International Journal of Pure and Applied Mathematics
Volume 78 No. 8 2012, 1091-1098
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

ON THE POSTULATION OF QUASI-STABLE CURVES IN PROJECTIVE SPACES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Here we study the postulation of general curves $C \cup D_1 \cup \cdots \cup D_z \subset \mathbb{P}^r$, where C is smooth, each D_i is a secant line of C and $D_i \cap D_j = \emptyset$ for all $i \neq j$.

AMS Subject Classification: 14H20, 14H51
Key Words: quasi-stable curve, embedded curve, postulation of curves, reducible curves

1. Introduction

Let $Y \subset \mathbb{P}^r$ any one-dimensional scheme. Let $a(Y)$ denote the minimal integer t such that $h^0(\mathcal{I}_Y(t)) > 0$. Obviously $h^0(\mathcal{I}_Y(t)) > 0$ if and only if $t \geq a(Y)$. Let $\delta(Y)$ be the minimal integer t such that $h^1(\mathcal{I}_Y(t)) > 0$ with the convention $\delta(Y) = -\infty$ if there is no such integer. Y is said to have *maximal rank* if for every $t \in \mathbb{Z}$ either $h^0(\mathcal{I}_Y(t)) = 0$ or $h^1(\mathcal{I}_Y(t)) = 0$, i.e. if $\delta(Y) < a(Y)$. For the curves Y we will use we have $h^1(Y, \mathcal{O}_Y(1)) = 0$ and $\delta(Y) \geq 2$. Hence for these curves Castelnuovo-Mumford's lemma says that the homogeneous ideal of Y is generated in degree $\leq \delta(Y) + 1$. Let Y be a reduced and connected

Received: February 14, 2012

projective curve. Fix an ordering $Y_1, \dots, Y_s$ of its irreducible components. For any $L \in \mathrm{Pic}(Y)$ set $\underline{\deg}(L) = (d_1, \dots, d_s) \in \mathbb{Z}^s$, where $d_i := \deg(L|Y_i))$ and call $\underline{deg}(L)$ the multidegree of L. If $Y \subset \mathbb{P}^r$ is reducible and with maximal rank, then quite ofter there are numerical obstructions for the multidegree $\underline{\deg}(\mathcal{O}_Y(1))$ (see Example 2 for some embedding of stable curves). As in [1] we consider nodal curves $Y = C \cup D_1 \cup D_z$ with C a smooth, connected and non-degenerate curve, each D_i a true secant line of C (i.e. $\sharp(C \cap D_i) = 2$ and D is not tangent to C) and $D_i \cap D_j = \emptyset$ for all $i \neq j$. Hence $\deg(Y) = \deg(C) + z$ and $p_a(Y) = p_a(C) + z$. The stable curve Y'' associated to C is a nodal curve of genus $p_a(C) + z$ with C as its normalization (Y'' is obtained from C gluing together the two points of $C \cap D_i$ to get a node of Y'' and make it for all $i = 1, \dots, z$). These reducible curves $Y \subset \mathbb{P}^r$ are obtained taking the very ample balanced line bundles on $C \cup D_1 \cdots \cup D_z$ (see [3], [4], [5]), i.e. the rank 1 torsion free sheaves on Y'', which are locally free at no point of $\mathrm{Sing}(Y'')$.

Question 1. Fix an integer $r \geq 3$. Let $C_r \subset \mathbb{P}^r$ be a rational normal curve and $X \subset \mathbb{P}^r$ a linearly normal elliptic curve. Let x_r be the largest integer $t \geq 0$ such that a general union of C_r and x_r secant lines of C has maximal rank. Let y_r be the maximal integer such that for each $1 \leq i \leq y_r$ a general union of C_r and s secant lines of C_r has maximal rank. Is it $x_r = y_r$? Similarly, define the integers $x_r(X)$ and $y_r(X)$ using X instead of C_r. Are these integers the same for all elliptic curves X? Compute these integers.

