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SIMPSON-LIKE TYPE INEQUALITIES FOR CO-ORDINATED (s_1, s_2)-CONVEX MAPPINGS IN THE SECOND SENSE

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Abstract: In this article, a new generalized identity for partial differentiable mappings on a bidimensional interval is derived. By using this equality, the author establish the generalizations of the Simpson-like type inequalities for differentiable co-ordinated (s_1, s_2)-convex mappings in the second sense on the rectangle from the plain.

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1. Introduction

The following definition is well-known in the literature as Simpson's inequality:

Theorem 1.1. *Let $f : I = [a, b] \to R$ be a four times continuously differentiable mapping on $[a, b]$ and $\| f^{(4)} \|_\infty = \sup_{x\in[a,b]} | f^{(4)}(x) | < \infty$. Then the following inequality holds:*

$$\Big|\frac{1}{3}\Big[\frac{f(a)+f(b)}{2}+2f(\frac{a+b}{2})\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Big| \leq \frac{1}{2880} \| f^{(4)} \|_\infty (b-a)^4.$$

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For recent results on Simpson type inequalities, you may see the papers [1, 4, 5, 6, 7, 15]. Özdemir [7, 8] and Park [9, 10, 13, 14] considered the class of mappings which are s-convex on the co-ordinates.

In sequel, in this article let $\Delta = [a, b] \times [c, d]$ be a bidimensional interval in R^2 with $a < b$ and $c < d$.

Definition 1.1. A mapping $f : \Delta \to R$ is said to be s-convex in the second sense on Δ if the following inequality

$$f(tx + (1-t)z, ty + (1-t)w) \leq t^s f(x, y) + (1-t)^s f(z, w) \tag{1}$$

holds, for all $(x, y), (z, w) \in \Delta$ and $s, t \in [0, 1]$.

Modifications for convex and s-convex mappings on Δ, which are also known as co-ordinated convex and co-ordinated s-convex mappings on Δ, respectively, were introduced by Dragomir [4, 5], Sarykaya [15] and Latif [3, 4, 5, 6].

Definition 1.2. A mapping $f : \Delta \to R^2$ will be called co-ordinated s-convex in the second sense on Δ if the partial mappings

$$f_y : [a, b] \to R, \quad f_y(u) = f(u, y)$$

and

$$f_x : [c, d] \to R, \quad f_x(v) = f(x, v)$$

are s-convex in the second sense, for all $x \in [a, b], y \in [c, d]$ and $s \in [0, 1]$.

A formal definition for co-ordinated s-convex mappings in the second sense may be stated as follows [7, 12, 13, 15]:

Definition 1.3. A mapping $f : \Delta \to R^2$ will be called co-ordinated s-convex in the second sense on Δ if the following inequality

$$\begin{aligned} & f(tx + (1-t)z, \lambda y + (1-\lambda)w) \\ & \leq t^s \lambda^s f(x, y) + (1-t)^s \lambda^s f(z, y) \\ & \qquad + t^s (1-\lambda)^s f(x, w) + (1-t)^s (1-\lambda)^s f(z, w) \end{aligned} \tag{2}$$

holds, for all $t, \lambda \in [0, 1]$ and $(x, y), (z, w) \in \Delta$, and for fixed $s \in (0, 1]$.

If we choose $s = 1$ in (2), the mapping f is said to be co-ordinated convex on Δ. The inequality in (2) is in reversed order if f a co-ordinated s-concave in the second sense on Δ.

