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NEW INEQUALITIES OF HERMITE-HADAMARD-LIKE TYPE FOR FUNCTIONS WHOSE DERIVATIVES IN ABSOLUTE VALUE ARE s-CONVEX

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Abstract: In this article we establish some new Hermite-Hadamard-like type inequalities for functions whose derivatives in absolute value are s-convex in the second sense.

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1. Introduction

The following double inequality is well-known in the literature as Hadamard's inequality for convex mappings: Let $f : I \subseteq R \to R$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

Both the inequalities hold in reversed direction if f is concave on I.

A number of papers have been written on this inequality providing new proofs, noteworthy extensions, generalizations, refinements and numerous applications. For the further results on this Hadamard' inequalities, you may see [1, 8, 10, 13].

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In [9], Orlicz introduced the definition for s-convexity of real valued mappings. In [3], Hudzik and Maligranda considers, among others, the class of mappings which are s-convex in the second sense. This class is defined as follows:

Definition 1.1. A mapping $f : I \subseteq R^+ = [0, \infty) \to R$ is said to be s-convex in the second sense on I if the inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{2}$$

holds for all $x, y \in I$ and for some fixed $s \in (0, 1]$.

We denote the class of s-convex mappings in the second sense by $K_s^2(I)$.

The inequality (2) holds in reversed direction if f is s-concave in the second sense.

It is easy to observe that for $s = 1$, the class of s-convex mappings in the second sense is merely the class of convex mappings defined on $[0, \infty)$. It was also proved in [3] that the mappings in $K_s^2(I)$ are nonnegative for $s \in (0, 1]$. For the further properties of $K_s^2(I)$, you may see [2, 3, 4, 5, 7, 11, 12].

The definition of s-convexity in the second sense was used in the theory of Orlicz spaces [9].

In [2], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex mapping in the second sense:

Theorem 1.1. *Suppose that $f : I \subseteq R^+ \to R$ is an s-convex mapping in the second sense on I, where $s \in (0, 1)$ and let $a, b \in I$ with $a < b$. If f is in $L([0, 1])$, then the following double inequality holds:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s+1}. \tag{3}$$

Both the inequalities (3) hold in reversed direction if f is s-concave in the second sense.

In [7, 8, 10, 11], Pachpatte, Park and Mevlüt Tunc proved variants of Hadamard's inequality for s-convex mappings in the second sense. In a recent paper [14], Tseng et al. established the following result which gives a refinement of (1):

Theorem 1.2. *Let $f, g : I \subset [0, b^*] \to R$ be a convex mapping on an interval I, where $b^* > 0$, $a, b \in I$ with $a < b$. If $\mid f' \mid$ is convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality holds:*

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})}{2} \leq \frac{1}{b-a} \int_a^b f(x)dx$$

$$\leq \frac{1}{2}\Big\{f\Big(\frac{a+b}{2}\Big)+\frac{f(a)+f(b)}{2}\Big\} \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

In a recent paper [6], Latif and Dragomir established the following theorem:

Theorem 1.3. *Let $f, g : I \subset [0, b^*] \to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $b^* > 0$, $a, b \in I$ with $a < b$. If $\mid f' \mid^q$ is convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality holds:*

$$\Big|\frac{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})}{2}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$
$$\leq \Big(\frac{1}{2}\Big)\Big(\frac{1}{2}\Big)^{\frac{1}{q}}\Big(\frac{b-a}{16}\Big)\Big\{\Big(\mid f'(a)\mid^q +2\mid f'(\frac{3a+b}{4})\mid^q\Big)^{\frac{1}{q}}$$
$$+\Big(\mid f'(\frac{a+b}{2})\mid^q +2\mid f'(\frac{3a+b}{4})\mid^q\Big)^{\frac{1}{q}}$$
$$+\Big(\mid f'(\frac{a+b}{2})\mid^q +2\mid f'(\frac{a+3b}{4})\mid^q\Big)^{\frac{1}{q}}$$
$$+\Big(2\mid f'(\frac{a+3b}{4})\mid^q +\mid f'(b)\mid^q\Big)^{\frac{1}{q}}\Big\}.$$

The main aim of this article is to establish some new generalized Hermite-Hadamard type inequalities which give an estimate between $\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}$ and $\frac{1}{b-a}\int_a^b f(x)dx$ for mappings whose derivatives in absolute value are s-convex in the second sense. As a consequence we get refinements of those results which have been established to estimate the difference between the middle and the leftmost terms in (1).