A priori these integers may be not well-defined, i.e. a priori for every integer $t > 0$ there could be an integer $s \geq t$ such that a general union of C_r (or X) and s secant lines of C_r (or X) has maximal rank. This is not the case: see Example 1 for the case $r \geq 4$ and Proposition 1 for the case $r = 3$.

For any integral curve $Y \subset \mathbb{P}^r$ let $E(Y)$ be the set of all lines $D \subset \mathbb{P}^r$ such that $\sharp(D \cap Y) = 2$, $D \cap \mathrm{Sing}(Y) = \emptyset$ and D is not tangent to Y at one of the points $Y \cap D$ (we call any line D as above a true secant line of Y). For all integers r, d, g, z such that $r \geq 3$, $0 \leq z \leq g$ and $d \geq g + r$ let $A(d-z, g-z, z; r)$ denote the set of all curves $Y \subset \mathbb{P}^r$ such that $Y = C \cup D_1 \cup \cdots \cup D_z$ with $C \subset \mathbb{P}^r$ a smooth, connected and non-degenerate curve of degree $d - z$ and genus $g - z$, $h^1(C, \mathcal{O}_C(1)) = 0$, $D_i \in E(C)$ for all i and $D_i \cap D_j = \emptyset$ for all $i \neq j$ (with the convention $Y = C$ if $z = 0$). A Mayer-Vietoris exact sequence gives $h^1(Y, \mathcal{O}_Y(1)) = 0$. Hence we are working in the range " non-special embeddings " of quasi-stable curves. In Section 3 we prove the following result, which improves the case $r = 4$ of the asymptotic [1], Theorem 1.

Theorem 1. *Fix integers d, g such that $d \geq g + 4$. Let k be the minimal integer such that $k(g+4) + 1 - g \leq \binom{k+4}{4}$. Then for every integer $z \geq 0$ such* *that* $g - z \geq 2k$ *a general* $Y \in A(d-z, g-z, z; 4)$ *has maximal rank.*

The proof of Theorem 1 closely follow [2].

2. Concerning the Question

Let $Y \subset \mathbb{P}^r$, $r \geq 3$ be an integral and non-degenerate curve. We have $E(Y) \neq \emptyset$ unless Y is strange (hence always in characteristic zero, while in positive characteristic if Y is smooth) (see [6]).

Example 1. Fix integers $r \geq 4$, $q \geq 0$ and $x \geq q + r$. Let $C \subset \mathbb{P}^r$ be a smooth, connected and non-degenerate curve of genus q and degree x such that $h^1(C, \mathcal{O}_C(1)) = 0$. Let $Y_z \subset \mathbb{P}^r$ be the union of C and z general secant lines of C. We have $\deg(Y_z) = q + x$, $p_a(Y_z) = q + z$. A Mayer-Vietoris exact sequence gives $h^1(Y_z, \mathcal{O}_{Y_z}(1)) = 0$. Hence $h^0(Y_z, \mathcal{O}_{Y_z}(t)) = tx + 1 - q + (t-1)z$ for all $t \geq 1$. Let $\sigma_2(C) \subset \mathbb{P}^r$ be the secant variety of C. We have $Y_z \subset \sigma_2(Y)$. The variety $\sigma_2(C)$ is an irreducible 3-dimensional variety of $\mathbb{P}^r$. Set $y := x - 1)(x-2)/2 - q$. Taking a linear projection of C from a general line and applying the genus formula for plane curves we get $\deg(\sigma_2(C)) = y$. Hence $\sigma_2(C)$ is contained in a hypersurface of degree y (take cones if $r > 4$). Hence $h^0(\mathbb{P}^r, \mathcal{I}_{Y_z}(y)) > 0$. Hence if $z \geq (\binom{y+r}{r} - yx - 1 + q)/(y-1)$, then Y_z has not maximal rank.