In [3], Dragomir proved the following Hermite-Hadamard type inequality for co-ordinated convex mappings on the rectangle from the plane:

Theorem 1.2. *If $f : \Delta \subseteq R^2 \to R$ is a co-ordinated convex partial differentiable mapping on a bidimensional interval $\Delta = [a,b] \times [c,d]$ with $a < b$ and $c < d$, then the following inequalities hold:*

$$f(\frac{a+b}{2}, \frac{c+d}{2}) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx$$
$$\leq \frac{1}{4}\{f(a,c) + f(a,d) + f(b,c) + f(b,d)\}. \qquad (3)$$

Theorem 1.3. *Let $f : \Delta \subseteq R^2 \to R$ be a partial differentiable mapping on a bidimensional interval $\Delta = [a,b] \times [c,d]$ with $a < b$ and $c < d$, and $r,t \in [0,1]$. If $\frac{\partial^2 f}{\partial t \partial r}$ is convex on the co-ordinates on Δ, then the following inequality holds:*

$$\Big|\frac{1}{9}\Big\{f(a, \frac{c+d}{2}) + f(b, \frac{c+d}{2}) + 4f(\frac{a+b}{2}, \frac{c+d}{2}) + f(\frac{a+b}{2}, c)$$
$$+ f(\frac{a+b}{2}, d)\Big\} + \frac{1}{36}\Big\{f(a,c) + f(b,c) + f(a,d) + f(b,d)\Big\}$$
$$- \frac{1}{6(b-a)} \int_a^b \Big\{f(x,c) + 4f(x, \frac{c+d}{2}) + f(x,d)\Big\}dx$$
$$- \frac{1}{6(d-c)} \int_c^d \Big\{f(a,y) + 4f(\frac{a+b}{2}, y) + f(b,y)\Big\}dy$$
$$+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)dydx\Big|$$
$$= (\frac{5}{72})^2(b-a)(d-c)$$
$$\times \Big\{\Big|\frac{\partial^2}{\partial r \partial t} f(a,c)\Big| + \Big|\frac{\partial^2}{\partial r \partial t} f(a,d)\Big| + \Big|\frac{\partial^2}{\partial r \partial t} f(b,c)\Big| + \Big|\frac{\partial^2}{\partial r \partial t} f(b,d)\Big|\Big\}.$$

In this article, firstly let us give the following definition:

Definition 1.4. A mapping $f : \Delta \to R^2$ will be called co-ordinated (s_1, s_2)-convex in the second sense on Δ if the following inequality

$$f(tx + (1-t)z, \lambda y + (1-\lambda)w)$$
$$\leq t^{s_1}\lambda^{s_2} f(x,y) + (1-t)^{s_1}\lambda^{s_2} f(z,y)$$
$$+ t^{s_1}(1-\lambda)^{s_2} f(x,w) + (1-t)^{s_1}(1-\lambda)^{s_2} f(z,w) \qquad (4)$$

holds, for all $t, \lambda \in [0,1]$ and $(x,y),(z,w) \in \Delta$, and for fixed $s_1, s_2 \in (0,1]$.

Özdemir et. al. [7, 8] and Park [9, 10, 11, 12, 13, 14] establish a generalizations of Simpson-like type inequalities for co-ordinated s-convex mappings in the second sense.

In this article, a new generalized identity for partial differentiable mappings on a bidimensional interval is derived. By using this equality the author establish the generalizations of Simpson-like type inequalities for co-ordinated (s_1, s_2)-convex in the second sense on a bidimensional interval.

2. Main Results

The following lemma is necessary and plays an important role in establishing our main results:

Lemma 1. Let $f : \Delta \to R$ be a partial differentiable mapping on a bidimensional interval $\Delta = [a, b] \times [c, d]$ where $a < b$ and $c < d$. If $\frac{\partial^2 f}{\partial t \partial \lambda}$ is in $L(\Delta)$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $\frac{1}{r_1} \leq h_1 \leq \frac{r_1 - 1}{r_1}$ and $\frac{1}{r_2} \leq h_2 \leq \frac{r_2 - 1}{r_2}$, the following identity holds:

$$
\begin{aligned}
&I(f)(h_1, h_2; r_1, r_2) \\
&\equiv \frac{(r_1 - 2)(r_2 - 2)}{r_1 r_2} f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
&\quad + \frac{r_2 - 2}{r_1 r_2}\Big\{f(h_1 a + (1 - h_1)b, c) + f(h_1 a + (1 - h_1)b, d)\Big\} \\
&\quad + \frac{r_1 - 2}{r_1 r_2}\Big\{f(a, h_2 c + (1 - h_2)d) + f(b, h_2 c + (1 - h_2)d)\Big\} \\
&\quad - \frac{1}{(b - a) r_2} \int_a^b \Big\{f(x, c) + (r_2 - 2) f(x, h_2 c + (1 - h_2)d) + f(x, d)\Big\} dx \\
&\quad - \frac{1}{(d - c) r_1} \int_c^d \Big\{f(a, y) + (r_1 - 2) f(h_1 a + (1 - h_1)b, y) + f(b, y)\Big\} dy \\
&\qquad + \frac{1}{r_1 r_2}\Big\{f(a, c) + f(a, d) + f(b, c) + f(b, d)\Big\} \\
&\qquad + \frac{1}{(b - a)(d - c)} \int_c^d \int_a^b f(x, y)\, dx dy \\
&\qquad = (b - a)(d - c) \int_0^1 \int_0^1 p(h_1, r_1, t) p(h_2, r_2, \lambda) \\
&\qquad\qquad \times \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1 - t)b, \lambda c + (1 - \lambda)d)\, dt d\lambda, \qquad (5)
\end{aligned}
$$

where

$$
p(h, r, z) = \begin{cases} z - \frac{1}{r} & z \in [0, h] \\ z - \frac{r - 1}{r} & z \in (h, 1]. \end{cases}
$$

Proof. Using the integration by parts, we get the desired result by substituting $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for $(\lambda, t) \in [0,1]^2$ and multiplying both sides with $(b-a)(d-c)$.

Remark 1. Lemma 1 is a generalization of the results which proved by M. Z. Sarikaya [15], J. Pecaric [2], M. E. Özdemir [7, 8] and S. S. Dragomir [4, 15].

Theorem 2.1. *Let $f : \Delta \subset R^2 \to R$ be a partial differentiable mapping on Δ. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|$ is in $L_1(\Delta)$ and is a co-ordinated (s_1, s_2)-convex mapping in the second sense on Δ, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0,1)$ with $\frac{1}{r_1} \leq h_1 \leq \frac{r_1-1}{r_1}$ and $\frac{1}{r_2} \leq h_2 \leq \frac{r_2-1}{r_2}$, the following inequality holds:*

$$
\begin{aligned}
&\frac{1}{(b-a)(d-c)}\Big|I(f)(h_1,h_2;r_1,r_2)\Big| \\
&\leq \Bigg[\Big\{(\mu_1+\mu_4)(0,r_2,s_2)+(\mu_2+\mu_3)(h_2,r_2,s_2)\Big\} \\
&\quad\times\Bigg\{\Big((\mu_1+\mu_4)(0,r_1,s_1)+(\mu_2+\mu_3)(h_1,r_1,s_1)\Big)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\right| \\
&\qquad+\Big((\mu_1+\mu_4)(0,r_1,s_1)+(\mu_1+\mu_4)(1-h_1,r_1,s_1,)\Big)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|\Bigg\} \\
&\quad+\Big\{(\mu_1+\mu_4)(0,r_2,s_2)+(\mu_2+\mu_3)(1-h_2,r_2,s_2)\Big\} \\
&\quad\times\Bigg\{\Big((\mu_1+\mu_4)(0,r_1,s_1)+(\mu_2+\mu_3)(h_1,r_1,s_1)\Big)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\right| \\
&\qquad+\Big((\mu_1+\mu_4)(0,r_1,s_1)+(\mu_2+\mu_3)(1-h_1,r_1,s_1)\Big)\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|\Bigg\}\Bigg], \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_1(h,r,s) &= \frac{1}{(s+1)(s+2)r^{s+2}}, \\
\mu_2(h,r,s) &= \frac{1+(hr)^{s+1}\{hr(s+1)-(s+2)\}}{(s+1)(s+2)r^{s+2}}, \\
\mu_3(h,r,s) &= \frac{(r-1)^{s+2}+(hr)^{s+1}\{hr(1+s)-(s+2)(r-1)\}}{(s+1)(s+2)r^{s+2}}, \\
\mu_4(h,r,s) &= \frac{(r-1)^{s+2}+r^{s+1}\{s+2-r\}}{(s+1)(s+2)r^{s+2}}.
\end{aligned}
$$

