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ON THE DISCOUNTED FACTORIAL MOMENTS OF THE DEFICIT IN THE DISCRETE TIME RENEWAL RISK MODEL

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Abstract: In this paper we consider a discrete time renewal risk model with arbitrary interclaim times. An explicit formula is derived for the discounted factorial moments of the deficit at ruin by using a recursive equation satisfied by the Gerber-Shiu discounted penalty function. Numerical illustrations are also given to show that the formula obtained is readily programmable and works well for special claim distributions.

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1. Introduction

In this paper we consider the discrete time Sparre Andersen risk model

$$U(n) = u + n - \sum_{i=1}^{N(n)} X_i, \tag{1}$$

where$u \in \mathbb{N}$ is the initial surplus. The number of claims process $\{N(n), n \in$

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$\mathbb{N}\}$ is defined as $N(n) = \max\{k : W_1 + W_2 + \cdots + W_k \leq n\}$, where the interclaim times $\{W_i, i \in \mathbb{N}^+\}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common probability function (p.f.) $k(x)$, $x \in \mathbb{N}^+$. The individual claim amounts $\{X_i, i \in \mathbb{N}^+\}$ are i.i.d. positive random variables with common p.f. $p(x)$, $x \in \mathbb{N}^+$ and distribution function (d.f.) $P(x) = 1 - \overline{P}(x)$. We assume that $\{X_i, i \in \mathbb{N}^+\}$ and $\{W_i, i \in \mathbb{N}^+\}$ are independent and $E(W_1) = (1+\theta)E(X_1)$ with $\theta > 0$ the relative security loading.

For the risk model (1), let $T = \min\{n \in \mathbb{N}^+, U(n) < 0\}$ be the time of ruin with $T = \infty$ if ruin does not occur. If ruin occurs, $|U(T)|$ is the deficit at ruin and $U(T-1)$ is the surplus immediately prior to ruin. The expected discounted penalty function introduced by Gerber and Shiu [1] is defined as

$$\phi_v(u) = \mathrm{E}\big[v^T \omega(U(T-1), |U(T)|) I(T < \infty) \,|\, U(0) = u\big], \tag{2}$$

where $0 < v < 1$ is the (constant) discount factor, $\omega(x, y) : \mathbb{N} \times \mathbb{N}^+ \to \mathbb{N}$ and $I(\cdot)$ is the indicator function.

In recent years, the analysis of the Gerber-Shiu function for particular choices of the distribution of the interclaim times has received considerable attention. See [2, 3, 4] and reference therein for details.

For the discrete time risk model, Pavlova and Willmot [5] expressed the Gerber-Shiu function in the stationary model in terms of the corresponding Gerber-Shiu function in the ordinary model. As the special case, they also examined the defective renewal equation for the well-known compound binomial model. Bao and Wang [6] further investigate the discrete time stationary risk model, an explicit expression for the discounted survival distribution of the deficit at ruin is derived. With claim waiting times have a discrete K_m distribution, Li [7] derived a recursive formula for the discounted penalty function by using the tool of probability generating function (p.g.f.). In a subsequent paper of [8], the discounted penalty function was explicitly expressed in terms of a compound geometric distribution function, and the explicit expressions for the p.g.f. of the time of ruin, the joint and marginal distributions of the surplus before ruin, the deficit at ruin, the claim causing ruin, as well as their moments were derived. Along the similar lines as [3], Wu and Li [9] obtained a recursive formula satisfied by the Gerber-Shiu function for the discrete renewal risk model with arbitrary interclaim times. See also [10, 11] for related analysis.

In this paper we consider the discounted factorial moments of the deficit at ruin in the discrete time Sparre Andersen risk model with general interclaim times. The rest of the paper is organized as follows: In Section 2, we present some notation and preliminaries. The main result is derived in Section 3. An explicit expression is derived for the discounted factorial moments of the deficit at ruin. In the last section we present two numerical examples to show that the formula is readily programmable in practice.