Example 2. Fix integers $r \geq 6$, $q \geq 0$, $x \geq q + r$ and $z \geq 0$. Let $C \subset \mathbb{P}^r$ be a non-degenerate smooth curve of genus q and degree x. Let $Y_z \subset \mathbb{P}^r$ be a general union of C and z smooth conics, each of them intersecting C at 3 points. The curve Y_z is a stable curve of genus $q + 2z$ and degree $x + 2z$ and $h^1(Y_z, \mathcal{O}_{Y_z}(1)) = 0$. Hence $h^0(Y_z, \mathcal{O}_{Y_z}(t)) = tx + 1 - q + (2t-2)z$ for all $t \geq 1$. Let $\sigma_3(C) \subset \mathbb{P}^r$ denote the closure of the union of all planes spanned by 3 points of C. We have $Y_z \subset \sigma_3(C)$. Set $y := \deg(\sigma_3(C))$. The variety $\sigma_3(Y)$ is an integral variety of dimension 5. Since $r > 5$, $\sigma_3(C)$ is contained in a degree y hypersurface. Hence Y_z has not maximal rank if $z \geq (\binom{r+y}{r} - yx - 1 + q)/(2y-2)$.

Proposition 1. *Fix an integral and non-degenerate curve* $Y \subset \mathbb{P}^r$,$r \geq 3$, *such that* $E(Y) \neq \emptyset$. *There is an integer* $\alpha > 0$ *such that for every integer* $z \geq \alpha$ *no* $T \in A(Y, z)$ *has maximal rank.*

Proof. For any closed subscheme $A \subset \mathbb{P}^r$, let $p_A(t) \in \mathbb{Q}[t]$ denote its Hilbert polynomial. Set $d := \deg(Y)$ and $q := p_a(Y)$. Fix disjoint lines $D_i \in E(Y)$, $i \geq 1$. Let $Y^{(1)} \subset \mathbb{P}^r$ be the closed subscheme of $\mathbb{P}^r$ with $(\mathcal{I}_Y)^2$ as its ideal sheaf. The scheme $Y^{(1)}$ has dimension 1 and it is supported by Y. If Y is singular, then $Y^{(1)}$ may have embedded points, but in any case the Hilbert polynomial $p_{Y^{(1)}}(t)$

has degree 1 with rdt has its leading monomial. Hence $p_{Y^{(1)}}(t) = rdt + \chi(\mathcal{O}_{Y^{(1)}})$. There is an integer β (depending only from Y) such that for each $t \geq \beta$ the Hilbert polynomials $p_Y(t)$ and $p_{Y^{(1)}}(t)$ are the Hilbert function of Y and $Y^{(1)}$, respectively, and $h^1(Y, \mathcal{O}_Y(\beta)) = 0$. Set $Y'_z := Y^{(1)} \cup D_1 \cup \cdots \cup D_z$ and $X_z := Y \cup D_1 \cup \cdots \cup D_z$. Notice that $h^0(X_z, \mathcal{O}_{X_z}(t)) = dt + 1 - q + z(t-1)$ for every $t \geq \beta$, while $h^0(\mathcal{I}_{Y^{(1)})}(t)) = \binom{r+t}{r} - rdt - \chi(\mathcal{O}_{Y^{(1)}})$. Since each $D_i \in E(Y)$ and $D_i \cap D_j = \emptyset$, $p_{Y'_z}(t) = p_{Y^{(1)}}(t) + z(t-3)$. Hence each D_i imposes at most $t-3$ independent conditions to $H^0(\mathcal{I}_{Y^{(1)})}(t))$. Set $\psi(t) := \binom{r+t}{r} - rdt - \chi\mathcal{O}_{Y^{(1)}})$ and $\phi(t) := \binom{r+t}{r} - dt - 1 + q$. Set $w_t := \lceil \psi(t)/(t-3) \rceil$ and $z_t := \lceil \phi(t)/(t-1) \rceil$. The integer w_t is the minimal integer c such that $\psi(t) \leq c(t-3)$, while z_t is the minimal integer c such that $\phi(t) \leq c(t-1)$. Since $r \geq 3$, we have $w_t > z_t$ for $t \gg 0$. Hence for $t \gg 0$ we have $h^0(\mathcal{I}_{Y'_{z_t}}(z_t)) > 0$. Hence $h^0(\mathcal{I}_A(z_t)) > 0$ for every $A \in A(Y, z_t)$. Thus no $A \in A(Y, z_t)$ has maximal rank. $\square$

