SIMPSON-LIKE TYPE INEQUALITIES FOR CO-ORDINATED \((s_1, s_2)\)-CONVEX MAPPINGS IN THE SECOND SENSE

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Abstract: In this article, a new generalized identity for partial differentiable mappings on a bidimensional interval is derived. By using this equality, the author establish the generalizations of the Simpson-like type inequalities for differentiable co-ordinated \((s_1, s_2)\)-convex mappings in the second sense on the rectangle from the plain.

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1. Introduction

The following definition is well-known in the literature as Simpson’s inequality:

**Theorem 1.1.** Let \( f : I = [a, b] \to R \) be a four times continuously differentiable mapping on \([a, b]\) and \( \| f^{(4)} \|_{\infty} = \sup_{x \in [a, b]} | f^{(4)}(x) | < \infty \). Then the following inequality holds:

\[
\left| \frac{1}{3} \left[ \frac{f(a) + f(b)}{2} + 2f \left( \frac{a + b}{2} \right) \right] - \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \right| \leq \frac{1}{2880} \| f^{(4)} \|_{\infty} (b - a)^4.
\]
For recent results on Simpson type inequalities, you may see the papers [1, 4, 5, 6, 7, 15]. Özdemir [7, 8] and Park [9, 10, 13, 14] considered the class of mappings which are $s$-convex on the co-ordinates.

In sequel, in this article let $\Delta = [a, b] \times [c, d]$ be a bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$.

**Definition 1.1.** A mapping $f : \Delta \to \mathbb{R}$ is said to be $s$-convex in the second sense on $\Delta$ if the following inequality

$$f(t x + (1 - t) z, t y + (1 - t) w) \leq t^s f(x, y) + (1 - t)^s f(z, w)$$

holds, for all $(x, y), (z, w) \in \Delta$ and $s, t \in [0, 1]$.

Modifications for convex and $s$-convex mappings on $\Delta$, which are also known as co-ordinated convex and co-ordinated $s$-convex mappings on $\Delta$, respectively, were introduced by Dragomir [4, 5], Sarykaya [15] and Latif [3, 4, 5, 6].

**Definition 1.2.** A mapping $f : \Delta \to \mathbb{R}^2$ will be called co-ordinated $s$-convex in the second sense on $\Delta$ if the partial mappings $f_y : [a, b] \to \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c, d] \to \mathbb{R}, f_x(v) = f(x, v)$ are $s$-convex in the second sense, for all $x \in [a, b], y \in [c, d]$ and $s \in [0, 1]$.

A formal definition for co-ordinated $s$-convex mappings in the second sense may be stated as follows [7, 12, 13, 15]:

**Definition 1.3.** A mapping $f : \Delta \to \mathbb{R}^2$ will be called co-ordinated $s$-convex in the second sense on $\Delta$ if the following inequality

$$f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \leq t^s \lambda^s f(x, y) + (1 - t)^s \lambda^s f(z, y) + t^s(1 - \lambda)^s f(x, w) + (1 - t)^s(1 - \lambda)^s f(z, w)$$

holds, for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$, and for fixed $s \in (0, 1]$.

If we choose $s = 1$ in (2), the mapping $f$ is said to be co-ordinated convex on $\Delta$. The inequality in (2) is in reversed order if $f$ a co-ordinated $s$-concave in the second sense on $\Delta$.

In [3], Dragomir proved the following Hermite-Hadamard type inequality for co-ordinated convex mappings on the rectangle from the plane:
Theorem 1.2. If \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) is a co-ordinated convex partial differentiable mapping on a bidimensional interval \( \Delta = [a, b] \times [c, d] \) with \( a < b \) and \( c < d \), then the following inequalities hold:

\[
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy \, dx \leq \frac{1}{4} \left\{ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right\}. \tag{3}
\]

