International Journal of Pure and Applied Mathematics
Volume 78 No. 8 2012, 1185-1191
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

SMALL SETS AND POSTULATION ON WEIGHTED PROJECTIVE SPACES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Let $\mathcal{L}$ be a line bundle on a weighted projective space $\mathbb{P}(q_0, \dots, q_r)$. Here we classify the sets $S \subset \mathbb{P}(q_0, \dots, q_r)$ with $h^1(\mathbb{P}(q_0, \dots, q_r), \mathcal{I}_S \otimes \mathcal{L}) > 0$ and small cardinality.

AMS Subject Classification: 14M25, 14N05
Key Words: weighted projective spaces

1. Introduction

Fix positive integers r, q_i, $0 \le i \le r$. We assume $0 < q_0 \le q_1 \le \cdots \le q_r$. Let $\mathbb{P}(q_0, \dots, q_r)$ denote the weighted projective space with weights $q_0, \dots, q_r$ (see [5], [6], [1]). We work over an algebraically closed base field $\mathbb{K}$. We always assume that either $\mathrm{char}(\mathbb{K}) = 0$ or that no q_i is divided by $\mathrm{char}(\mathbb{K})$. This assumption is essential to see $\mathbb{P}(q_0, \dots, q_r)$ as a quotient of $\mathbb{P}^r$ by the action of a finite abelian group with order not divided by $\mathrm{chark}(\mathbb{K})$ and hence to use many foundational results lifted from (see [6], [1]). In Theorem 1 we assume $\mathrm{char}(\mathbb{K}) = 0$, because it uses [7], Proposition 1.27.

From now on we assume that the weights $(q_0, \dots, q_r)$ are reduced, i.e. no prime divides r of them. Every weighted projective space is isomorphic (as an

Received: May 26, 2012

algebraic variety) to a weighted projective space with reduced weights (see [5], Proposition 1.3, [6] §1.3, [1], Proposition 3C.5). With this assumption for each $t \in \mathbb{Z}$ there is a unique rank 1 reflexive sheaf $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)$ and $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)$ and $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(y)$ are isomorphic if and only if $t = y$. Let m be the minimal common multiple of the integers $q_0, \dots, q_r$. Each rank 1 reflexive sheaf $\mathcal{F}$ on $\mathbb{P}(q_0, \dots, q_r)$ is isomorphic to $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)$ for a unique $t \in \mathbb{Z}$ (see [1], Theorem 7.1 (a), (b)). The sheaf $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)$ is locally free (i.e. it is a line bundle) if and only if $t \equiv 0 \pmod{m}$ (see [1], Theorem 7.1 (c)). The line bundle $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(m)$ is ample (see [5], Proposition 2.3 (a)). We fix weighted homogeneous coordinates $x_0, \dots, x_r$ on $\mathbb{P}(q_0, \dots, q_r)$ with weight $w(x_i) = q_i$ for all i. For each integer $t \geq 0$ let S_t denote the set of all $f \in \mathbb{K}[x_0, \dots, x_r]$ which are weighted homogeneous of weight degree t. Set $S_t := \{0\}$ if $t < 0$. For each $t \in \mathbb{Z}$ we have $H^i(\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)) \cong S_t$, $H^i(\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)) = 0$ if $1 \leq i \leq r-1$ and $H^r(\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(t)) \cong S_{-t-q_0-\dots-q_r}$ (see [5], §3, [6], §1.4). Hence $\mathcal{O}_{\mathbb{P}(q_0,\dots,q_r)}(m)$ is spanned. This line bundle is very ample , at least if $\mathrm{char}(\mathbb{K}) = 0$ (see [7], Proposition 1.27). Let $\phi_m : \mathbb{P}(q_0, \dots, q_r) \to \mathbb{P}^N$, $N := h^0(\mathbb{P}(q_0, \dots, q_r), \mathcal{I}_S(m)) - 1$, denote the embedding induced by the complete linear system $|\mathcal{I}_S(m)|$. Let $\alpha(q_0, \dots, q_r)$, denote the minimal integer t such that $\phi_m(\mathbb{P}(q_0, \dots, q_r)) \subset \mathbb{P}^N$ is set-theoretically cut out by hypersurfaces of degree $\leq t$.