The proof of Proposition 1 just given shows that the integer α may be computed only in term of $\deg(Y)$, $p_a(Y)$ and the singularities of Y (if any). A better lower bound for α in the case $r \geq 4$ may be obtained as in Example 1 using $\sigma_2(Y)$ (only the explicit formula must be modified using the integer $\deg(\sigma_2(Y))$).

3. Proof of Theorem 1

Set $A(x,q,z) := A(x,q,z;4)$. Let $A'(x,q,z)$ denote the closure of $A(x,q,z)$ in $\mathrm{Hilb}(\mathbb{P}^4)$. Set $B(x,q) := A(x,q,0)$, i.e. let $B(x,q)$ denote the set of all $C \subset \mathbb{P}^4$ with C smooth, connected and non degenerate, $\deg(C) = x$, $p_a(C) = q$ and $h^1(C, \mathcal{O}_C(1)) = 0$.

For all integers $t > 0$, $g \geq 0$ define the integers $g(t)$, $f(t)$, $d_g(t)$ and $h_g(t)$ by the relations

$$t(g(t)+4) + 1 - g(t) + f(t) = \binom{t+4}{4}, \; 0 \leq f(t) \leq \max\{0, t-2\} \quad (1)$$

$$td_g(t) + 1 - g + h_g(t) = \binom{t+4}{4} \; 0 \leq h_g(t) \leq t-1 \quad (2)$$

Remark 1. We have $g(1) = 0$, $f(1) = 0$, $g(2) = 6$, $f(2) = 0$, $g(3) = 11$, $f(3) = 0$, $g(4) = 17$, $f(4) = 2$, $g(5) = 26$, $f(5) = 1$.

For each integer $t \geq 1$ we define the following assertion H_t:

H_t: There is a triple (Y, D, S), where $Y \in A(2t+2, 2t-2, g(t) - 2t + 6)$, say $Y = C \cup C_1$ with $C \in B(2t+2, 2t-2)$ and C_1 a disjoint union of $g(t) - 2t + 6$

secant lines of C, D is a secant line of C not intersecting C_1, $S \subset D \setminus D \cap C$, $\sharp(S) = f(t)$ and $h^0(\mathcal{I}_{Y \cup S}(t)) = 0$.

By Riemann-Roch the condition " $h^0(\mathcal{I}_{Y \cup S}(t)) = 0$ " in H_t is equivalent to the condition " $h^1(\mathcal{I}_{Y \cup S}(t)) = 0$ ".

Remark 2. Fix a hyperplane $H \subset \mathbb{P}^4$. For any closed subscheme $E \subset \mathbb{P}^4$ let $\mathrm{Res}_H(E$ denote the residual scheme of Z with respect to H, i.e. the closed subscheme of $\mathbb{P}^4$ with $\mathcal{I}_E : \mathcal{I}_H$ as its ideal sheaf. If E is reduced, then $\mathrm{Res}_H(Z)$ is the closure of $E \setminus E \cap H$ in $\mathbb{P}^4$. For any integer t we have an exact sequence of coherent sheaves:

$$0 \to \mathcal{I}_{\mathrm{Res}_{H(E)}}(t-1) \to \mathcal{I}_E(t) \to \mathcal{I}_{E \cap H, H}(t) \to 0 \qquad (3)$$

Now take a closed subscheme Z of H and a quadric $Q \subset H$. The residual scheme $\mathrm{Res}_Q(Z)$ of Z with respect to Q is the closed subscheme of H with $\mathcal{I}_{Z,H} : \mathcal{I}_{Q,H}$ as its ideal sheaf. For each $t \in \mathbb{Z}$ we have an exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_{Q(Z)},H}(t-2) \to \mathcal{I}_{Z,H}(t) \to \mathcal{I}_{Q \cap Z, Q}(t) \to 0 \qquad (4)$$

Remark 3. Here we check H_t for all $t \leq 5$. Obviously H_1 is true. We will silently use that $f(t) = 0$ if $t = 1, 2, 3$.

