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FINITE NON-SOLVABLE GROUPS HAVING A UNIQUE IRREDUCIBLE CHARACTER OF A GIVEN DEGREE

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Abstract: It has been conjectured that $\mathrm{PSL}(2, q)$, the projective special linear group of 2×2 matrices over a field of order q, is the only non-solvable group satisfying the property that it has a unique irreducible complex character χ of degree $m > 1$ and every other irreducible complex character is such that its degree is relatively prime to m. (Such a χ is a particular case of the Steinberg character of finite Chevalley groups.) In this paper, we consider finite non-solvable groups satisfying the above property and show that the derived group G' is a non-abelian simple group and that when $\chi(1) = p$, p an odd prime, G itself is a non-abelian simple group, and is such that its p-sylow subgroup P is a cyclic group of order p and equals its centralizer and that all involutions in G are conjugate.

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1. Notations and Preliminary Results

Our notations are quite standard and agree with those in [4]. In what follows, group means finite group and all characters are considered characters over com-

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plex numbers. For a character ψ of a subgroup H of a group G, ψ^G denotes the character of G induced by ψ (see[4] for a definition). Induction has the following properties:

Property 1. If $H \subset K \subset G$, then $(\psi^K)^G = \psi^G$.

Property 2. Frobenius reciprocity: If φ is a character of G, then $(\psi^G, \varphi)_G = (\psi, \varphi|_H)_H$.

For $x \in G$, ψ^x is defined on H^x by $\psi^x(h^x) = \psi(h)$, $h \in H$, and is a character of H^x. If H is normal in G, the stabilizer of $\psi = \{x \in G \mid \psi^x = \psi\}$ is denoted by $T_{\psi,H}$ or simply by T_ψ if the domain of ψ is clear from the context.

If Φ is a matrix representation, then $\det \Phi$ is a linear character. Equivalent representations have the same determinant, so we may put $\det \Phi = \det \varphi$, where φ is the character of Φ.

If π is a set of prime numbers, then by a π-number b we mean that each prime factor of b is in π. If $|G|$ is a π-number then we say that G is a π-group.

The following result of Clifford will be used repeatedly.

Lemma 3 ([6], Theorem 9.10)**.** *Let H be a normal subgroup of G. If η is an irreducible character of G, then there exists an irreducible character θ of H and a positive integer a such that*

$$\eta|_H = a \sum_{i=1}^{t} \theta^{g_i}, \text{ where } t = [G : T_\theta]$$

and $\{g_i\}$ is a complete system of coset representatives of T_θ in G.

2. Main Results

Theorem 4. *Suppose G is a non-solvable group having exactly one irreducible character χ of degree m, and every other irreducible character θ of G is such that $(\theta(1), m) = 1$. Further, suppose that χ is faithful. Then G', the derived group of G, is a non-abelian simple group.*

Remark. It follows from the above hypotheses and the main theorem in [10] that G has at least one non-linear irreducible character different from χ.

Proof. The proof is carried out in five steps.

Step 1. Every abelian, normal subgroup A of G is cyclic, and is contained in $Z(G)$.

By 3, we can write

$$\chi|_A = a\sum_{i=1}^{t} \psi^{g_i}$$

where ψ^{g_i} are irreducible characters of A, $t = [G : T_\psi]$, and a is some positive integer. Since A is abelian, $\psi(1) = 1$ and so $\chi(1) = a \cdot t$ so that a and t are π-numbers, where π is the set of primes dividing $\chi(1)$. First suppose that $t > 1$. If $\eta \neq \chi$ is any irreducible character of G, then $(\eta|_A, \psi)_A = 0$, for otherwise all conjugates of ψ appear in η_A (Lemma 3) and so $t|\eta(1)$, contradicting our hypotheses. This and Frobenius reciprocity (2) imply that $\psi^G = a\cdot\chi$. Comparing degrees, we obtain

$$[G : A]\psi(1) = [G : A] = a\chi(1) = a^2 t.$$

Therefore, G/A is a π-group. But then the hypothesis on the degrees of irreducible characters of G implies that G/A cannot have any non-linear irreducible characters. Hence G/A is abelian and this is a contradiction to our assumption that G is non-solvable. Therefore, $t = 1$ and $\chi|_A = \chi(1)\cdot\psi$. Then, ψ is faithful since χ is, and so A is cyclic. Let $A = \langle x\rangle$ and y be arbitrary in G. Since ψ has no conjugates different from ψ, $\psi^x = \psi$ and this implies $x^y = x$. Hence $A \subseteq Z(G)$. □

Step 2. Let $N/Z(G)$ be a minimal normal subgroup of $G/Z(G)$. Then G/N is abelian.