Theorem 1.3. Let \( f : \Delta \subseteq \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on a bidimensional interval \( \Delta = [a, b] \times [c, d] \) with \( a < b \) and \( c < d \), and \( r, t \in [0, 1] \). If \( \frac{\partial^2 f}{\partial r \partial t} \) is convex on the co-ordinates on \( \Delta \), then the following inequality holds:

\[
\left| \left| f\left(\frac{a}{2}, \frac{c+d}{2}\right) + f\left(\frac{b}{2}, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) \right| \right| \\
+ f\left(\frac{a+b}{2}, d\right) \right\} + \frac{1}{36} \left\{ f(a, c) + f(b, c) + f(a, d) + f(b, d) \right\} \right| dx \\
- \frac{1}{6(b-a)} \int_a^b \left\{ f\left(\frac{a+b}{2}, y\right) + 4f\left(\frac{a+b}{2}, y\right) + f\left(\frac{a+b}{2}, y\right) \right\} dy \\
+ \frac{1}{6(d-c)} \int_c^d \left\{ f(x, \frac{a+b}{2}) + f(x, \frac{c+d}{2}) + f(x, \frac{a+b}{2}) \right\} dy \\
= \left( \frac{5}{72} \right)^2 (b-a)(d-c) \\
\times \left\{ \left| \frac{\partial^2}{\partial r \partial t} f(a, c) \right| + \left| \frac{\partial^2}{\partial r \partial t} f(a, d) \right| + \left| \frac{\partial^2}{\partial r \partial t} f(b, c) \right| + \left| \frac{\partial^2}{\partial r \partial t} f(b, d) \right| \right\}.
\]

In this article, firstly let us give the following definition:

Definition 1.4. A mapping \( f : \Delta \to \mathbb{R}^2 \) will be called co-ordinated \((s_1, s_2)\)-convex in the second sense on \( \Delta \) if the following inequality

\[
f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\
\leq t^{s_1} \lambda^{s_2} f(x, y) + (1-t)^{s_1} \lambda^{s_2} f(z, y) \\
+ t^{s_1} (1-\lambda)^{s_2} f(x, w) + (1-t)^{s_1} (1-\lambda)^{s_2} f(z, w) \tag{4}
\]

holds, for all \( t, \lambda \in [0, 1] \) and \( (x, y), (z, w) \in \Delta \), and for fixed \( s_1, s_2 \in (0, 1] \).

Özdemir et. al. \cite{7, 8} and Park \cite{9, 10, 11, 12, 13, 14} establish a generalizations of Simpson-like type inequalities for co-ordinated \( s \)-convex mappings in the second sense.
In this article, a new generalized identity for partial differentiable mappings on a bidimensional interval is derived. By using this equality the author establish the generalizations of Simpson-like type inequalities for co-ordinated \((s_1, s_2)\)-convex in the second sense on a bidimensional interval.

2. Main Results

The following lemma is necessary and plays an important role in establishing our main results:

**Lemma 1.** Let \( f : \Delta \to R \) be a partial differentiable mapping on a bidimensional interval \( \Delta = [a, b] \times [c, d] \) where \( a < b \) and \( c < d \). If \( \frac{\partial^2 f}{\partial t \partial \lambda} \) is in \( L(\Delta) \), then, for \( r_1, r_2 \geq 2 \) and \( h_1, h_2 \in (0, 1) \) with \( \frac{1}{r_1} \leq h_1 \leq \frac{r_1 - 1}{r_1} \) and \( \frac{1}{r_2} \leq h_2 \leq \frac{r_2 - 1}{r_2} \), the following identity holds:

\[
I(f)(h_1, h_2; r_1, r_2) = \left( \frac{r_1 - 2}{r_1} - 2 \right) \left( \frac{r_2 - 2}{r_2} \right) f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
+ \frac{r_2 - 2}{r_1 r_2} \left\{ f(h_1 a + (1 - h_1)b, c) + f(h_1 a + (1 - h_1)b, d) \right\} \\
+ \frac{r_1 - 2}{r_1 r_2} \left\{ f(a, h_2 c + (1 - h_2)d) + f(b, h_2 c + (1 - h_2)d) \right\} \\
- \frac{1}{(b-a)r_2} \int_a^b \left\{ f(x, c) + (r_2 - 2)f(x, h_2 c + (1 - h_2)d) + f(x, d) \right\} dx \\
- \frac{1}{(d-c)r_1} \int_c^d \left\{ f(a, y) + (r_1 - 2)f(h_1 a + (1 - h_1)b, y) + f(b, y) \right\} dy \\
+ \frac{1}{r_1 r_2} \left\{ f(a, c) + f(a, d) + f(b, c) + f(b, d) \right\} \\
+ \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dxdy \\
= (b - a)(d - c) \int_0^1 \int_0^1 p(h_1, r_1, t) p(h_2, r_2, \lambda) \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1 - t)b, \lambda c + (1 - \lambda)d) dtd\lambda, \tag{5}
\]