Proof. From Lemma 1, we can write

$$
\left|\frac{1}{(b-a)(d-c)}I(f)(h_1,h_2;r_1,r_2)\right|
$$

$$\leq \int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| \times \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \Big| dt d\lambda. \tag{7}$$

Since $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|$ is in $L_1(\Delta)$ and is a co-ordinated (s_1, s_2)-convex mapping in the second sense on Δ, the following inequality

$$\begin{aligned} &\frac{\partial^2}{\partial t \partial \lambda} f(ta + (1-t)b, \lambda c + (1-\lambda)d) \\ &\leq t^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \Big| + (1-t)^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \Big| \\ &\quad + t^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \Big| + (1-t)^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \Big| \end{aligned} \tag{8}$$

holds, for all $t, \lambda \in [0, 1]$ and for fixed $s_1, s_2 \in (0, 1]$.

By the inequalities (7) and (8), we have

$$\begin{aligned} &\Big| \frac{1}{(b-a)(d-c)} I(f)(h_1, h_2; r_1, r_2) \Big| \\ &\leq \int_0^1 \Big| p(h_1, r_1, t) \Big| \Big[\int_0^1 \Big| p(h_2, r_2, \lambda) \Big| \Big\{ t^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \Big| \\ &\quad + (1-t)^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \Big| + t^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \Big| \\ &\quad + (1-t)^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \Big| \Big\} d\lambda \Big] dt \\ &= \Big(\int_0^1 \Big| p(h_1, r_1, t) \Big| t^{s_1} dt \Big) \Big(\int_0^1 \Big| p(h_2, r_2, \lambda) \Big| \lambda^{s_2} d\lambda \Big) \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \Big| \\ &\quad + \Big(\int_0^1 \Big| p(h_1, r_1, t) \Big| (1-t)^{s_1} dt \Big) \Big(\int_0^1 \Big| p(h_2, r_2, \lambda) \Big| \lambda^{s_2} d\lambda \Big) \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \Big| \\ &\quad + \Big(\int_0^1 \Big| p(h_1, r_1, t) \Big| t^{s_1} dt \Big) \Big(\int_0^1 \Big| p(h_2, r_2, \lambda) \Big| (1-\lambda)^{s_2} d\lambda \Big) \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \Big| \\ &\quad + \Big(\int_0^1 \Big| p(h_1, r_1, t) \Big| (1-t)^{s_1} dt \Big) \\ &\qquad \times \Big(\int_0^1 \Big| p(h_2, r_2, \lambda) \Big| (1-\lambda)^{s_2} d\lambda \Big) \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \Big|. \end{aligned} \tag{9}$$

By the simple calculations, we have the following equalities:

(a) $\displaystyle\int_0^1 \Big| p(h_1, r_1, t) \Big| t^{s_1} dt$

$$= \mu_1(0,r_1,s_1) + \mu_2(h_1,r_1,s_1) + \mu_3(h_1,r_1,s_1) + \mu_4(0,r_1,s_1),$$

$$(b) \int_0^1 \Big|p(h_2,r_2,\lambda)\Big|\lambda^{s_2}d\lambda$$
$$= \mu_1(0,r_2,s_2) + \mu_2(h_2,r_2,s_2) + \mu_3(h_2,r_2,s_2) + \mu_4(0,r_2,s_2),$$

$$(c) \int_0^1 \Big|p(h_1,r_1,t)\Big|(1-t)^{s_1}dt$$
$$= \mu_1(0,r_1,s_1) + \mu_2(1-h_1,r_1,s_1) + \mu_3(1-h_1,r_1,s_1) + \mu_4(0,r_1,s_1),$$