2. Notation and Preliminaries

Given $U(0) = u$, we consider the joint p.f. of the surplus just before ruin, the deficit at ruin, and the time of ruin

$$f(x, y, t \,|\, u) = \Pr\big\{U(T-1) = x, |U(T)| = y, T = t \,|\, U(0) = u\big\},$$

for $x \in \mathbb{N}$, $y \in \mathbb{N}^+$. Denote by

$$p_x(y) = \frac{p(x+y+1)}{\overline{P}(x+1)}, \quad y \in \mathbb{N}^+ \tag{3}$$

the conditional p.f. of the deficit (y) given both the surplus prior to ruin (x) and the time of ruin (t). It is easy to see that

$$f(x, y, t \,|\, u) = p_x(y) f(x, t \,|\, u), \quad x \in \mathbb{N}, \, y \in \mathbb{N}^+,$$

where $f(x, t \,|\, u)$ is the joint p.f. of $U(T-1)$ and T given $U(0) = u$. For $0 < v < 1$, we define

$$f(x \,|\, u) = \sum_{t=1}^{\infty} v^t f(x, t \,|\, u),$$

as the discounted p.f. of $U(T-1)$. Further, let $\xi_v = \sum_{x=0}^{\infty} f(x \,|\, 0)$ and

$$g_v(y) = \sum_{x=0}^{\infty} p_x(y) f(x \,|\, 0)/\xi_v, \quad y \in \mathbb{N}^+.$$

By the definition of ξ_v and (3) we know that $g_v(y)$ is a proper p.f.. Denote by $G_v(y) = 1 - \overline{G}_v(y)$ the corresponding d.f..

Now define $\mu_{(n)} := \sum_{y=1}^{\infty} y^{(n)} g_v(y)$ to the n-th factorial moment of $g_v(y)$, where $y^{(n)} = y(y-1)\cdots(y-n+1)$ denotes the n-th factorial power of y and $y^{(0)} = 1$. For $n \in \mathbb{N}^+$, $x, z \in \mathbb{N}$ and $x \le z$, it is well-known (see e.g. [12] p.182) that

$$\sum_{k=x}^{z} k^{(n)} = \frac{(z+1)^{(n+1)} - x^{(n+1)}}{n+1}.$$

Let $G_{0,v}(y) = 1 - \overline{G}_{0,v}(y) = G_v(y)$, define the n-th equilibrium distribution $G_{n,v}(y) = 1 - \overline{G}_{n,v}(y)$ of $g_v(y)$ with $g_{n,v}(y)$ the corresponding p.f. as follows

$$g_{n,v}(y) = \frac{\overline{G}_{n-1,v}(y)}{\sum_{z=0}^{\infty} \overline{G}_{n-1,v}(z)}, \quad n \in \mathbb{N}^+,\ y \in \mathbb{N}.$$

By Lemma 2 in [11] we know that

$$\overline{G}_{n,v}(y) = \frac{1}{\mu_{(n)}} \sum_{z=y+1}^{\infty} (z - (y+1))^{(n)} g_v(z). \tag{4}$$

By (4), the mean of $G_{n,v}(y)$ can be expressed as

$$\begin{aligned}\sum_{y=0}^{\infty} \overline{G}_{n,v}(y) &= \frac{1}{\mu_{(n)}} \sum_{z=1}^{\infty} g_v(z) \sum_{y=0}^{z-1} (z - (y+1))^{(n)} \\ &= \frac{1}{(n+1)\mu_{(n)}} \sum_{z=1}^{\infty} z^{(n+1)} g_v(z) = \frac{\mu_{(n+1)}}{(n+1)\mu_{(n)}}.\end{aligned} \tag{5}$$

With $w(x,y) = (y-1)^{(n)}$ in (2), we denote by

$$\psi_{n,v}(u) = \mathrm{E}\big[v^T (|U(T)| - 1)^{(n)} I(T < \infty) \,|\, U(0) = u\big]. \tag{6}$$

It follows by (4) and Eq. (2.7) in [9] that

$$\psi_{n,v}(u) = \xi_v \sum_{y=1}^{u} \psi_{n,v}(u-y) g_v(y) + \xi_v \mu_{(n)} \overline{G}_{n,v}(u). \tag{7}$$

If $n = 0$, we denote by $\overline{D}_v(u) = \psi_{0,v}(u)$ the p.g.f. of ruin time with definition $\overline{D}_v(u) = \mathrm{E}\big[v^T I(T < \infty) \,|\, U(0) = u\big]$. Note that $\mu_{(0)} = 1$, by (7) we have

$$\overline{D}_v(u) = \xi_v \sum_{y=1}^{u} \overline{D}_v(u-y) g_v(y) + \xi_v \overline{G}_v(u). \tag{8}$$

Now we define a new random variable L as $\overline{D}_v(u) = \Pr(L > u)$ and denote by $d_v(u), u \in \mathbb{N}$ the p.f. of L throughout the entire paper.

3. Explicit Expression for $\psi_{n,v}(u)$

In this section, we show that $\psi_{n,v}(u)$ satisfies an explicit formula. To prove the main result, we need the following lemmas.