where

\[ p(h, r, z) = \begin{cases} 
  z - \frac{1}{r} & z \in [0, h] \\
  z - \frac{r-1}{r} & z \in (h, 1].
\end{cases} \]
Proof. Using the integration by parts, we get the desired result by substituting \( x = ta + (1 - t)b \) and \( y = \lambda c + (1 - \lambda)d \) for \((\lambda, t) \in [0, 1]^2\) and multiplying both sides with \((b - a)(d - c)\).

Remark 1. Lemma 1 is a generalization of the results which proved by M. Z. Sarikaya [15], J. Pecaric [2], M. E. Özdemir [7, 8] and S. S. Dragomir [4, 15].

Theorem 2.1. Let \( f : \Delta \subset R^2 \rightarrow R \) be a partial differentiable mapping on \( \Delta \). If \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right| \) is in \( L_1(\Delta) \) and is a co-ordinated \((s_1, s_2)\)-convex mapping in the second sense on \( \Delta \), then, for \( r_1, r_2 \geq 2 \) and \( h_1, h_2 \in (0, 1) \) with \( \frac{1}{r_1} \leq h_1 \leq \frac{r_1 - 1}{r_1} \) and \( \frac{1}{r_2} \leq h_2 \leq \frac{r_2 - 1}{r_2} \), the following inequality holds:

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2; r_1, r_2) \right| \\
\leq \left\{ (\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(h_2, r_2, s_2) \right\} \\
\times \left\{ \left( (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(h_1, r_1, s_1) \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| \\
+ \left( (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(1 - h_2, r_2, s_2) \right) \right\} \\
\times \left\{ \left( (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(h_1, r_1, s_1) \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \\
+ \left( (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(1 - h_1, r_1, s_1) \right) \right\},
\]

(6)

where

\[
\mu_1(h, r, s) = \frac{1}{(s + 1)(s + 2)r^{s+2}},
\]

\[
\mu_2(h, r, s) = \frac{1 + (hr)^{s+1}\{hr(s + 1) - (s + 2)\}}{(s + 1)(s + 2)r^{s+2}},
\]

\[
\mu_3(h, r, s) = \frac{(r - 1)^{s+2} + (hr)^{s+1}\{hr(1 + s) - (s + 2)(r - 1)\}}{(s + 1)(s + 2)r^{s+2}},
\]

\[
\mu_4(h, r, s) = \frac{(r - 1)^{s+2} + r^{s+1}\{s + 2 - r\}}{(s + 1)(s + 2)r^{s+2}}.
\]

Proof. From Lemma 1, we can write

\[
\frac{1}{(b - a)(d - c)} \left| I(f)(h_1, h_2; r_1, r_2) \right|
\]
\[
\leq \int_0^1 \int_0^1 \left| p(h_1, r_1, t)p(h_2, r_2, \lambda) \right| \times \left| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right| dt d\lambda.
\]

Since \( \frac{\partial^2 f}{\partial t \partial \lambda} \) is in \( L_1(\Delta) \) and is a co-ordinated \((s_1, s_2)\)-convex mapping in the second sense on \( \Delta \), the following inequality

\[
\frac{\partial^2 f}{\partial t \partial \lambda} f(ta + (1 - t)b, \lambda c + (1 - \lambda)d) \\
\leq t^{s_1} \lambda^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right| + (1 - t)^{s_1} \lambda^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \\
+ t^{s_1} (1 - \lambda)^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \\
+ (1 - t)^{s_1} (1 - \lambda)^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right| 
\]

holds, for all \( t, \lambda \in [0, 1] \) and for fixed \( s_1, s_2 \in (0, 1] \).