Theorem 1. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix an integer* $t \geq \alpha(q_0, ..., q_r) - 1$. *Let* $Z \subset \mathbb{P}(q_0, \dots, q_r)$ *be a zero-dimensional subscheme such that* $\deg(Z) \leq 2t + 1$.

We have $h^1(\mathbb{P}(q_0, \dots, q_r), \mathcal{I}_S(tm)) > 0$ *if and only if there is a curve* $C \subset \mathbb{P}(q_0, \dots, q_r)$ *such that* $\deg(\mathcal{O}_C(m)) = 1$ *and* $\deg(Z \cap C) \geq t + 2$.

In the case $r = 2$ a miracle occurs (under our default assumption that the weights are normalized): $\mathcal{O}_{\mathbb{P}(q_0,q_1,q_2)}(m)$ is projectively normal (see [1], Remark 3 at page 138) and hence it is very ample. Since the weights are normalized, we have $(q_i, q_j) = 1$ for all $i \neq j$. Hence $m = q_0q_1q_2$. If $(q_0, q_1, q_2) \neq (1, 1, 1)$, then $q_2 > q_1$ and either $q_1 > q_0$ or $q_0 = q_0 = 1$

Theorem 2. *Assume* $r = 2$ *and* $q_0 > 1$. *Fix an integer* $t \geq 1$ *and a finite set* $S \subset \mathbb{P}(q_0, q_1, q_2)$. *Set* $E := \{x_0 = 0\} \subset \mathbb{P}(q_0, q_1, q_2)$.

(a) Assume $t \geq 2$ *and* $\sharp(S) \leq tq_1 + 1$. *We have* $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_S(tm)) > 0$ *if and only if* $\sharp(S \cap E) \geq tq_0 + 2$.

(b) Assume $t = 1$.

(ii) Assume $q_1 = 2$ *and* $\sharp(S) < 9$. *We have* $h^1(\mathcal{I}_S(m)) > 0$ *if and only if* $\sharp(S \cap E) \geq 6$.

(iii) Assume $q_1 > 2$ *and* $\sharp(S) < 9$. *We have* $h^1(\mathcal{I}_S(m)) = 0$.

See Proposition 1 and 2 for the case $q_0 = 1$.

2. The Proofs

Remark 1. Fix any projective scheme W, any coherent sheaf $\mathcal{L}$ on W and zero-dimensional schemes $Z' \subset Z \subset W$. If $h^1(W, \mathcal{I}_{Z'} \otimes \mathcal{L}) > 0$, then $h^1(W, \mathcal{I}_Z \otimes \mathcal{L}) > 0$.

Proof of Theorem 2. Since $q_0 > 1$ and $(q_i, q_j) = 1$ for all $i \neq j$, we have $q_2 > q_1 > q_0$. We have $E \cong \mathbb{P}^1$, because any weighted projective line is isomorphic to $\mathbb{P}^1$ as an abstract variety. Let $S_{tm}|E$ denote the restriction of S_{tm} to E. Since $(q_0, q_1) = 1$, we may identify $S_{tm}|E$ with the set of all weight homogeneous $g \in \mathbb{K}[x_1, x_2]$ with weight $tm = tq_0q_1q_2$. Take an exponent (i_1, i_2) with weight $tq_0q_1q_2$, i.e. with $i_1q_1 + i_2q_2 = tq_0q_1q_2$. Since $(q_1, q_2) = 1$, there are $j_1, j_2 \in \mathbb{N}$ such that $i_1 = j_1q_2$, $i_2 = j_2q_1$ and $j_1 + j_2 = tq_0$. Hence $\dim(S_{tm}|E) = tq_0 + 1$. Since $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_E(tm)) = 0$, we also get that $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_{S \cap E}(tm)) > 0$ if and only if $\sharp(S \cap E) \geq tq_0 + 2$. Now assume $k := \sharp(S \cap E) \leq tq_0 + 1$. Set $F := \{x_1 = 0\}$. As above we see that for any finite set $A \subset F$ we have $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_A(tm)) = 0$ if and only if $\sharp(A) \geq tq_1 + 2$.