(a) Here we check H_2. Fix (Y, D, S) satisfying H_1 with $Y = C \cup C_1$ $C \in B(4, 0)$ and $C_1 = \emptyset$. Since $f(1) = 0$, we have $S = \emptyset$. Fix a general hyperplane $H \subset \mathbb{P}^4$ and set $H \cap C = \{P_1, P_2, P_3, P_4\}$. We have $h^i(\mathcal{I}_Y(1)) = 0$, $i = 0, 1$. Call D_1, D_2 the lines of H spanned by $\{P_1, P_2\}$ and $\{P_3, P_4\}$, respectively. Since C is a rational normal curve, $\{P_1, P_2, P_3, P_4\}$ spans H. Hence $C \cup D_1 \cup D_2 \in A(4, 0, 2)$. Fix a smooth quadric $Q \subset H$ and take 3 different lines D_3, D_4, D_5 in the ruling of Q not containing D_1. Let $D_6 \subset H$ be a general line intersecting both D_1 and D_2. We have $C \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \in A(4, 0, 5)$ and hence $E := C \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5 \cup D_6 \in A(4, 0, 6)$. Since Q is the only quadric surface containing $D_3 \cup D_4 \cup D_5$ and $D_6 \nsubseteq Q$, we have $h^0(\mathcal{I}_{E \cap H}(2)) = 0$. Since $\mathrm{Res}_H(E) = C$ and $h^0(\mathcal{I}_C(1)) = 0$, The case $t = 2$ of (3) gives $h^0(\mathcal{I}_E(2)) = 0$. We may smooth $C \cup D_1 \cup D_2$ (say in a family $\{Y_\lambda\}_{\lambda\Delta}$, Δ irreducible, $o \in \Delta$ and $Y_o = C \cup D_1 \cup D_2$. Since any two points of a projective space span a unique line, we may follow the smoothing $\{Y_\lambda\}$ and get a flat family $\{(D_3 \cup D_4 \cup D_5 \cup D_6)_\lambda\}$ with a flat family of secant lines. The family $\{Y_\lambda \cup (D_3 \cup D_4 \cup D_5 \cup D_6)_\lambda\}_{\lambda \in \Delta}$ shows that $E \in A'(6, 2, 4)$. HenceH_2 is true by semicontinuity.

(b) Here we check H_3. Fix (Y, D, S) satisfying H_2, with $Y = C \cup C_1$, and a general hyperplane H. Take two disjoint lines $D_1, D_2 \subset H$ spanned by 2 points of $C \cap H$ and 3 general lines D_3, D_4, D_5 intersecting D_1 and D_2 (and hence contained in H). Set $E := Y \cup D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$. Deforming $C \cup D_1 \cup D_2$ as in (a) we get $E \in A'(8,4,7)$. The scheme $E \cap H$ is the union of the lines D_i, $1 \leq i \leq 5$, and of 6 points of H. Since $D_1 \cup \cdots \cup D_5$ is contained in a quadric surface, Q, we have $h^0(H, \mathcal{I}_{D_1 \cup \cdots \cup D_5}(3)) = 6$. We may find Y with the additional condition that exactly 2 of the points of $C_1 \cap H$ are contained in Q, while the other 4 points of $Y \cap H$ not coplanar. The first condition implies $h^0(Q, \mathcal{I}_{Q \cap E}(3)) = 0$. Then taking the residual in H with respect to Q we get $h^0(H, \mathcal{I}_{E \cap H}(3)) = 0$. Apply (3) with $t = 3$.