Lemma 1. *The function $\psi_{n,v}(u)$ defined by (6) satisfies*

$$\psi_{n,v}(u) = \frac{\xi_v \mu_{(n)}}{1-\xi_v} \left\{ \sum_{y=0}^{u} \overline{D}_v(u-y) g_{n,v}(y) + \overline{G}_{n,v}(u) - \overline{D}_v(u) \right\}. \tag{9}$$

Proof. Let $\widehat{g}_{n,v}(z) = \sum_{u=0}^{\infty} z^u g_{n,v}(u)$ be the p.g.f. of $g_{n,v}(y)$ with $\widehat{g}_v(z) = \widehat{g}_{0,v}(z)$. Also, let $\widehat{d}_v(z) = \sum_{u=0}^{\infty} z^u d_v(u)$. It is not hard to show that

$$\sum_{u=0}^{\infty} z^u \overline{D}_v(u) = \frac{1-\widehat{d}_v(z)}{1-z},$$

then from (8) we know that

$$\widehat{d}_v(z) = \frac{1-\xi_v}{1-\xi_v \widehat{g}_v(z)}. \tag{10}$$

By (7) and (10), the p.g.f. of $\psi_{n,v}(u)$ can be expressed as

$$\begin{aligned} &\widehat{\psi}_{n,v}(z) \\ &= \frac{\xi_v \mu_{(n)}}{1-\xi_v \widehat{g}_v(z)} \frac{1-\widehat{g}_{n,v}(z)}{1-z} = \frac{\xi_v \mu_{(n)}}{1-\xi_v} \left\{ \frac{1-\widehat{g}_{n,v}(z)}{1-z} \widehat{d}_v(z) \right\} \\ &= \frac{\xi_v \mu_{(n)}}{1-\xi_v} \left\{ \widehat{g}_{n,v}(z) \frac{1-\widehat{d}_v(z)}{1-z} + \frac{1-\widehat{g}_{n,v}(z)}{1-z} - \frac{1-\widehat{d}_v(z)}{1-z} \right\}. \end{aligned} \tag{11}$$

Then (9) follows from (11) by inversion of the p.g.f.. □

Lemma 1 demonstrates that we have to explicitly compute $g_{n,v}(y)$ in order to evaluate $\psi_{n,v}(u)$. We now give an alternative representation. We define

$$\overline{H}_{n,v}(u) = \sum_{y=0}^{u} \overline{D}_v(u-y) g_{n,v}(y) + \overline{G}_{n,v}(u), \tag{12}$$

and $h_{n,v}(u)$ is the corresponding p.f. of $\overline{H}_{n,v}(u)$. Let $\widehat{h}_{n,v}(z) = \sum_{u=0}^{\infty} z^u h_{n,v}(u)$. By (12) we have

$$\sum_{u=0}^{\infty} z^u \overline{H}_{n,v}(u) = \widehat{g}_{n,v}(z) \frac{1-\widehat{d}_v(z)}{1-z} + \frac{1-\widehat{g}_{n,v}(z)}{1-z}. \tag{13}$$

By (13) we know that

$$\widehat{h}_{n,v}(z) = 1 - (1-z)\sum_{u=0}^{\infty} z^u \overline{H}_{n,v}(u) = \widehat{g}_{n,v}(z)\widehat{d}_v(z). \tag{14}$$

In terms of $\overline{H}_{n,v}(u)$, we can rewritten $\psi_{n,v}(u)$ as

$$\psi_{n,v}(u) = \frac{\xi_v \mu_{(n)}}{1-\xi_v}\left(\overline{H}_{n,v}(u) - \overline{D}_v(u)\right). \tag{15}$$

The following Lemma aims to give a recursive formula for $\overline{H}_{n,v}(u)$.

Lemma 2. *For $n \in \mathbb{N}$, $\overline{H}_{n,v}(u)$ defined by (12) satisfies*

$$\overline{H}_{n+1,v}(u) = \frac{(n+1)\mu_{(n)}}{\mu_{(n+1)}}\sum_{y=u+1}^{\infty}\left(\overline{H}_{n,v}(y) - \overline{D}_v(y)\right). \tag{16}$$

Proof. First, Eq. (12) implies that

$$\sum_{y=0}^{\infty}\overline{H}_{n,v}(y) = \sum_{y=0}^{\infty}\overline{D}_v(y) + \sum_{y=0}^{\infty}\overline{G}_{n,v}(y), \tag{17}$$

and the mean of L can be expressed explicitly by using (10)

$$\sum_{y=0}^{\infty}\overline{D}_v(y) = \widehat{d}'_v(z)|_{z=1} = \frac{\xi_v\mu_{(1)}}{1-\xi_v}. \tag{18}$$