2. Main Results

We need the following lemma which deals with the simple characterization of s-convex mappings:

Lemma 1. Let $f, g : I \subset R \to R$ be a differentiable mapping on I^0, the interior of I, where $a, b \in I$ with $a < b$. If $f' \in L([a, b])$, then for $r > 1$ the following equality holds:

$$\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}-\frac{1}{b-a}\int_a^b f(u)du$$
$$=\frac{b-a}{r^2}\Big\{\int_0^1 tf'(t\frac{(r-1)a+b}{r}+(1-t)a)dt$$

$$+\Big(\frac{r-2}{2}\Big)^2\int_0^1 (t-1)f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)dt$$
$$+\Big(\frac{r-2}{2}\Big)^2\int_0^1 tf'\big(t\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2}\big)dt$$
$$+\int_0^1 (t-1)f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)dt\Big\}. \tag{5}$$

Proof. By Integration by parts and by making use of the substitution $u = t\frac{(r-1)a+b}{r} + (1-t)a$, we have

$$(a)\quad \frac{b-a}{r^2}\int_0^1 tf'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)dt$$
$$=\frac{1}{r}f\big(\frac{(r-1)a+b}{r}\big)-\frac{1}{b-a}\int_a^{\frac{(r-1)a+b}{r}} f(u)du. \tag{6}$$

Analogously, also by similar substitutions we have

$$(b)\quad \Big(\frac{r-2}{2r}\Big)^2(b-a)\int_0^1 (t-1)f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)dt$$
$$=\frac{r-2}{2r}f\big(\frac{(r-1)a+b}{r}\big)-\frac{1}{b-a}\int_{\frac{(r-1)a+b}{r}}^{\frac{a+b}{2}} f(u)du, \tag{7}$$
$$(c)\quad \Big(\frac{r-2}{2r}\Big)^2(b-a)\int_0^1 tf'\big(t\frac{(r-1)a+b}{r}+(1-t)\frac{a+b}{2}\big)dt$$
$$=\frac{r-2}{2r}f\big(\frac{a+(r-1)b}{r}\big)-\frac{1}{b-a}\int_{\frac{a+b}{2}}^{\frac{a+(r-1)b}{r}} f(u)du,$$
$$(d)\quad \Big(\frac{1}{r^2}\Big)(b-a)\int_0^1 (t-1)f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)dt$$
$$=\frac{1}{r}f\big(\frac{a+(r-1)b}{r}\big)-\frac{1}{b-a}\int_{\frac{a+(r-1)b}{r}}^{b} f(u)du. \tag{8}$$

By adding (6)-(9), we get the desired equality (5).

Theorem 2.1. *Let $f, g : I \subset R \to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $\mid f' \mid$ is s-convex in the second sense on $[a,b]$, then for $r > 1$ the following inequality holds:*

$$\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$

$$\leq \frac{b-a}{r^2(s+2)}\Big\{\frac{1}{s+1}\Big(\Big|f'(a)\Big|+\Big|f'(b)\Big|\Big)+\frac{(r-2)^2}{2(s+1)}\Big|f'(\frac{a+b}{2})\Big|$$

$$+(\frac{r^2-4r+8}{4})\Big(\Big|f'(\frac{(r-1)a+b}{r})\Big|+\Big|f'(\frac{a+(r-1)b}{r})\Big|\Big)\Big\}. \tag{9}$$