(c) Here we check H_4. Fix (Y, D, S) satisfying H_3, with $Y = C \cup C_1$, and a general hyperplane H. Take two disjoint lines $D_1, D_2 \subset H$ spanned by 2 points of $C \cap H$ and take a general $S' \subset D$ such that $\sharp(S') = 2$. Fix a smooth quadric $Q \subset H$ containing $D_1 \cup D_2$ and 3 other points of $Y \cap (H \setminus Q)$. For general Y we may assume that Q contains no other point of $Y \cap H$ and that no two of the 3 points of $Y \cap (Q \setminus D_1 \cup D_2)$ are contained in a line of Q. Call $(0,1)$ the type of the ruling of Q to which D_1 belongs. Let $D_i \subset Q$, $1 \leq i \leq 4$, be general lines intersecting D_1 (i.e. of type $(1,0)$). Set $E := Y \cup D_1 \cup \cdots \cup D_6$. We have $E \in A'(10,6,11)$. Notice that $E \cap Q$ is the union of $D_1 \cup \cdots \cup D_6$ and 3 points not on a curve of type $(0,2)$. Hence $h^0(Q, \mathcal{I}_{Q \cap E}(4)) = 0$. Fix a general line D' intersecting both D_1 and D_2 and such that $\sharp(S') = 2$. Set $S_1 := Y \cap (H \setminus Q)$. We may arrange so that S_1 is formed by 8 points, no 4 of them coplanar and with $h^0(\mathcal{I}_{S_1}(2)) = 2$ (as in [2] use that $\deg(\sigma_2(C))$ is large to handle the points of $C_1 \cap H$). Take a general $Q_1 \in H^0(H, \mathcal{I}_{S_1}(2))$. For a general $P \in Q_1$ there is a line meeting D_1 and D_2 and containing P. Hence for general D we may assume that one of the points, P, of $Q_1 \cap D'$, satisfies $h^0(Q_1, \mathcal{I}_{S_1 \cup P}(Q_1)) = 0$, i.e. $h^0(H, \mathcal{I}_{S_1 \cup \{P\}}(2)) = 1$. Hence for a general $P' \in D$ we have $h^0(H, \mathcal{I}_{S_1 \cup \{P, P'\}}(2)) = 0$. Hence $h^0(H, \mathcal{I}_{E \cap H \cup S'}(4)) = 0$, where $S' := \{P, P'\}$. Use (E, D', S') and semicontinuity.

(d) Here we check H_5. Fix (Y, D_1, S) satisfying H_4, with $Y = C \cup C_1$, and a general hyperplane H containing D_1. For general H and C we may assume that no 4 of the points of $C \cap H$ are coplanar. We have $\sharp(S) = 2$. For each $P \in S$ fix a 3-dimensional linear subspace V_P of $\mathbb{P}^4$ containing D_1, but $\neq H$. Set $\tau := \cup_{P \in S} \chi_{V_P}(P)$. Notice that $\mathrm{Res}_H(\tau) = S$. Fix a line D_2 spanned by two of the points $C \cap (H \setminus D_1)$ and a smooth quadric $Q \subset H$ containing $D_1 \cup D_2$ and 2 other points of $C \cap H$ Call $(0,1)$ the system of lines of Q containing D_1 and D_2. For general C we may assume that C contains no other point of $C \cap H$. Deforming C_1 we may assume that Q contains 4 points of $C_1 \cap H$ (use that $\sigma_2(C) \cap H$ is a degree 30 surface). Let D_i, $i = 3, 4$, the lines of type $(1,0)$ of Q containing the two points of $C \cap (Q \setminus (D_1 \cup D_2))$ (we have $D_3 \neq D_4$, because no 4 of the points of $C \cap H$ are coplanar). Let $D_i \subset Q$, $5 \leq i \leq 7$, be general lines of type $(1,0)$ on Q. Take 2 general lines D_8, D_9 intersecting both D_1 and D_2. Set $E := Y \cup D_1 \cup \cdots D_9$. We have $E \in A'(12,8,18)$. Since $E \cap Q$ contains 5 lines of type $(1,0)$, 2 lines of type $(0,1)$ and 3 points not on a line on Q, we have $h^0(Q, \mathcal{I}_{E\cap Q}(5)) = 0$. The scheme $\mathrm{Res}_Q(E \cap H)$ is the union of $D_8 \cup D_9$, 4 points of $C \cap H$ and 7 points of $C_1 \cap H$. Since these 7 points may be chosen one at each time and general in the degree 30 surface $H \cap \sigma_2(C)$, we get $h^0(H, \mathcal{I}_{\mathrm{Res}_Q(H\cap E)}(3)) = 1$. From (4) we get $h^0(H, \mathcal{I}_{E\cap H}(5)) \le 1$. From (3) we get $h^0(\mathcal{I}_E(5)) \le 1$. Take a general secant line of C and a general $P \in D$. Set $S' := \{P\}$. Since $\deg(\sigma_2(C)) > 5$, the generality of D and P implies $h^0(\mathcal{I}_{E\cup S'}(5)$. Deform (E, D, S') to get a solution of H_5.