$$(d) \int_0^1 \Big|p(h_2,r_2,\lambda)\Big|\lambda^{s_2}d\lambda$$
$$= \mu_1(0,r_2,s_2) + \mu_2(h_2,r_2,s_2) + \mu_3(h_2,r_2,s_2) + \mu_4(0,r_2,s_2),$$

$$(e) \int_0^1 \Big|p(h_1,r_1,t)\Big|t^{s_1}dt$$
$$= \mu_1(0,r_1,s_1) + \mu_2(h_1,r_1,s_1) + \mu_3(h_1,r_1,s_1) + \mu_4(0,r_1,s_1),$$

$$(f) \int_0^1 \Big|p(h_2,r_2,\lambda)\Big|(1-\lambda)^{s_2}d\lambda$$
$$= \mu_1(0,r_2,s_2) + \mu_2(1-h_2,r_2,s_2) + \mu_3(1-h_2,r_2,s_2) + \mu_4(0,r_2,s_2),$$

$$(g) \int_0^1 \Big|p(h_1,r_1,t)\Big|(1-t)^{s_1}dt$$
$$= \mu_1(0,r_1,s_1) + \mu_2(1-h_1,r_1,s_1) + \mu_3(1-h_1,r_1,s_1) + \mu_4(0,r_1,s_1),$$

$$(h) \int_0^1 \Big|p(h_2,r_2,\lambda)\Big|(1-\lambda)^{s_2}d\lambda$$
$$= \mu_1(0,r_2,s_2) + \mu_2(1-h_2,r_2,s_2) + \mu_3(1-h_2,r_2,s_2) + \mu_4(0,r_2,s_2). \tag{10}$$

By (9) and (10), we have the desired result.

Corollary 2.2. *In Theorem 2.1,*
(a) if we choose $r_1 = r_2 = 2$, $h_1 = h_2 = \frac{1}{2}$ *and* $s_1 = s_2 = 1$, *then we have*

$$\frac{64}{(b-a)(d-c)}\Big|I(f)(\frac{1}{2},\frac{1}{2};2,2)\Big|_{s_1=s_2=1}$$
$$\leq \Big\{\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big\},$$

which implies that Lemma 1 is a generalization of Lemma 1.2 in [4].

(b) if we choose $r_1 = r_2 = 6$, $h_1 = h_2 = \frac{1}{2}$ and $s_1 = s_2 = 1$, then we have

$$\frac{(\frac{72}{5})^2}{(b-a)(d-c)}\Big|I(f)(\frac{1}{2},\frac{1}{2};6,6)\Big|_{s_1=s_2=1}$$
$$\leq \Big\{\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big\},$$

which implies that Theorem 2.1 is a generalization of Theorem 1.2 in [7].

Theorem 2.3. *Let $f : \Delta \subset R^2 \to R$ be a partial differentiable mapping on a bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$ and $c < d$, and $\lambda, t \in [0,1]$. For $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ if $\left|\frac{\partial^2 f}{\partial t\partial\lambda}\right|^q$ is in $L(\Delta)$ and is a co-ordinated (s_1,s_2)-convex mapping on Δ, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in [0,1]$ with $\frac{1}{r_1} \leq h_1 \leq \frac{r_1-1}{r_1}$ and $\frac{1}{r_2} \leq h_2 \leq \frac{r_2-1}{r_2}$, the following inequality holds:*

$$\Big|\frac{1}{(b-a)(d-c)}I(f)(h_1,h_2;r_1,r_2)\Big|$$
$$\leq \mu_5^{\frac{1}{p}}(h_1,r_1)\mu_5^{\frac{1}{p}}(h_2,r_2)\Big\{\frac{1}{(s_1+1)(s_2+1)}\Big\}^{\frac{1}{q}}$$
$$\times \Big\{\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(a,d)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\Big| + \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\Big|\Big\}^{\frac{1}{q}}, \tag{11}$$

where

$$\mu_5(h,r) = \frac{2 + (r - rh - 1)^{p+1} + (rh-1)^{p+1}}{r^{p+1}(p+1)}.$$

Proof. From Lemma 1 and using the Hölder inequality for double integrals, we obtain