Proof. Using Lemma 1 and taking the modulus, we have

$$\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$

$$\leq \frac{b-a}{r^2}\Big\{\int_0^1 t\Big|f'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)\Big|dt$$

$$+\Big(\frac{r-2}{2}\Big)^2\int_0^1 (1-t)\Big|f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)\Big|dt$$

$$+\Big(\frac{r-2}{2}\Big)^2\int_0^1 t\Big|f'\big(t\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2}\big)\Big|dt$$

$$+\int_0^1 (1-t)\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)dt\Big|\Big\}. \tag{10}$$

Using the s-convexity of $|\ f'\ |$ in the second sense on $[a,b]$, we observe that the following inequality holds:

$$(i)\int_0^1 t\Big|f'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)\Big|dt$$

$$\leq \frac{1}{s+2}\Big|f'(\frac{(r-1)a+b}{r})\Big|+\frac{1}{(s+1)(s+2)}\Big|f'(a)\Big|. \tag{11}$$

Analogously, we also have that the following inequalities hold:

$$(ii)\int_0^1 (1-t)\Big|f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)\Big|dt$$

$$\leq \frac{1}{(s+1)(s+2)}\Big|f'(\frac{a+b}{2})\Big|+\frac{1}{s+2}\Big|f'(\frac{(r-1)a+b}{r})\Big|, \tag{12}$$

$$(iii)\int_0^1 t\Big|f'\big(t\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2}\big)\Big|dt$$

$$\leq \frac{1}{s+2}\Big|f'(\frac{a+(r-1)b}{r})\Big|+\frac{1}{(s+1)(s+2)}\Big|f'(\frac{a+b}{2})\Big|, \tag{13}$$

and

$$(iv)\int_0^1 (1-t)\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)dt$$

$$\leq \frac{1}{(s+1)(s+2)}\Big|f'(b)\Big| + \frac{1}{s+2}\Big|f'(\frac{a+(r-1)b}{r})\Big|. \tag{14}$$

By using (12)-(15) in (11), we have

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\leq \frac{b-a}{r^2}\frac{1}{s+2}\Big[\Big\{\Big|f'(\frac{(r-1)a+b}{r})\Big| + \frac{1}{s+1}\Big|f'(a)\Big|\Big\} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big\{\frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big| + \Big|f'(\frac{(r-1)a+b}{r})\Big|\Big\} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big\{\Big|f'(\frac{a+(r-1)b}{r})\Big| + \frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big|\Big\} \\
&\quad + \Big\{\frac{1}{s+1}\Big|f'(b)\Big| + \Big|f'(\frac{a+(r-1)b}{r})\Big|\Big|\Big\}\Big],
\end{aligned}$$

which implies that the inequality (10) holds.

Corollary 2.2. *Suppose all the conditions of Theorem 2.1 are satisfied. If we choose $r = 4$ and $s = 1$ in Theorem 2.1, then we have:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\leq \frac{b-a}{96}\Big[\Big\{\Big|f'(a)\Big| + 4\Big|f'(\frac{3a+b}{4})\Big| + \Big|f'(\frac{a+b}{2})\Big|\Big\} \\
&\qquad + \Big\{\Big|f'(\frac{a+b}{2})\Big| + 4\Big|f'(\frac{a+3b}{4})\Big| + \Big|f'(b)\Big|\Big\}\Big].
\end{aligned}$$

Theorem 2.3. *Let $f, g : I \subset R \to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f' \in L([a,b])$, where $a, b \in I$ with $a < b$. If $\mid f' \mid^q$ is s-convex in the second sense on $[a,b]$ for some fixed $q > 1$, then for $r > 1$ the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\leq \frac{b-a}{r^2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{s+1}\Big)^{\frac{1}{q}}\Big\{\Big(\Big|f'(a)\Big|^q + \Big|f'(\frac{(r-1)a+b}{r})\Big|^q\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big(\Big|f'(\frac{(r-1)a+b}{r})\Big|^q + \Big|f'(\frac{a+b}{2})\Big|^q\Big)^{\frac{1}{q}} \\
&\qquad + \Big(\frac{r-2}{2}\Big)^2\Big(\Big|f'(\frac{a+b}{2})\Big|^q + \Big|f'(\frac{a+(r-1)b}{r})\Big|^q\Big)^{\frac{1}{q}}
\end{aligned}$$

$$+\Big(\Big|f'(\frac{a+(r-1)b}{r})\Big|^q+\Big|f'(b)\Big|^q\Big)^{\frac{1}{q}}\Big\}. \tag{15}$$