Lemma 1. *H_t is true for all $t \ge 1$.*

Proof. Remark 3 covers the cases $t \le 5$. Hence we may assume $t \ge 6$. We give the following modification of the proof that $H_{t-1} \Longrightarrow H_t$ given in [2], Proposition 1. Take (Y, D, S) satisfying H_{t-1} with $Y = C \cup M_1 \cup \cdots \cup M_z$ and each M_i a true secant of C. Let D_1, D_2 be secant lines of C. Then we make the construction in the proof of [2], Proposition 1; if in that proof a line L_i meets Y, then we impose that it meet C, but no line M_i. $\square$

We need the following modification $R_t(g)$ of Assertion R_t of [2], §2:

$R_t(g)$, $g \ge 0$: Let k be the maximal integer such that $g(k) \le g$. Fix any integer $t > k$. There is a triple (Y, D, S), where $Y \in A(d_g(t) - (g - 2t + 2), 2t - 2, g - 2t + 2)$, say $Y = C \cup C_1$ with $C \in B(d_g(t) - 2t + 2, 2t - 2)$ and C_1 disjoint union of $g - 2t + 2$ secant lines of C, D is a secant line of C not intersecting C_1, $S \subset D \setminus D \cap C$, $\sharp(S) = h_g(t)$ and $h^0(\mathcal{I}_{Y\cup S}(t)) = 0$.

Proof of Theorem 1. Fix $g \ge 0$. Let $k \ge 0$ be the maximal integer such that $g(k) > g$. As in [2] to prove Theorem 1 for the genus g it is sufficient to prove $R_t(g)$ for all $t > k$. The proofs that $H_k \Longrightarrow R_{k+1}(g)$ and that $R_{t-1}(g) \Longrightarrow R_r(g)$ if $t \ge k+2$ is done as in [2], §2, adding only lines intersecting C, but not C_1. To control at each step the postulation of $Y \cap H$ and of certain of its subset one uses that $\deg(\sigma_2(C)) > t$. $\square$

Acknowledgments

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] E. Ballico, Unions of curves and secant lines with maximal rank, *Int. J. Pure Appl. Math.*, **55**, No. 2 (2009), 231-235.

[2] E. Ballico, Ph. Ellia, On postulation of curves in $\mathbb{P}^4$, *Math. Z.*, **188** (1985), 215-223.

[3] L. Caporaso, A compactification of the universal Picard variety over the moduli space of stable curves, *J. Amer. Math. Soc.*, **7**, No. 3 (1994), 589-660.

[4] L. Caporaso, Linear series on semistable curves, *Int. Math. Res. Not. IMRN*, No. 13 (2011), 2921-2969.

[5] M. Melo, Compactified Picard stacks over $\overline{\mathcal{M}}_g$, *Math. Z.*, **263**, No. 4 (2009), 939-957.

[6] J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276**, No. 4 (1987), 565-579.