By (10), (14), (17) and (18), the equilibrium distribution of $H_{n,v}(u)$ has p.g.f.

$$\begin{aligned}
&\sum_{u=0}^{\infty} z^u \frac{\overline{H}_{n,v}(u)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)} \\
&= \frac{1}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}\frac{1-\widehat{h}_{n,v}(z)}{1-z} \\
&= \frac{1}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}\frac{1-\widehat{g}_{n,v}(z)\widehat{d}_v(z)}{1-z} \\
&= \frac{1}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}\frac{\xi_v(1-\widehat{g}_v(z)) + (1-\xi_v)(1-\widehat{g}_{n,v}(z))}{(1-z)(1-\xi_v\widehat{g}_v(z))} \\
&= \frac{1}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}\frac{\xi_v\mu_{(1)}\widehat{g}_{1,v}(z) + (1-\xi_v)\widehat{g}_{n+1,v}(z)\sum_{y=0}^{\infty}\overline{G}_{n,v}(y)}{(1-\xi_v\widehat{g}_v(z))}
\end{aligned}$$

$$= \widehat{d}_v(z)\frac{(1-\xi_v)^{-1}\xi_v\mu_{(1)}\widehat{g}_{1,v}(z)+\widehat{g}_{n+1,v}(z)\sum_{y=0}^{\infty}\overline{G}_{n,v}(y)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}$$

$$= \widehat{d}_v(z)\frac{\sum_{y=0}^{\infty}\overline{D}_v(y)\widehat{g}_{1,v}(z)+\widehat{g}_{n+1,v}(z)\left(\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)-\sum_{y=0}^{\infty}\overline{D}_v(y)\right)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}$$

$$= \widehat{d}_v(z)\left(b_v\widehat{g}_{1,v}(z)+(1-b_v)\widehat{g}_{n+1,v}(z)\right), \tag{19}$$

where

$$b_v = \frac{\sum_{y=0}^{\infty}\overline{D}_v(y)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}. \tag{20}$$

By (17) we know that $0 < b_v < 1$ and (19) implies

$$\sum_{u=0}^{\infty} z^u \frac{\sum_{y=u+1}^{\infty}\overline{H}_{n,v}(y)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)} = \frac{1-b_v\widehat{d}_v(z)\widehat{g}_{1,v}(z)-(1-b_v)\widehat{d}_v(z)\widehat{g}_{n+1,v}(z)}{1-z}$$
$$= \frac{1-\widehat{d}_v(z)}{1-z} + b_v\widehat{d}_v(z)\frac{1-\widehat{g}_{1,v}(z)}{1-z}$$
$$+ (1-b_v)\widehat{d}_v(z)\frac{1-\widehat{g}_{n+1,v}(z)}{1-z}. \tag{21}$$

By inversion of the p.g.f., Eq. (21) results in the presentation

$$\frac{\sum_{y=u+1}^{\infty}\overline{H}_{n,v}(y)}{\sum_{y=0}^{\infty}\overline{H}_{n,v}(y)}$$
$$= \overline{D}_v(u) + \sum_{y=0}^{u}\left(b_v\overline{G}_{1,v}(u-y)+(1-b_v)\overline{G}_{n+1,v}(u)\right)d_v(y)$$
$$= b_v\overline{H}_{1,v}(u)+(1-b_v)\overline{H}_{n+1,v}(u). \tag{22}$$

Now, (22) with $n = 0$ we have

$$\overline{H}_{1,v}(u) = \frac{\sum_{y=u+1}^{\infty}\overline{H}_{0,v}(y)}{\sum_{y=0}^{\infty}\overline{H}_{0,v}(y)}. \tag{23}$$

On the other hand, (12) with $n = 0$ together with (8) we obtain

$$\overline{H}_{0,v}(u) = \frac{\overline{D}_v(u)}{\xi_v}. \tag{24}$$

It follows from (23) and (24) that

$$\overline{H}_{1,v}(u) = \frac{\sum_{y=u+1}^{\infty}\overline{D}_v(y)}{\sum_{y=0}^{\infty}\overline{D}_v(y)}. \tag{25}$$

By (5), (17), (20) and (25), (22) can be rewritten as

$$\sum_{y=u+1}^{\infty}\overline{H}_{n,v}(y)=\sum_{y=u+1}^{\infty}\overline{D}_v(y)+\frac{\mu_{(n+1)}}{\mu_{(n)}(n+1)}\overline{H}_{n+1,v}(u). \tag{26}$$

(16) then follows from (26) directly. □

We are now ready to give an explicit formula for $\psi_{n,v}(u)$.