Proof. Using Lemma 1 and using the well-known Hölder integral inequality, we have

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|\\
&=\frac{b-a}{r^2}\Big\{\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{(r-1)a+b}{r}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1(1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r})\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1 t^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'(\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}\\
&+\Big(\int_0^1(1-t)^p dt\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{16}$$

Using the s-convexity of $|\,f'\,|$ in the second sense on $[a,b]$, we observe that the following inequality holds:

$$\begin{aligned}
(i)\ &\int_0^1\Big|f'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)\Big|^q dt\\
&\le\frac{1}{s+1}\Big|f'(\frac{(r-1)a+b}{r})\Big|^q+\frac{1}{s+1}\Big|f'(a)\Big|^q,
\end{aligned} \tag{17}$$

$$\begin{aligned}
(ii)\ &\int_0^1\Big|f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)\Big|^q dt\\
&\le\frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big|^q+\frac{1}{s+1}\Big|f'(\frac{(r-1)a+b}{r})\Big|^q,
\end{aligned} \tag{18}$$

$$\begin{aligned}
(iii)\ &\int_0^1\Big|f'\big(t\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2}\big)\Big|^q dt\\
&\le\frac{1}{s+1}\Big|f'(\frac{a+(r-1)b}{r})\Big|^q+\frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big|^q,
\end{aligned} \tag{19}$$

and

$$(iv)\ \int_0^1\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)\Big|^q dt$$

$$\leq \frac{1}{s+1}\Big|f'(b)\Big|^q + \frac{1}{s+1}\Big|f'(\frac{a+(r-1)b}{r})\Big|^q. \tag{20}$$

By using (18)-(21) in (17), we get the desired result (16).

From Theorem 2.2, by using the s-convexity in the second sense of $| \ f' \ |^q$ and the fact that

$$\Sigma_{k=1}^n (u_k + v_k)^s \leq \Sigma_{k=1}^n (u_k)^s + \Sigma_{k=1}^n (v_k)^s,$$

for $u_k, v_k \geq 0, 1 \leq k \leq n, 0 < s < 1$, we have the following corollary:

Corollary 2.4. *Suppose all the conditions of Theorem 2.2 are satisfied. If we choose $r = 4$ and $s = 1$ in Theorem 2.1, then we have:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\leq \frac{b-a}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big(\frac{1}{2}\Big)^{\frac{1}{q}} \\
&\quad \times \Big\{1 + 2\big(\frac{1}{2}\big)^s + 2\big(\frac{1}{4}\big)^s + 2\big(\frac{3}{4}\big)^s\Big\}\Big\{ \mid f'(a) \mid + \mid f'(b) \mid \Big\}.
\end{aligned}$$

Theorem 2.5. *Let $f, g : I \subset R \to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $| \ f' \ |^q$ is s-convex in the second sense on $[a, b]$ for some fixed $q \geq 1$, then for $r > 1$ the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\leq \frac{b-a}{r^2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{s+2}\Big)^{\frac{1}{q}} \\
&\quad \times \Big\{\Big(\frac{1}{s+1}\Big|f'(a)\Big|^q + \Big|f'(\frac{(r-1)a+b}{r})\Big|^q\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big(\Big|f'(\frac{(r-1)a+b}{r})\Big|^q + \frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big|^q\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big(\frac{1}{s+1}\Big|f'(\frac{a+b}{2})\Big|^q + \Big|f'(\frac{a+(r-1)b}{r})\Big|^q\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{1}{s+1}\Big|f'(b)\Big|^q + \Big|f'(\frac{a+(r-1)b}{r})\Big|^q\Big)^{\frac{1}{q}}\Big\}.
\end{aligned} \tag{21}$$