By the inequalities (7) and (8), we have

\[
\left| \int_0^1 \left[ t \int_0^1 p(h_1, r_1, t) \left| t^{s_1} \lambda^{s_2} d\lambda \right| \right] \frac{\partial^2 f}{\partial t \partial \lambda} (a, \lambda) \right| dt \\
+ (1 - t)^{s_1} \lambda^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, \lambda) \right| \\
+ (1 - t)^{s_1} (1 - \lambda)^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right| \right| d\lambda \right| dt \\
+ \left( \int_0^1 \left| p(h_1, r_1, t) \right| (1 - t)^{s_1} dt \left( \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1 - \lambda)^{s_2} d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right| \\
+ \left( \int_0^1 \left| p(h_1, r_1, t) \right| (1 - t)^{s_1} dt \right) \right| \right| d\lambda \right| \\
\times \left( \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1 - \lambda)^{s_2} d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|. \tag{9}
\]

By the simple calculations, we have the following equalities:

\[
(a) \int_0^1 \left| p(h_1, r_1, t) \right| t^{s_1} dt
\]
\[
= \mu_1(0, r_1, s_1) + \mu_2(h_1, r_1, s_1) + \mu_3(h_1, r_1, s_1) + \mu_4(0, r_1, s_1),
\]

\((b)\) \[
\int_0^1 p(h_2, r_2, \lambda) \lambda^s d\lambda
= \mu_1(0, r_2, s_2) + \mu_2(h_2, r_2, s_2) + \mu_3(h_2, r_2, s_2) + \mu_4(0, r_2, s_2),
\]

\((c)\) \[
\int_0^1 p(h_1, r_1, t) (1 - t)^s d\lambda
= \mu_1(0, r_1, s_1) + \mu_2(1 - h_1, r_1, s_1) + \mu_3(1 - h_1, r_1, s_1) + \mu_4(0, r_1, s_1),
\]

\((d)\) \[
\int_0^1 p(h_2, r_2, \lambda) \lambda^s d\lambda
= \mu_1(0, r_2, s_2) + \mu_2(h_2, r_2, s_2) + \mu_3(h_2, r_2, s_2) + \mu_4(0, r_2, s_2),
\]

\((e)\) \[
\int_0^1 p(h_1, r_1, t) t^s d\lambda
= \mu_1(0, r_1, s_1) + \mu_2(h_1, r_1, s_1) + \mu_3(h_1, r_1, s_1) + \mu_4(0, r_1, s_1),
\]

\((f)\) \[
\int_0^1 p(h_2, r_2, \lambda) (1 - \lambda)^s d\lambda
= \mu_1(0, r_2, s_2) + \mu_2(1 - h_2, r_2, s_2) + \mu_3(1 - h_2, r_2, s_2) + \mu_4(0, r_2, s_2),
\]

\((g)\) \[
\int_0^1 p(h_1, r_1, t) (1 - t)^s d\lambda
= \mu_1(0, r_1, s_1) + \mu_2(1 - h_1, r_1, s_1) + \mu_3(1 - h_1, r_1, s_1) + \mu_4(0, r_1, s_1),
\]

\((h)\) \[
\int_0^1 p(h_2, r_2, \lambda) (1 - \lambda)^s d\lambda
= \mu_1(0, r_2, s_2) + \mu_2(1 - h_2, r_2, s_2) + \mu_3(1 - h_2, r_2, s_2) + \mu_4(0, r_2, s_2).
\]

(10)

By (9) and (10), we have the desired result.

**Corollary 2.2.** In Theorem 2.1,

\((a)\) if we choose \(r_1 = r_2 = 2, h_1 = h_2 = \frac{1}{2}\) and \(s_1 = s_2 = 1\), then we have

\[
\frac{64}{(b - a)(d - c)} \left| I(f)(\frac{1}{2}, \frac{1}{2}; 2, 2) \right|_{s_1 = s_2 = 1}
\leq \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\},
\]

which implies that Lemma 1 is a generalization of Lemma 1.2 in [4].
(b) if we choose \( r_1 = r_2 = 6, \ h_1 = h_2 = \frac{1}{2} \) and \( s_1 = s_2 = 1, \) then we have

\[
\frac{(\frac{72}{5})^2}{(b-a)(d-c)} \left| I(f)(\frac{1}{2}, \frac{1}{2}; 6, 6) \right|_{s_1=s_2=1} \leq \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\},
\]

which implies that Theorem 2.1 is a generalization of Theorem 1.2 in [7].