$$\Big|\frac{1}{(b-a)(d-c)}I(f)(h_1,h_2;r_1,r_2)\Big|$$
$$\leq \int_0^1\int_0^1 \Big|p(h_1,r_1,t)p(h_2,r_2,\lambda)\Big|\Big|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b, \lambda c + (1-\lambda)d)\Big|dtd\lambda$$
$$\leq \Big\{\int_0^1\int_0^1 \Big|p(h_1,r_1,t)\Big|^p\Big|p(h_2,r_2,\lambda)\Big|^p dtd\lambda\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\int_0^1\int_0^1 \Big|\frac{\partial^2 f}{\partial t\partial\lambda}(ta+(1-t)b, \lambda c+(1-\lambda)d)\Big|^q dtd\lambda\Big\}^{\frac{1}{q}}. \tag{12}$$

Since $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is an (s_1, s_2)-convex mapping on the co-ordinates on Δ for $q > 1$, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(tb + (1-t)a, \lambda d + (1-\lambda)c)\Big|^q dt d\lambda \\
&\le \int_0^1 \int_0^1 \Big\{ t^{s_1}\lambda^{s_2}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\Big|^q + t^{s_1}(1-\lambda)^{s_2}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\Big|^q \\
&\quad + (1-t)^{s_1}\lambda^{s_2}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\Big|^q + (1-t)^{s_1}(1-\lambda)^{s_2}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\Big|^q \Big\} dt d\lambda \\
&= \frac{\Big\{\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,c)\Big|^q + \Big|\frac{\partial^2 f}{\partial t \partial \lambda}(b,d)\Big|^q\Big\}}{(s_1+1)(s_2+1)}. \qquad (13)
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^1 \Big|p(h_1, r_1, t)\Big|^p dt &= \mu_5(h_1, r_1), \\
\int_0^1 \Big|p(h_2, r_2, \lambda)\Big|^p d\lambda &= \mu_5(h_2, r_2). \qquad (14)
\end{aligned}$$

The substitution of (13) and (14) in (12) gives the desired result.

Theorem 2.4. *Let $f : \Delta \subset R^2 \to R$ be a partial differentiable mapping on a bidimensional interval $\Delta = [a,b] \times [c,d]$ in R^2 with $a < b$, $c < d$ and $t, \lambda \in [0,1]$. If $\left|\frac{\partial^2 f}{\partial t \partial \lambda}\right|^q$ is in $L(\Delta)$ and is a co-ordinated (s_1, s_2)-convex mapping on Δ, for $q \ge 1$, then, for $r_1, r_2 \ge 2$ and $h_1, h_2 \in [0,1]$ with $\frac{1}{n_1} \le h_1 \le \frac{n_1 - 1}{n_1}$ and $\frac{1}{n_2} \le h_2 \le \frac{n_2 - 1}{n_2}$, the following inequality holds:*

$$\begin{aligned}
&\Big|\frac{1}{(b-a)(d-c)} I(f)(h_1, h_2; r_1, n_r)\Big| \\
&\le \Big[\Big\{\frac{2h_1^2 - 2h_1 + 1}{2} + \frac{2 - r_1}{r_1^2}\Big\}\Big\{\frac{2h_2^2 - 2h_2 + 1}{2} + \frac{2 - r_2}{r_2^2}\Big\}\Big]^{1-\frac{1}{q}} \\
&\quad \times \Big[\Big\{(\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_1 + \mu_4)(h_1, r_1, s_1)\Big\} \\
&\qquad \times \Big\{\big\{(\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(h_2, r_2, s_2)\big\}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,c)\Big|^q \\
&\qquad + \big\{(\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(1 - h_2, r_2, s_2)\big\}\Big|\frac{\partial^2 f}{\partial t \partial \lambda}(a,d)\Big|^q\Big\} \\
&\quad + \Big\{(\mu_1 + \mu_4)(0, r_1, s_1) + (\mu_2 + \mu_3)(1 - h_1, r_1, s_1)\Big\}
\end{aligned}$$