Proof. Suppose that $q \geq 1$. From Lemma 1 and using the well-known power-mean inequality, we have

$$\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big|$$

$$\leq \frac{b-a}{r^2}\Big(\frac{1}{2}\Big)^{1-\frac{1}{q}}\Big\{\Big(\int_0^1 t\Big|f'(\frac{(r-1)a+b}{r}+(1-t)a)\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1 (1-t)\Big|f'(\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r})\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1 t\Big|f'(\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 (1-t)\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r})\Big|^q dt\Big)^{\frac{1}{q}}\Big\}. \tag{22}$$

Since $| f' |^q$ is s-convex in the second sense on $[a, b]$, we can observe that the following inequality holds:

$$\int_0^1 t\Big|f'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)\Big|^q dt$$
$$\leq \frac{1}{s+2}\Big|f'(\frac{(r-1)a+b}{r})\Big|^q+\frac{1}{(s+1)(s+2)}\Big|f'(a)\Big|^q. \tag{23}$$

Analogously, we also have that the following inequalities hold:

$$\int_0^1 (1-t)\Big|f'\big(t\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r})\Big|^q dt$$
$$\leq \frac{1}{(s+1)(s+2)}\Big|f'(\frac{a+b}{2})\Big|^q+\frac{1}{s+2}\Big|f'(\frac{(r-1)a+b}{r})\Big|^q, \tag{24}$$

$$\int_0^1 t\Big|f'\big(t\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2})\Big|^q dt$$
$$\leq \frac{1}{s+2}\Big|f'(\frac{a+(r-1)b}{r})\Big|^q+\frac{1}{(s+1)(s+2)}\Big|f'(\frac{a+b}{2})\Big|^q, \tag{25}$$

and

$$\int_0^1 \Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r})\Big|^q dt$$
$$\leq \frac{1}{(s+1)(s+2)}\Big|f'(b)\Big|^q+\frac{1}{s+2}\Big|f'(\frac{a+(r-1)b}{r})\Big|^q. \tag{26}$$

By using (24)-(27) in (23), we get the desired result (22) which is a generalization of Theorem 1.3.

Corollary 2.6. *Suppose all the conditions of Theorem 2.3 are satisfied. If we choose $r = 4$ and $s = 1$ in Theorem 2.3, then we have:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{3a+b}{4}) + f(\frac{a+3b}{4})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&\le \frac{b-a}{32}\Big(\frac{1}{3}\Big)^{\frac{1}{q}}\Big[\Big\{\Big|f'(a)\Big|^q + 2\Big|f'(\frac{3a+b}{4})\Big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{2\Big|f'(\frac{a+3b}{4})\Big|^q + \Big|f'(b)\Big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big|f'(\frac{a+b}{2})\Big|^q + 2\Big|f'(\frac{3a+b}{4})\Big|^q\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\Big|f'(\frac{a+b}{2})\Big|^q + 2\Big|f'(\frac{a+3b}{4})\Big|^q\Big\}^{\frac{1}{q}}\Big].
\end{aligned}$$

Theorem 2.7. *Let $f, g : I \subset R \to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f' \in L([a, b])$, where $a, b \in I$ with $a < b$. If $|\ f'\ |^q$ is s-concave in the second sense on $[a, b]$ for some fixed $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $r > 1$ the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&= \frac{b-a}{r^2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}} 2^{\frac{s-1}{q}} \\
&\times \Big\{\Big|f'\Big(\frac{(2r-1)a+b}{2r}\Big)\Big| + \Big(\frac{r-2}{2}\Big)^2\Big|f'\Big(\frac{(3r-2)a+(r+2)b}{4r}\Big)\Big| \\
&+ \Big(\frac{r-2}{2}\Big)^2\Big|f'\Big(\frac{(r+2)a+(3r-2)b}{4r}\Big)\Big| + \Big|f'\Big(\frac{a+(2r-1)b}{2r}\Big)\Big|\Big\}. \qquad (27)
\end{aligned}$$