Let $\chi|_N = a\sum_{i=1}^{t}\psi^{g_i}$, ψ^{g_i} irreducible characters of N. If $\psi(1) = 1$, then $N' \subseteq \ker\psi^{g_i}\forall i$, and so $N' \subseteq \ker\chi = 1$. But N cannot be abelian by step 1. Therefore, $\psi(1) \neq 1$ and clearly $\psi(1)$ is a π-number. If $\eta \neq \chi$ is any irreducible character of G, then $(\eta|_N, \psi)_N = 0$, for otherwise, $\psi(1)$ would divide $\eta(1)$, contrary to our hypotheses. Therefore, $\psi^G = a\cdot\chi$. Comparing degrees again, we obtain

$$[G : N]\cdot\psi(1) = a\chi(1) = a^2 t\cdot\psi(1)$$

so that G/N is a π-group and hence cannot have any non-linear irreducible characters. Thus G/N is abelian. □

Step 3. $|Z(G)| \leq 2$.

Let $Z(G) = \langle z\rangle$ and $\chi|_{Z(G)} = \chi(1)\cdot\varphi$ for some irreducible character φ of $Z(G)$. Since χ is unique of its degree, it is fixed under field automorphisms. Therefore, χ is integer valued and hence φ is too. This, together with the fact that $\varphi(z)$ is a root of unity implies that $|Z(G)| \leq 2$. □

Step 4. If $|Z(G)| = 2$ then $N' = N$.

If $N' \supseteq Z(G)$, this is obvious. If $N' \not\subseteq Z(G)$, then since $N/Z(G)$ is minimal normal in $G/Z(G)$, we obtain that $N = N' \times Z(G)$. We will show that this cannot happen.

The characters $\{\psi^{g_i}\}$ in the equation $\chi|_N = a\sum_{i=1}^{t} \psi^{g_i}$ are all the non-linear irreducible characters of N whose degrees are π-numbers, for if θ is any non-linear irreducible character of N such that $\theta(1)$ is a π-number, then $(\theta^G, \eta)_G \geq 1$ for some irreducible character η of G. This implies $(\theta, \eta|_N)_N \geq 1$ and so $\theta(1) \mid \eta(1)$. Hence, by hypotheses, we have $\eta = \chi$ so that $\theta = \psi^{g_i}$ for some i. In particular, all non-linear irreducible characters of N whose degrees are π-numbers have the same degree and are conjugate. But if $N = N' \times Z(G)$, then the irreducible characters of N are products of irreducible characters of N' and $Z(G)$ (Theorem 3.7.1, p. 100, [7]) and if $\zeta \neq 1_N$ is an irreducible character of N', then $\zeta \times 1_{Z(G)}$ is not conjugate to $\zeta \times \varphi$, where φ is as in Step 3. This contradiction proves Step 4. □

Step 5. $Z(G) = 1$ and N is non-abelian simple.

Suppose $|Z(G)| = 2$ so that $\varphi(z) = -1$ and $N' = N = G'$. If $\chi(1)$ is odd, then $\det\chi(z) = -1$, a contradiction since $G' \subset \ker\det\chi$. Hence, $2 \mid \chi(1)$ and so every irreducible character θ of N different from any ψ^{g_i} has odd degree. (There is at least one such non-linear θ since G has non-linear characters different from χ.) For any such θ, we must have $Z(G) \subseteq \ker\theta$, for otherwise $\det\theta|_{N'} \neq 1_{N'}$; i.e., ψ^{g_i} for $i = 1, 2, \ldots t$ are all irreducible characters of N which don't have $Z(G)$ in their kernel. Thus, we have on one hand,

$$|N| = 2 \cdot |N/Z(G)| = 2\sum \theta(1)^2,$$

where θ ranges over all irreducible characters of $N/Z(G)$, and on the other hand,

$$|N| = t\psi(1)^2 + \sum \theta(1)^2.$$

Hence, $t\psi(1)^2 = \sum\theta(1)^2 = |N/Z(G)|$. This implies that $\theta(1) \mid t\psi(1)^2$ since $\theta(1) \mid |N/Z(G)|$. This is a contradiction since $(\theta(1), \chi(1)) = 1$ and so $Z(G) = 1$.

Since N is non-solvable and minimal normal in G, we obtain that $N = N_1 \times N_2 \times \cdots \times N_r$, where the N_i's are non-abelian simple and isomorphic (Theorem 2.1.4, p. 16, [7]) and r is some positive integer. If $r > 1$, then then non-linear irreducible characters of N whose degrees are π-numbers cannot all have the same degree (Theorem 3.7.1, p. 100, [7]). Hence, N is non-abelian simple and the proof of Theorem 4 is complete. □

□

Theorem 5. *Suppose that G and χ are as in Theorem 4 and that $\chi(1) = p$, p an odd prime. Let P be a p-sylow subgroup of G. Then the following hold:*

(a) G is a non-abelian simple group;

(b) $P = C_G(P)$ and $|P| = p$

(c) All involutions in G are conjugate.

Proof. (a) In step 2 of the proof of Theorem 4, we had $[G : N] = a^2t$, where a and t are as in the equation

$$\chi|N = a\sum_{i=1}^{t}\psi^{g_i}$$

Also, $\psi(1) \neq 1$ since N is a non-abelian simple group. Thus, if $\chi(1) = p$, then we obtain, by comparing degrees in the above equation, that $a = t = 1$. Therefore, $G = N$. This proves part (a).