(a) For any $a \in \mathbb{K}$ set $C_a := \{x_1^{q_0} + ax_0^{q_1} = 0\}$. If $a \neq 0$, then C_a is an effective Weil divisor with weighted degree q_0q_1. C_0 is just F counted with multiplicity q_1. We have $C_a \cap C_b = (0, 0, 1)$ for all $a \neq b$. The curve C_a is not a Cartier divisor of $\mathbb{P}(q_0, q_1, q_2)$, but for all, except at most the first one, we look at sets $S_i \cap C_a$ contained in the subset of C_a in which it is a Cartier divisor. For each $a \neq 0$ let $\rho_{C_a,tm} : S_{tm} \to H^0(C_a, \mathcal{O}_{C_a}(tm)$ denote the restriction map. The vector space $\mathrm{Im}(\rho_{C_a,tm})$ contains the set Δ of all weighted homogeneous $f \in \mathbb{K}[x_0, x_2]$ with weight tm. Take $(i_0, i_2) \in \mathbb{N}$ such that $i_0q_0 + i_2q_2 = tq_0q_1q_2$. Since $(q_0, q_2) = 1$, there are integers j_0, j_2 such that $i_0 = j_0q_2$, $i_2 = j_0q_0$ and $j_0 + j_2 = tq_1$. Hence Δ is $\mathbb{K}$-vector space of dimension $tq_1 + 1$. For any $a \neq 0$ let $u_a : \mathbb{P}^1 \to C_a$ denote the normalization map. and $v_a : \mathbb{P}^1 \to \mathbb{P}(q_0, q_1, q_2)$ the composition of v_a with the inclusion $C_a \hookrightarrow \mathbb{P}(q_0, q_1, q_2)$. For any integer c let $\mathcal{L}_a(c)$ denote the quotient of $v_a^*(\mathcal{O}_{\mathbb{P}(q_0,q_1,q_2)}(cq_0q_1))$ by its torsion (the torsion is zero if $c \equiv 0 \pmod{q_2}$). We have $\deg(\mathcal{L}_a(c)) = c + 1$ and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(c))$ is spanned by $v^*(H^0(\mathbb{P}(q_0, q_1, q_2), \mathcal{O}_{\mathbb{P}(1,1,q_2)}(cq_0q_1q_2)))$. Hence for any $A \subset C_a$ such that $\sharp(A) \leq c + 1$ we have $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_A(cq_0q_1)) = 0$. Notice that each $P \in \mathbb{P}(q_0, q_1, q_2) \setminus (E \cup F)$ is contained in a unique curve C_a, $a \neq 0$.

(b) In this step we assume $t \geq 2$. Since $\sharp(S \cap F) \leq \sharp(S) \leq tq_1 + 1$, we saw that $h^1(\mathcal{I}_{F \cap S}(tq_0q_1q_2)) = 0$. Set $S_0 := S \setminus S \cap (E \cup F)$ and $k'' := \sharp(S \cap (E \cup F)$.

Fix $a_1 \in \mathbb{K} \setminus \{0\}$ such that $b_1 := \sharp(S_0 \cap C_{a_1})$ is maximal. Set $S_1 := S \setminus S \cap C_{a_1}$. For each integer $i \geq 2$ define recursively $a_i \in \mathbb{K} \setminus \{0\}$, the integer b_i and the set S_i in the following way. Fix any $a_i \in \mathbb{K}$ such that $b_i := \sharp(C_{a_i} \cap S_{i-1})$ is maximal and set $S_i := S_{i-1} \setminus C_{a_i} \cap S_{i-1}$. Notice that the sequence $\{b_i\}_{i \geq 1}$ is non-decreasing and that $b_i = 0$ if and only if $S_{i-1} = \emptyset$, i.e. $S_0 \subset \cup_{h=1}^{i-1} C_h$. Let e be the last integer such that $b_e > 0$. We have $e \leq \sharp(S) - k''$. In particular we have $e \leq tq_1 + 1$.

Claim 1. *We have $h^1(\mathcal{I}_{S_{i-1} \cap C_{a_i}}((t-1)q_2 + 1 - i)q_0q_1)) = 0$ for all $i \in \{1, \dots, e\}$.*

Proof of Claim 1. By assumption $a_i \neq 0$. By step (a) it is sufficient to prove that $b_i \leq (t-1)q_2 + 1 - i$ for all i. Assume that this is not true at take the first integer $j \in \{1, \dots, e\}$ such that $b_j \geq (t-1)q_2 + 2 - j$. Since the sequence $\{b_i\}$ is non-decreasing, we get $\sharp(S_0) \geq j(2 + (t-1)q_2 + 2 - j) \geq (rt-1)tq_2 + 2$. Since $t \geq 2$ and $q_2 \geq 2q_1$, we get a contradiction, concluding the proof of Claim 1.