$$\times \Big\{ \big\{ (\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(h_2, r_2, s_2) \big\} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \Big|^q + \big\{ (\mu_1 + \mu_4)(0, r_2, s_2) + (\mu_2 + \mu_3)(1 - h_2, r_2, s_2) \big\} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \Big|^q \Big\} \Big]^{\frac{1}{q}},$$

where $\mu_i(r, s, h)(i = 1, 2, 3, 4)$ *are defined as in Theorem 2.1.*

Proof. From Lemma 1 and using the power mean inequality for double integrals, we obtain

$$\begin{aligned} &\Big| \frac{1}{(b-a)(d-c)} I(f)(h_1, h_2; n_1, n_2) \Big| \\ &\le \int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| \\ &\qquad \times \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \Big| dt d\lambda \\ &\le \Big\{ \int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| dt d\lambda \Big\}^{1 - \frac{1}{q}} \\ &\qquad \times \Big\{ \int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| \\ &\qquad\qquad \times \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \Big|^q dt d\lambda \Big\}^{\frac{1}{q}}. \end{aligned} \tag{15}$$

Since $\left| \frac{\partial^2 f}{\partial t \partial \lambda} \right|^q$ is in $L(\Delta)$ and is a co-ordinated (s_1, s_2)-convex mapping on Δ for $q \ge 1$, we have

$$\begin{aligned} &\int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| \\ &\quad \times \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \Big|^q dt d\lambda \\ &\le \int_0^1 \int_0^1 \Big| p(h_1, r_1, t) p(h_2, r_2, \lambda) \Big| \\ &\quad \times \Big\{ t^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \Big|^q + t^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \Big|^q \\ &\qquad + (1-t)^{s_1} \lambda^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \Big|^q + (1-t)^{s_1} (1-\lambda)^{s_2} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \Big|^q \Big\} dt d\lambda \\ &= \Big\{ \int_0^1 \Big| p(h_1, r_1, t) \Big| t^{s_1} dt \Big\} \Big\{ \int_0^1 \Big| p(h_2, r_2, \lambda) \Big| \lambda^{s_2} d\lambda \Big\} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \Big|^q \\ &\quad + \Big\{ \int_0^1 \Big| p(h_1, r_1, t) \Big| t^{s_1} dt \Big\} \Big\{ \int_0^1 \Big| p(h_2, r_2, \lambda) \Big| (1-\lambda)^{s_2} d\lambda \Big\} \Big| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \Big|^q \end{aligned}$$

$$+\left\{\int_0^1 \Big|p(h_1,r_1,t)\Big|(1-t)^{s_1}dt\right\}\left\{\int_0^1 \Big|p(h_2,r_2,\lambda)\Big|\lambda^{s_2}d\lambda\right\}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,c)\right|^q$$
$$+\left\{\int_0^1 \Big|p(h_1,r_1,t)\Big|(1-t)^{s_1}dt\right\}$$
$$\times\left\{\int_0^1 \Big|p(h_2,r_2,\lambda)\Big|(1-\lambda)^{s_2}d\lambda\right\}\left|\frac{\partial^2 f}{\partial t\partial\lambda}(b,d)\right|^q. \qquad (16)$$

By noting that

$$\int_0^1 \Big|p(h,r,z)\Big|dz = \frac{2h^2-2h+1}{2}+\frac{2-r}{r^2},$$

we have

$$\int_0^1 \Big|p(h_1,r_1,t)\Big|dt = \frac{2h_1^2-2h_1+1}{2}+\frac{2-r_1}{r_1^2},$$
$$\int_0^1 \Big|p(h_2,r_2,\lambda)\Big|d\lambda = \frac{2h_2^2-2h_2+1}{2}+\frac{2-r_2}{r_2^2}. \qquad (17)$$

By the sustitutions of (10),(16) and (17) in (15), we have the desired result.

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