Proof. Suppose that $q > 1$. From Lemma 1 and using the well-known Hölder integral inequality for $q > 1$, we have

$$\begin{aligned}
&\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big| \\
&= \frac{b-a}{r^2}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big(\int_0^1 \Big|f'(\frac{(r-1)a+b}{r} + (1-t)a)\Big|^q dt\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1 \Big|f'(\frac{a+b}{2} + (1-t)\frac{(r-1)a+b}{r})\Big|^q dt\Big)^{\frac{1}{q}} \\
&\quad + \Big(\frac{r-2}{2}\Big)^2\Big(\int_0^1 \Big|f'(\frac{a+(r-1)b}{r} + (1-t)\frac{a+b}{2})\Big|^q dt\Big)^{\frac{1}{q}}
\end{aligned}$$

$$+\Big(\int_0^1 \Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)\Big|^q dt\Big)^{\frac{1}{q}}\Big\}. \tag{28}$$

Since $| f' |^q$ is s-convex in the second sense on $[a,b]$, we can observe that the following inequality holds:

$$(i) \int_0^1 \Big|f'\big(t\frac{(r-1)a+b}{r}+(1-t)a\big)\Big|^q dt$$
$$\le 2^{s-1}\Big|f'\Big(\frac{(2r-1)a+b}{2r}\Big)\Big|^q. \tag{29}$$

$$(ii) \int_0^1 \Big|f'\big(\frac{a+b}{2}+(1-t)\frac{(r-1)a+b}{r}\big)\Big|^q dt$$
$$\le 2^{s-1}\Big|f'\Big(\frac{(3r-2)a+(r+2)b}{4r}\Big)\Big|^q, \tag{30}$$

$$(iii) \int_0^1 \Big|f'\big(\frac{a+(r-1)b}{r}+(1-t)\frac{a+b}{2}\big)\Big|^q dt$$
$$\le 2^{s-1}\Big|f'\Big(\frac{(r+2)a+(3r-2)b}{4r}\Big)\Big|^q, \tag{31}$$

$$(iv) \int_0^1 \Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)\Big|^q dt$$
$$\le 2^{s-1}\Big|f'\Big(\frac{a+(2r-1)b}{2r}\Big)\Big|^q. \tag{32}$$

By using (30)-(33) in (29), we get the desired result (28).

Corollary 2.8. *Suppose all the conditions of Theorem 2.4 are satisfied. If we choose $r=4$ and $s=1$ in Theorem 2.4, then we have:*

$$\Big|\frac{1}{2}\Big\{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$
$$=\frac{b-a}{16}\Big(\frac{1}{p+1}\Big)^{\frac{1}{p}}\Big\{\Big|f'\Big(\frac{7a+b}{8}\Big)\Big|+\Big|f'\Big(\frac{5a+3b}{8}\Big)\Big|$$
$$+\Big|f'\Big(\frac{3a+5b}{8}\Big)\Big|+\Big|f'\Big(\frac{a+7b}{8}\Big)\Big|\Big\}.$$

Theorem 2.9. *Let $f,g: I\subset R\to R$ be a differentiable mapping on the interior I^0 of an interval I such that $f'\in L([a,b])$, where $a,b\in I$ with $a<b$. If $| f' |^q$ is concave in the second sense on $[a,b]$ for some fixed $q\ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then for $r>1$ the following inequality holds:*

$$\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r})+f(\frac{a+(r-1)b}{r})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$

$$\leq \frac{b-a}{2r^2}\Big\{\Big|f'(\frac{(3r-2)a+2b}{3r})\Big| + \Big|f'(\frac{2a+(3r-2)b}{3r})\Big|$$
$$+ \Big(\frac{r-2}{2}\Big)^2\Big(\Big|f'(\frac{(5r-4)a+(r+4)b}{6r})\Big|$$
$$+ \Big|f'(\frac{(r+4)a+(5r-4)b}{6r})\Big|\Big)\Big\}. \quad (33)$$