First assume $(t-1)q_2 - e \leq k'' - 2$. Since $b_i > 0$ for all $i \in \{1, \dots, e\}$, we get $\sharp(S) \geq t(t-1)q_2 + 2 > tq_1 + 1$, a contradiction.

Now assume $(t-1)q_2 - e \geq k'' - 1$. We apply e times Claim 1, first with $i = 1$, then with $i = 2$, and so on. Since $h^1(\mathcal{I}_{F \cap S}(tq_0q_1q_2)) = 0$, step (a) gives $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_S((tq_0q_1q_2) \leq h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_{S \cap E}((((t-1)q_2 - e)q_1q_2))$. Since $\sharp(S \cap E) \leq k''$ and we assumed $k'' \leq (t-1)q_2 - e$, we may use again step (a) and get $h^1(\mathbb{P}(q_0, q_1, q_2), \mathcal{I}_S(tq_0q_1q_2)) = 0$.

(c) Now assume $t = 1$ and $q_1 > q_0$. The curve $\phi_m(\{x_0 = 0\})$ is an irreducible curve of degree q_0, while all other irreducible curves of $\phi_m(\mathbb{P}(q_0, q_1, q_2))$ have degree $\geq q_1 > q_0$. Apply [4], Theorem 3.8. □

Now we look at the case $r = 2$ and $q_0 = 1$. Since $(\mathbb{P}(1,1,1), \mathcal{O}_{\mathbb{P}(1,1,1)}(m)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, it is sufficient to study the case $q_2 > 1$.

Remark 2. Fix an integer $q_2 \geq 2$. We take $r = 2$ and $q_0 = q_1 = 1$. Hence $m = q_2$. We have $h^0(\mathbb{P}(1,1,q_2), \mathcal{O}_{\mathbb{P}(1,1,q_2)}(q_2)) = q_2 + 2$. Let $\phi_{q_2} : \mathbb{P}(1,1,q_2) \to \mathbb{P}^{q_2+1}$ denote the morphism associated to the complete linear system $|\mathcal{O}_{\mathbb{P}(1,1,q_2)}(q_2)|$. Call T_{q_2} its image. The normal surface T_{q_2} is a cone with vertex $O = \phi_{q_2}((0,0,1))$ and with bases a rational normal curve of $\mathbb{P}^{q_2}$. The variety T_{q_2} is projectively normal and scheme-theoretically cut out by quadrics.

Proposition 1. *Fix integers $q_2 \geq 2$ and $t \geq 1$. We take $q_0 = q_1 = 1$ and take the set-up of Remark 2. Hence we identify $\mathcal{O}_{T_{q_2}}(t)$ with $\mathcal{O}_{\mathbb{P}(1,1,q_2)}(tq_2)$. Let $S \subset T_{q_2}$ be a finite set such that $\sharp(S) \leq 3t$. If $\sharp(S) \geq 2t+2$, then assume $t \geq 3$. We have $h^1(T_{q_2}, \mathcal{I}_S(t)) > h^1(T_{q_2}, \mathcal{I}_{S'}(t))$ for each $S' \subsetneq S$ if and only if S and q_2 are as follows:*

(i) $\sharp(S) = t+2$ *and* S *is contained in a line through* O*;*

(ii) $\sharp(S) = 2t+2$, $O \notin S$ *and there are two lines* $L_1, L_2 \subset T_{q_2}$ *through* O *such that* $\sharp(S \cap L_1) = \sharp(S \cap L_2) = t+1$*;*

(iii) $q_2 = 2$, $\sharp(S) = 2t+2$ *and* S *is contained in a smooth hyperplane section of the quadric cone* T_2*.*