**Theorem 2.3.** Let \( f : \Delta \subset \mathbb{R}^2 \to \mathbb{R} \) be a partial differentiable mapping on a bidimensional interval \( \Delta = [a, b] \times [c, d] \) in \( \mathbb{R}^2 \) with \( a < b \) and \( c < d, \) and \( \lambda, t \in [0, 1] \). For \( q > 1 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \) if \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q \) is in \( L(\Delta) \) and is a co-ordinated \((s_1, s_2)\)-convex mapping on \( \Delta, \) then, for \( r_1, r_2 \geq 2 \) and \( h_1, h_2 \in [0, 1] \) with \( \frac{1}{r_1} \leq h_1 \leq \frac{r_1-1}{r_1} \) and \( \frac{1}{r_2} \leq h_2 \leq \frac{r_2-1}{r_2}, \) the following inequality holds:

\[
\left| \frac{1}{(b-a)(d-c)} I(f)(h_1, h_2; r_1, r_2) \right| \leq \mu_5^{\frac{1}{p}}(h_1, r_1)\mu_5^{\frac{1}{p}}(h_2, r_2) \left\{ \frac{1}{(s_1+1)(s_2+1)} \right\}^{\frac{1}{q}} \times \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\}^{\frac{1}{q}},
\]

(11)

where

\[
\mu_5(h, r) = \frac{2 + (r - rh - 1)^{p+1} + (rh - 1)^{p+1}}{r^{p+1}(p+1)}.
\]

**Proof.** From Lemma 1 and using the Hölder inequality for double integrals, we obtain

\[
\left| \frac{1}{(b-a)(d-c)} I(f)(h_1, h_2; r_1, r_2) \right| \leq \left\{ \int_0^1 \int_0^1 \left| p(h_1, r_1, t) p(h_2, r_2, \lambda \right| \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \right\}^{\frac{1}{p}} \times \left\{ \int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt d\lambda \right\}^{\frac{1}{q}}.
\]

(12)
Since $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q$ is an $(s_1, s_2)$-convex mapping on the co-ordinates on $\Delta$ for $q > 1$, we have

$$
\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda} (tb + (1 - t)a, \lambda d + (1 - \lambda)c) \right|^q dtd\lambda \\
\leq \int_0^1 \int_0^1 \left\{ t^{s_1} \lambda^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q + t^{s_1} (1 - \lambda)^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q \\
+ (1 - t)^{s_1} \lambda^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|^q + (1 - t)^{s_1} (1 - \lambda)^{s_2} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|^q \right\} dtd\lambda \\
= \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial \lambda} (b, d) \right|^q \right\} \\
= \left( \frac{1}{(s_1 + 1)(s_2 + 1)} \right) (s_1 + 1)(s_2 + 1).
$$

Note that

$$
\int_0^1 \left| p(h_1, r_1, t) \right|^p dt = \mu_5(h_1, r_1), \\
\int_0^1 \left| p(h_2, r_2, \lambda) \right|^p d\lambda = \mu_5(h_2, r_2).
$$

The substitution of (13) and (14) in (12) gives the desired result.

**Theorem 2.4.** Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on a bidimensional interval $\Delta = [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$ and $t, \lambda \in [0, 1]$. If $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q$ is in $L(\Delta)$ and is a co-ordinated $(s_1, s_2)$-convex mapping on $\Delta$, for $q \geq 1$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in [0, 1]$ with $\frac{1}{n_1} \leq h_1 \leq \frac{n_1 - 1}{n_1}$ and $\frac{1}{n_2} \leq h_2 \leq \frac{n_2 - 1}{n_2}$, the following inequality holds:

$$
\left| \frac{1}{(b - a)(d - c)} I(f)(h_1, h_2; r_1, n_r) \right| \\
\leq \left\{ \left( \frac{2h_1^2 - 2h_1 + 1}{2} + \frac{2 - r_1}{r_1^2} \right) \left( \frac{2h_2^2 - 2h_2 + 1}{2} + \frac{2 - r_2}{r_2^2} \right) \right\}^{1 - \frac{1}{q}} \\
\times \left\{ \left( (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_1 + \mu_4)(h_1, r_1, s_1) \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, c) \right|^q \right. \\
+ \left\{ (\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(1 - h_2, r_2, s_2) \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda} (a, d) \right|^q \\
+ \left\{ (\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(1 - h_1, r_1, s_1) \right\}
$$
\[
\times \left\{ \{(\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(h_2, r_2, s_2)\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| \right\}^{q} \frac{1}{q}
\]
\[
+ \left\{ (\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(1 - h_2, r_2, s_2) \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^{q} \frac{1}{q},
\]
where \( \mu_i(r, s, h)(i = 1, 2, 3, 4) \) are defined as in Theorem 2.1.

**Proof.** From Lemma 1 and using the power mean inequality for double integrals, we obtain

\[
\left| \frac{1}{(b - a)(d - c)} I(f)(h_1, h_2; n_1, n_2) \right|
\]
\[
\leq \int_0^1 \int_0^1 \left| p(h_1, r_1, t) p(h_2, r_2, \lambda) \right| dt d\lambda \\
\times \left\{ \int_0^1 \int_0^1 \left| p(h_1, r_1, t) p(h_2, r_2, \lambda) \right| dt d\lambda \right\}^{1 - \frac{1}{q}}
\]
\[
\times \left\{ \int_0^1 \int_0^1 \left| p(h_1, r_1, t) p(h_2, r_2, \lambda) \right| dt d\lambda \right\}^{\frac{1}{q}}
\]
\[
\times \left\{ \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right\}^{q} dt d\lambda \right\}^{\frac{1}{q}}.
\]

Since \( \left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^{q} \) is in \( L(\Delta) \) and is a co-ordinated \((s_1, s_2)\)-convex mapping on \( \Delta \) for \( q \geq 1 \), we have

\[
\int_0^1 \int_0^1 \left| p(h_1, r_1, t) p(h_2, r_2, \lambda) \right| \times \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1 - t)b, \lambda c + (1 - \lambda)d) \right|^{q} dt d\lambda
\]
\[
+ \left\{ \int_0^1 \left| p(h_1, r_1, t) \right|^{q} dt \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right|^{q} d\lambda \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1 - \lambda)^{s_2} d\lambda \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^{q}
\]
\[
= \left\{ \int_0^1 \left| p(h_1, r_1, t) \right|^{q} dt \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right|^{q} d\lambda \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1 - \lambda)^{s_2} d\lambda \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^{q}
\]
\[
\left\{ \int_0^1 \left| p(h_1, r_1, t) \right|^{q} dt \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right|^{q} d\lambda \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1 - \lambda)^{s_2} d\lambda \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^{q}
\]
\[
\begin{align*}
&\quad + \left\{ \int_0^1 \left| p(h_1, r_1, t) \right| (1-t)^{s_1} dt \right\} \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right| \lambda^{s_2} d\lambda \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q \\
&\quad + \left\{ \int_0^1 \left| p(h_1, r_1, t) \right| (1-t)^{s_1} dt \right\} \\
&\quad \times \left\{ \int_0^1 \left| p(h_2, r_2, \lambda) \right| (1-\lambda)^{s_2} d\lambda \right\} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q.
\end{align*}
\tag{16}
\]

By noting that
\[
\int_0^1 \left| p(h, r, z) \right| dz = \frac{2h^2 - 2h + 1}{2} + \frac{2 - r}{r^2},
\]
we have
\[
\begin{align*}
\int_0^1 \left| p(h_1, r_1, t) \right| dt &= \frac{2h_1^2 - 2h_1 + 1}{2} + \frac{2 - r_1}{r_1^2}, \\
\int_0^1 \left| p(h_2, r_2, \lambda) \right| d\lambda &= \frac{2h_2^2 - 2h_2 + 1}{2} + \frac{2 - r_2}{r_2^2}.
\end{align*}
\tag{17}
\]

By the substitutions of (10),(16) and (17) in (15), we have the desired result.

References