Proof. By the concavity of $| f' |^q$ on $[a, b]$ and the power-mean inequality, we have that $| f' |$ is also concave on $[a, b]$. Suppose that $q \geq 1$. From Lemma 1 and using the well-known Jensen's integral inequality for $q \geq 1$, we have

$$\Big|\frac{1}{2}\Big\{f(\frac{(r-1)a+b}{r}) + f(\frac{a+(r-1)b}{r})\Big\} - \frac{1}{b-a}\int_a^b f(u)du\Big|$$
$$\leq \frac{b-a}{r^2}\Big\{\int_0^1 t\Big|f'(\frac{(r-1)a+b}{r} + (1-t)a)\Big|dt$$
$$+ \Big(\frac{r-2}{2}\Big)^2\int_0^1 (1-t)\Big|f'(\frac{a+b}{2} + (1-t)\frac{(r-1)a+b}{r})\Big|dt$$
$$+ \Big(\frac{r-2}{2}\Big)^2\int_0^1 t\Big|f'(\frac{a+(r-1)b}{r} + (1-t)\frac{a+b}{2})\Big|dt$$
$$+ \int_0^1 (1-t)\Big|f'\big(tb + (1-t)\frac{a+(r-1)b}{r}\big)\Big|dt\Big\}. \quad (34)$$

Since $| f' |^q$ is s-concave in the second sense on $[a, b]$, we can observe that the following inequality holds:

$$(i) \int_0^1 t\Big|f'\big(t\frac{(r-1)a+b}{r} + (1-t)a\big)\Big|^q dt$$
$$\leq \Big(\int_0^1 tdt\Big)\Big|f'\Big(\frac{\int_0^1 t(\frac{(r-1)a+b}{r}t + (1-t)a)dt}{\int_0^1 tdt}\Big)\Big| = \frac{1}{2}\Big|f'\Big(\frac{(3r-2)a+2b}{3r}\Big)\Big|. \quad (35)$$

Analogously, we also have that the following inequalities hold:

$$(ii) \int_0^1 (1-t)\Big|f'(\frac{a+b}{2} + (1-t)\frac{(r-1)a+b}{r})\Big|dt$$
$$\leq \frac{1}{2}\Big|f'\Big(\frac{(5r-4)a+(r+4)b}{6r}\Big)\Big|, \quad (36)$$
$$(iii) \int_0^1 t\Big|f'(\frac{a+(r-1)b}{r} + (1-t)\frac{a+b}{2})\Big|dt$$
$$\leq \frac{1}{2}\Big|f'\Big(\frac{(r+4)a+(5r-4)b}{6r}\Big)\Big|, \quad (37)$$

and

$$(iv)\int_0^1 (1-t)\Big|f'\big(tb+(1-t)\frac{a+(r-1)b}{r}\big)\Big|dt$$
$$\leq \frac{1}{2}\Big|f'\Big(\frac{2a+(3r-2)b}{3r}\Big)\Big|. \tag{38}$$

By using (36)-(39) in (35), we get the desired result (34).

Corollary 2.10. *Suppose all the conditions of Theorem 2.5 are satisfied. If we choose $r=4$, then we have:*

$$\Big|\frac{1}{2}\Big\{f(\frac{3a+b}{4})+f(\frac{a+3b}{4})\Big\}-\frac{1}{b-a}\int_a^b f(u)du\Big|$$
$$\leq \frac{b-a}{32}\Big\{\Big|f'\Big(\frac{13a+3b}{12}\Big)\Big|+\Big|f'\Big(\frac{11a+5b}{12}\Big)\Big|$$
$$+\Big|f'\Big(\frac{5a+13b}{12}\Big)\Big|+\Big|f'\Big(\frac{3a+13b}{12}\Big)\Big|\Big\}.$$

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