Proof. Since $T_{q_2} \subset$ is projectively normal, for any finite set $A \subset T_{q_2}$ we have $h^1(T_{q_2}, \mathcal{I}_A(t)) = h^1(\mathbb{P}^{q_2+1}, \mathcal{I}_A(t))$. Since T_{q_2} is cut out by quadrics, if $L \subset \mathbb{P}^{q_2+1}$ is a line and $\sharp(L \cap T_{q_2}) \geq 3$, then $L \subset T_{q_2}$. Since T_{q_2} is cut out by quadrics, if $L \subset \mathbb{P}^{q_2+1}$ is a smooth conic and $\sharp(L \cap T_{q_2}) \geq 5$, then $L \subset T_{q_2}$. Since T_{q_2} is cut out by quadrics, but it is not a plane, it contains no plane cubic. Moreover, if $L \subset \mathbb{P}^{q_2+1}$ is an irreducible plane cubic we have $\sharp(L \cap T_{q_2}) \leq 6$. Now take a reduced but not irreducible cubic plane curve $C \subset \mathbb{P}^{q_2+1}$ and take an irreducible component T of C; we just saw that if $\deg(T \cap L_{q_2})) \geq 2 \cdot \deg(T) + 1$, then $T \subset T_{q_2}$. Hence Proposition 1 follows from [4], Theorem 3.8, after identifying the lines, the reduced conics of T_{q_2} (in the case $\sharp(S) = 3t$ and S contained in a plane cubic C, we use the assumption $t \geq 3$, because we need $\sharp(S \cap C) > 6$). □

Remark 3. Take $q_0 = 1$ and $q_2 > q_1 > 1$, $(q_1, q_2) = 1$. Hence $m = q_1 q_2$. Set $T_{q_1,q_2} := \phi_m(\mathbb{P}(1, q_1, q_2)$. The singular variety $\mathbb{P}(1, q_1, q_2)$ is a compactification of $\mathbb{A}^2$ adding a curve $E \cong \mathbb{P}(q_1, q_2) \cong \mathbb{P}^1$ (see [1], Example 3 at page 124). In this case E is a generator of the class group of $\mathbb{P}(1, q_1, q_2)$ and $\deg(\mathcal{O}_{\mathbb{P}(1,q_1,q_2)}(q_1q_2)|E) = \deg(\mathcal{O}_{\mathbb{P}(q_1,q_2)}(q_1q_2)) = 1$. For any other irreducible curve $C \subset \mathbb{P}(1, q_1, q_2)$ we have $\deg(\mathcal{O}_{\mathbb{P}(1,q_1,q_2)}(q_1q_2)|C) \geq q_1$. Hence E is the only line of T_{q_1,q_2} and T_{q_1,q_2} contains no other curve of degree $< q_1$.

Proposition 2. *Take* $q_0 = 1$ *and* $q_2 > q_1 > 1$, $(q_1, q_2) = 1$*. Take* T_{q_1,q_2} *as in Remark 3. Hence* $m = q_1q_2$*. We identify* $\mathcal{O}_{T_{q_1,q_2}}(t)$ *with* $\mathcal{O}_{\mathbb{P}(1,q_1,2)}(tq_1q_2)$*. Let* $S \subset T_{q_0,q_1}$ *be finite set such that* $\sharp(S) \leq 2t+1$*. If* $\sharp(S) \geq 2t+1$*, then assume* $q_1 \neq 2$*. We have* $h^1(T_{q_0}, \mathcal{I}_S(t)) > 0$ *if and only if* $\sharp(S \cap E) \geq t+2$*.*

Proof. Recall that T_{q_1,q_2} is projectively normal. Hence

$$h^1(\mathbb{P}(q_1, q_2), \mathcal{I}_S(tm)) = h^1(\mathbb{P}^N, \mathcal{I}_{\phi_m(S)}(t)).$$

Since $\sharp(S) < 3t$, we have $h^1(\mathbb{P}^N, \mathcal{I}_{\phi_m(S)}(t)) > 0$ if and only if there is a line $L \subset \mathbb{P}^N$ such that $\sharp(L \cap \phi_m(S)) \geq t+2$. The singular variety is a compactification of $\mathbb{A}^2$ adding a curve $E \cong \mathbb{P}(q_1, q_2) \cong \mathbb{P}^1$ (see [1], Example 3 at page 124). This curve is embedded in T_{q_0,q_1} as a line, because $\deg(\mathcal{O}_{\mathbb{P}(1,q_1,q_2)}(q_1q_2)|E) =$

1. Hence if $\sharp(S \cap E) \geq t+2$, then $h^1(T_{q_0}, \mathcal{I}_S(t)) > 0$. To get the converse it is sufficient to check that if $A \subset T_{q_1,q_2}$, $\sharp(A) = 3$ and A spans a line L, then $L = \phi_m(E)$, i.e. that if $B \subset \mathbb{P}(1, q_{1,2})$, $\sharp(B) = 3$ either $B \subset E$ or $h^1(\mathbb{P}(1, q_{1,2}), \mathcal{I}_B(q_1 q_2)) = 0$. This is done as in the proof of Theorem 2. □

Proof of Theorem 1. We first check the " only if " part.