(b) Since G is simple, we obtain, by a result of Burnside (Theorem 18.4, p. 94, [6]), that $\chi|P = \mu_P$, where μ_P is the regular character of P, and so $|P| = p$.

Let $P = \langle x\rangle$ and suppose that y is an element of prime order q, $q \neq p$ and that $y \in C_G(P)$. Since $\chi|_{\langle x\rangle} = \mu_{\langle x\rangle}$, we obtain that

$$\chi|_{\langle x\rangle\times\langle y\rangle} = \sum_{j=1}^{p}\varphi_j\omega_j,$$

where φ_j's are all the linear characters of $\langle x\rangle$, and ω_j's are among the linear characters of $\langle y\rangle$. Thus, we can write

$$\chi(xy) = \omega_1(y)\lambda + \omega_2(y)\lambda^2 + \cdots + \omega_p(y)\lambda^p,$$

where λ is a primitive p^{th} root of unity. Then, we obtain

$$\omega_1(y)\lambda + \omega_2(y)\lambda^2 + \cdots + \omega_{p-1}(y)\lambda^{p-1} = \alpha$$

for some $\alpha \in \mathbb{Q}[\sqrt[q]{1}]$, the field obtained by adjoining $\sqrt[q]{1}$ to the field of rationals. Also, $\lambda + \lambda^2 + \cdots + \lambda^{p-1} = -1$ and so $(-\alpha)\lambda + (-\alpha)\lambda^2 + \cdots + (-\alpha)\lambda^{p-1} = \alpha$ as well. Thus, if $\lambda, \lambda^2, \ldots, \lambda^{p-1}$ are linearly independent over $\mathbb{Q}[\sqrt[q]{1}]$, then all $\omega_i(y)$'s are equal and $\chi(y) = p \cdot \omega_1(y)$, so that $y \in Z(G)$, a contradiction. Hence, it is enough to show that $\lambda, \lambda^2, \dots, \lambda^{p-1}$ are linearly independent over $\mathbb{Q}[\sqrt[q]{1}]$. Clearly, they span $\mathbb{Q}[\sqrt[q]{1}][\sqrt[p]{1}]$ as a $\mathbb{Q}[\sqrt[q]{1}]$-space. Also $\mathbb{Q}[\sqrt[q]{1}][\sqrt[p]{1}] = \mathbb{Q}[\sqrt[pq]{1}]$, and $|\mathbb{Q}[\sqrt[pq]{1}] : \mathbb{Q}| = (p-1)(q-1)$. The proof of (b) is complete.

(c) A result of Brauer (Theorem 3, [2]), together with parts (a) and (b) above, implies that G has exactly two p-blocks, and that the principal block $B_0(p)$ contains all the irreducible characters of G except χ. Part (c), now, follows from a lemma of Richen (p. 299, [9]).

□

Remark. The assumption that p is odd is never used in the proof Theorem 5. But then, using Theorem 5, one can easily show that there are no groups satisfying the hypotheses of Theorem 3.2 when $p = 2$ or 3.

Remark. The conjecture that $\mathrm{PSL}(2, p)$ is the only group satisfying the hypotheses of Theorem 5 is still open. However, Theorem 5 enables us to apply Brauer's results ([1] and [2]) and obtain the following information about p-blocks and irreducible characters of G.

Let $|N_G(P)| = e \cdot p$, where e is some positive integer dividing $p - 1$ and let $t = (p-1)/e$. Let $\lambda_1, \lambda_2, \dots, \lambda_t$ denote the irreducible characters of $N_G(P)$ which are induced from linear characters of P. There are $1 + e + t$ irreducible characters of G. There is a unique p-block $B(p)$ of defect zero and $B(p) = \{\chi\}$. All other irreducible characters belong to the principal block $B_0(p)$, and so let

$$B_0(p) = \{\zeta = 1_G, \zeta_2, \dots, \zeta_e, \theta_1, \theta_2, \dots, \theta_t\},$$

where the θ_i's are exceptional characters and satisfy $\lambda_i^G = \lambda_j^G = \varepsilon(\theta_i - \theta_j)$, $\varepsilon = \pm 1$, for $i, j = 1, 2, \dots, t$. Each ζ_i has the property that ζ_i has constant value a_i on all p-elements and $a_i = \pm 1$. As a result, $\zeta_i(1) = k_i p \pm 1$ for some positive integer k_i. If $g \in G$ has order prime to p then $\theta_i(g) = \theta_j(g) \forall i, j$. Also, $\sum_{i=1}^t \theta_i$ has constant value a on the p-elements and $a = \pm 1$. Hence, $t \cdot \theta_i(1) = \sum_{i=1}^t \theta_i(1) = kp \pm 1$ for some positive integer k. (These θ_i's are algebraically conjugate.)

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