Theorem 3. *For $n\in\mathbb{N}$, $\psi_{n,v}(u)$ can be expressed as*

$$\begin{aligned}\psi_{n,v}(u)=&\frac{1}{1-\xi_v}\sum_{y=u+1}^{\infty}(y-(u+1))^{(n)}d_v(y)\\&-\frac{\xi_v}{1-\xi_v}\sum_{j=0}^{n}\binom{n}{j}\mu_{(n-j)}\sum_{y=u+1}^{\infty}(y-(u+1))^{(j)}d_v(y).\end{aligned} \tag{27}$$

Proof. Equating (15) and (27), it is sufficient to prove that

$$\begin{aligned}\overline{H}_{n,v}(u)=&\overline{D}_v(u)+\frac{1}{\xi_v\mu_{(n)}}\sum_{y=u+1}^{\infty}(y-(u+1))^{(n)}d_v(y)\\&-\sum_{j=0}^{n}\binom{n}{j}\frac{\mu_{(n-j)}}{\mu_{(n)}}\sum_{y=u+1}^{\infty}(y-(u+1))^{(j)}d_v(y).\end{aligned} \tag{28}$$

We use induction to prove (28). First, Eq. (24) implies that (28) holds for $n=0$ immediately. Now suppose that (28) holds for some $n>1$.

By interchanging the order of summation, for $j=0,1,2,\cdots$

$$\begin{aligned}\sum_{y=u+1}^{\infty}\sum_{t=y+1}^{\infty}(t-(y+1))^{(j)}d_v(t)&=\sum_{t=u+2}^{\infty}d_v(t)\sum_{y=u+1}^{t-1}(t-(y+1))^{(j)}\\&=\frac{1}{j+1}\sum_{t=u+2}^{\infty}d_v(t)(t-(u+1))^{(j+1)}\\&=\frac{1}{j+1}\sum_{t=u+1}^{\infty}d_v(t)(t-(u+1))^{(j+1)}.\end{aligned} \tag{29}$$

Thus, using (29) and assuming (28) holds, Eq. (10) yields

$$\overline{H}_{n+1,v}(u)=(n+1)\frac{\sum_{y=u+1}^{\infty}\sum_{t=y+1}^{\infty}(t-(y+1))^{(n)}d_v(t)}{\xi_v\mu_{(n+1)}}$$

$$
\begin{aligned}
&- \frac{n+1}{\mu_{(n+1)}} \sum_{j=0}^{n} \binom{n}{j} \mu_{(n-j)} \sum_{y=u+1}^{\infty} \sum_{t=y+1}^{\infty} (t-(y+1))^{(j)} d_v(t) \\
&= \frac{\sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(n+1)}}{\xi_v \mu_{(n+1)}} \\
&\quad - \sum_{j=0}^{n} \binom{n}{j} \frac{n+1}{j+1} \frac{\mu_{(n-j)}}{\mu_{(n+1)}} \sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(j+1)} \\
&= \frac{\sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(n+1)}}{\xi_v \mu_{(n+1)}} \\
&\quad - \sum_{j=0}^{n} \binom{n+1}{j+1} \frac{\mu_{(n-j)}}{\mu_{(n+1)}} \sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(j+1)} \\
&= \frac{\sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(n+1)}}{\xi_v \mu_{(n+1)}} \\
&\quad - \sum_{j=1}^{n+1} \binom{n+1}{j} \frac{\mu_{(n+1-j)}}{\mu_{(n+1)}} \sum_{t=u+1}^{\infty} d_v(t)(t-(u+1))^{(j)},
\end{aligned}
$$

which is (28) with n replaced by $n+1$. □

We remark that Eq. (27) presents a convenient way to compute $\psi_{n,v}(u)$ because the quantities $\overline{D}_v(y)$ and ξ_v can have explicit expression in the case of some particular claim size distributions. We will illustrate this point in the following section.