Since $h^1(\phi_m(\mathbb{P}(q_0, \dots, q_r)), \mathcal{I}_{\phi_m(Z)}(t)) > 0$, we have $h^1(\mathbb{P}^N, \mathcal{I}_{\phi_m(Z)}(t)) > 0$. By [3], Lemma 34, there is a line $D \subset \mathbb{P}^N$ such that $\deg(D \cap \phi_m(Z)) \geq t+2$. Since $t+2 > \alpha(q_0, \dots, q_r)$, we have $\phi_m(Z) \subset D$. Now we check the " if " part. It is sufficient to prove $h^1(\phi_m(\mathbb{P}(q_0, \dots, q_r)), \mathcal{I}_{D \cap \phi_m(Z)}(t)) > 0$ (Remark 1). Let $\rho : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(t)) \to H^0(D, \mathcal{O}_D(t))$ denote the restriction map. Let $\rho' : H^0(\phi_m(\mathbb{P}(q_0, \dots, q_r)), \mathcal{O}_{\phi_m(\mathbb{P}(q_0, \dots, q_r))}(t)) \to H^0(D, \mathcal{O}_D(t))$ denote the restriction map. Since ρ is surjective, ρ' is surjective. Since ρ' is surjective, we have $h^1(\phi_m(\mathbb{P}(q_0, \dots, q_r)), \mathcal{I}_{\phi_m(Z) \cap D}(t)) > 0$ if and only $h^1(D, \mathcal{I}_{\phi_m(Z) \cap D}(t)) > 0$, i.e. if and only if $\deg(Z \cap D) \geq t+2$. □

Question 1. Take $r > 2$ and $q_0, \dots, q_r$ normalized. Let

$$\phi_m : \mathbb{P}(q_0, \dots, q_r) \to \mathbb{P}^N$$

denote the embedding associated to the complete linear system $|\mathcal{O}_{\mathbb{P}(q_0, \dots, q_r)}(m)|$. Extend the proof of [7], Proposition , to find the maximal degree.

$\deg(L \cap \phi_m(\mathbb{P}(q_0, \dots, q_r)))$ and the maximal number of points.

$\sharp(L \cap \phi_m(\mathbb{P}(q_0, \dots, q_r)))$, where $L \subset \mathbb{P}^N$ is a line not contained in $\phi_m(\mathbb{P}(q_0, \dots, q_r))$. Compute the largest integer k such that $\mathcal{O}_{\mathbb{P}(q_0, \dots, q_r)}(tm)$ is k-spanned and/or k-very ample in the sense of [2].

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] M. Beltrametti, L. Robbiano, Introduction to the theory of weighted projective spaces, *Exp. Math.*, **4** (1986), 111-162.

[2] M.C. Beltrametti, A.J. Sommese, *The Adjunction Theory of Complex Projective Varieties*, De Gruyter Expositions in Mathematics, **16**, Walter de Gruyter & Co., Berlin (1995).

[3] A. Bernardi, A. Gimigliano, M. Idà, On the stratification of secant varieties of Veronese varieties via symmetric rank, *J. Symbolic. Comput.*, **46**, No. 1 (2011), 34-53.

[4] A. Couvreur, The dual minimum distance of arbitrary dimensional algebraic-geometric codes, *J. Algebra*, **350**, No. 1 (2012), 84-107.

[5] C. Delorme, Espaces projectifs anisotropes, *Bull. Soc. Math. France*, **103** (1975), 203-223.

[6] I. Dolgachev, Weighted projective varieties, In: *Group Actions and Vector Fields, Proceedings, Vancouver 1981*, 34-71; Lect. Notes Math., **956**, Springer-Verlag, Berlin (1982).

[7] M. Rossi, L. Terracini, Weighted projective spaces from the toric point of view with computational applications, *ArXiv:* 1112.1677v2.