Ferrara and Guerrini [4-6], Guerrini [7-14]).

2. The Model

We consider the Solow-Swan model with AK technology [2] in which there is a delay of $T > 0$ periods before capital can be used for production. At time t, the productive capital stock is a function of the productive capital stock at time $t - T$. By assuming that a proportion of the capital stock depreciates during production and a constant population normalized to unity, the resulting delay model of economic growth model becomes

$$\dot{k}_t = (sA - \delta)\, k_{t-T}, \quad k_t = \phi_t,\ t \in [-T, 0], \tag{1}$$

where k_t is per capita capital stock, $s \in (0, 1)$ is the saving rate, $\delta > 0$ is capital depreciation and $A > 0$ is a parameter reflecting the level of the technology. Notice that instead of an initial point value for an ordinary differential equation, the initial function ϕ_t is required, which is defined over the range of time delimited by the delay.

Remark 1. In case there is no delay, i.e. $T = 0$, then it immediate from Eq. (1) that the per-capita capital stock remains constant over time at its initial level if $sA = \delta$, it monotonically converges over time toward zero if $sA < \delta$, and it monotonically diverges over time at infinity if $sA > \delta$.

3. Stability and Hopf Bifurcation

Eq. (1) has the equilibrium point $k_* = k_0$ if $sA - \delta \neq 0$, whose stability depends on the location of the roots of the associated characteristic equation:

$$\lambda - (sA - \delta)\, e^{-\lambda T} = 0, \tag{2}$$

where λ is a complex number. It is known that the stability analysis of all solutions of (1) concerns with whether all roots lie in the left half of the complex plane (see, e.g., Bellman and Cooke [3]). Therefore, we have that the trivial solution is asymptotically stable if $Re\lambda < 0$ for all the eigenvalues λ and it is unstable if there is a root with positive real part. A change in stability can occur only when a root of Eq. (2) crosses the imaginary axis.

Theorem 1.

1. *Let $sA > \delta$. Then Eq.* (1) *is unstable for all positive delay T.*
2. *Let $sA < \delta$. Then Eq.* (1) *is uniformly asymptotically stable when $T < \pi/2\omega$ and unstable when $T > \pi/2\omega$, where $\omega = \delta - sA$.*
3. *Let $sA = \delta$. Then Eq.* (1) *is always stable.*

Proof. Let $sA = \delta$. In this case, Eq. (1) becomes $\dot{k}_t = 0$ and Eq. (2) has $\lambda = 0$ as its only root. Hence the solution is the constant k_0. Assume $sA - \delta \neq 0$. We seek conditions on T such that $Re\lambda$ changes from negative to positive. Since $\lambda = 0$ is not a root of the characteristic equation, we can assume that $\lambda = i\omega$, with $\omega > 0$, is a root for $T = T_0 \geq 0$. Substituting into Eq. (2), we obtain

$$-(sA - \delta)\cos\omega T = 0, \quad \omega + (sA - \delta)\sin\omega T = 0. \tag{3}$$

The sum of the squares of these two equations yields $\omega^2 = (sA - \delta)^2$, implying $\omega = |sA - \delta| > 0$. Hence, purely imaginary roots of Eq. (2) exist and are simple. Differentiating Eq. (2) with respect to λ, and using (2), we get

$$\frac{d\lambda}{dT} = -\frac{\lambda^2}{1 + T\lambda}.$$

Thus,

$$sign\left[\frac{d\,(Re\lambda)}{dT}\right]_{\lambda=i\omega} = sign\left[Re\left(\frac{d\lambda}{dT}\right)^{-1}\right]_{\lambda=i\omega} = \frac{1}{\omega^2} > 0. \tag{4}$$

This implies that all the roots that cross the imaginary axis at $i\omega$ cross from left to right as T increases and thus results in the loss of stability. Let $sA > \delta$. If $T = 0$, then Eq. (2) yields $\lambda_0 = sA - \delta > 0$, i.e. the trivial solution of Eq. (1) is unstable when there is no delay. By (4) it will remain unstable for all $T > 0$. Let $sA < \delta$. In this case, $\lambda_0 = sA - \delta < 0$, i.e. Eq. (1) is asymptotically stable when there is no delay. From (4) we have

$$\cos\omega T = 0, \quad \sin\omega T = \frac{\omega}{\delta - sA} = 1. \tag{5}$$

Hence, there is a unique θ, $0 < \theta \leq 2\pi$ such that $\omega T = \theta$ makes (5) to hold. We have $\theta = \pi/2$, so that $T_0 = \pi/2\omega$. The preceding arguments show that the trivial solution of Eq. (1) is asymptotically stable when $0 < T < T_0$, while it is unstable when $T > T_0$. □

Corollary 2. *Eq.* (1) *undergoes a Hopf bifurcation when $sA < \delta$ and $T = \pi/2\omega$.*

References

[1] P.K. Asea, P. Zak, Time-to-build and cycles, *Journal of Economic Dynamics and Control*, **23** (1999), 1155-1175.

[2] R.J. Barro, X. Sala-i-Martin, *Economic Growth*, McGraw-Hill (1995).

[3] R. Bellman, K.L. Cooke, *Differential-Difference Equations*, Academic Press (1963).

[4] M. Ferrara, L. Guerrini, Economic development and sustainability in a Solow model with natural capital and logistic population change, *International Journal of Pure and Applied Mathematics*, **48** (2008), 435-450.

[5] M. Ferrara, L. Guerrini, The Ramsey model with logistic population growth and benthamite felicity function revisited, *WSEAS Transactions on Mathematics*, **8** (2009), 97-106.

[6] M. Ferrara, L. Guerrini, A note on the Uzawa-Lucas model with unskilled labor, *Applied Sciences*, **12** (2010), 90-95.

[7] L. Guerrini, The Solow-Swan model with AK technology and bounded population growth rate, *International Journal of Pure and Applied Mathematics*, **60** (2010), 211-215.

[8] L. Guerrini, The dynamic of the AK Ramsey growth model with quadratic utility and logistic population change, *International Journal of Pure and Applied Mathematics*, **62** (2010), 221-225.

[9] L. Guerrini, The Ramsey model with AK technology and a bounded population growth rate, *Journal of Macroeconomics*, **32** (2010), 1178-1183.

[10] L. Guerrini, A note on the Ramsey growth model with the von Bertalanffy population law, *Applied Mathematical Sciences*, **4** (2010), 3233-3238.

[11] L. Guerrini, The AK Ramsey growth model with the von Bertalanffy population law, *Applied Mathematical Sciences*, **4** (2010), 3245-3249.

[12] L. Guerrini, The AK Ramsey model with von bertalanffy population law and Benthamite function, *Far East Journal of Mathematical Sciences*, **45** (2010), 187-192.

[13] L. Guerrini, Kaldor and Classical savings in the Solow-Swan model with a bounded population growth rate, *International Journal of Pure and Applied Mathematics*, **60** (2010), 193-199.

[14] L. Guerrini, Logistic population change and the Mankiw-Romer-Weil model, *Applied Sciences*, **12** (2010), 96-101.

[15] L.E. Jones, R.E. Manuelli, A convex model of equilibrium growth: theory and policy implications, *Journal of Political Economy*, **98** (1990), 1008-1038.

[16] M. Kalecki, A macroeconomic theory of business cycles, *Econometrica*, **3** (1935), 327-344.

[17] S. Rebelo, Long-run policy analysis and long-run growth, *Journal of Political Economy*, **99** (1991), 500-521.

[18] P.M. Romer, Increasing returns and long-run growth, *Journal of Political Economy*, **94** (1986), 1002-1037.