Define the expected discount value of $|U(T)|$ as

$$\varphi_v(u) = \mathrm{E}[v^T |U(T)| I(T<\infty) \,|\, U(0)=u],$$

then we have the following result:

Corollary 4. *The function $\varphi_v(u)$ satisfies*

$$\varphi_v(u) = \sum_{z=u+1}^{\infty} z d_v(z) - \left(u + \frac{\mu_{(1)}\xi_v}{1-\xi_v}\right) \overline{D}_v(u). \tag{30}$$

Proof. With $n=1$ in the Eq. (27) we have

$$\varphi_v(u) = \psi_{1,v}(u) + \overline{D}_v(y), \tag{31}$$

then (30) follows from (31) after some calculations, and the details are omitted. □

4. Geometric Claim Amounts

In this section we assume that claim amounts are geometrically distributed with $p(x) = (1-q)q^{x-1}$, $x \in \mathbb{N}^+$, $0 < q < 1$. Then it can be easily verified that

$$p_x(y) = p(y) = (1-q)q^{y-1}, \quad y \in \mathbb{N}^+,$$
$$g_v(y) = p(y) = (1-q)q^{y-1}, \quad y \in \mathbb{N}^+,$$

and consequently

$$\mu_{(n)} = \sum_{y=1}^{\infty} y^{(n)} g_v(y) = (1-q)\sum_{y=n}^{\infty} y^{(n)} q^{y-1}, \quad n \in \mathbb{N}^+, \tag{32}$$

with $\mu_{(0)} = 1$. Moreover, it is shown by [9] that

$$\overline{D}_v(u) = \xi_v(q + \xi_v(1-q))^u, \tag{33}$$

where ξ_v is determined by

$$\xi_v = \widehat{k}(vq + v\xi_v(1-q)), \tag{34}$$

and the Eq. (34) has a unique solution between 0 and 1. Furthermore, the Eq. (33) implies that

$$d_v(u) = (1-q)(1-\xi_v)\xi_v(q + \xi_v(1-q))^{u-1}. \tag{35}$$

Substitution (35) into the Eq. (27) we have

$$\begin{aligned}\psi_{n,v}(u) = \xi_v(1-q)\Bigg\{ &\sum_{y=u+1}^{\infty} (y-(u+1))^{(n)}(q+\xi_v(1-q))^{y-1} \\ &- \xi_v \sum_{j=0}^{n} \binom{n}{j} \mu_{(n-j)} \sum_{y=u+1}^{\infty} (y-(u+1))^{(j)}(q+\xi_v(1-q))^{y-1} \Bigg\},\end{aligned}$$

where $\mu_{(n)}$ can be calculated by (32).

Example 1. Suppose that the claim waiting times are also geometrically distributed with $k(x) = (1-\alpha)\,\alpha^{x-1}$, $x \in \mathbb{N}^+$, then $\widehat{k}(s) = s(1-\alpha)/(1-\alpha s)$. In this case, model (1) is the classical compound binomial risk process. From (34) we know that ξ_v is the solution of

$$\xi_v = \frac{(1-\alpha)v(q+\xi_v(1-q))}{1-\alpha v(q+\xi_v(1-q))}.$$

Let $v = 0.9$, $q = 0.2$ and $\alpha = 0.3$, Table 1 shows the numerical results for $\psi_{n,v}(u)$.

	$n=0$	$n=1$	$n=2$	$n=3$
$u=0$	0.342341	0.085585	0.042793	0.032094
$u=1$	0.162226	0.040556	0.020278	0.015209
$u=2$	0.076874	0.019219	0.009609	0.007207
$u=3$	0.036429	0.009107	0.004554	0.003415
$u=4$	0.017263	0.004316	0.002158	0.001618
$u=5$	0.008180	0.002045	0.001023	0.000767

Table 1: Geometric interclaim times

Example 2. We further assume that the claim waiting times are shifted negative binomial distributed with $k(x) = x\beta^{x-1}(1-\beta)^2$, $x \in \mathbb{N}^+$, then $\widehat{k}(s) = s(1-\beta)^2/(1-\beta s)^2$. It follows from (34) that ξ_v is the solution to

$$\xi_v = v(q+\xi_v(1-q))\left\{\frac{1-\beta}{1-\beta v(q+\xi_v(1-q))}\right\}^2.$$

Let $v = 0.9$, $q = 0.2$ and $\beta = 0.4$, Table 2 shows the numerical results for $\psi_{n,v}(u)$.

	$n=0$	$n=1$	$n=2$	$n=3$
$u=0$	0.120243	0.030061	0.015030	0.011273
$u=1$	0.035615	0.008904	0.004452	0.003339
$u=2$	0.010549	0.002637	0.001319	0.000989
$u=3$	0.003125	0.000781	0.000391	0.000293
$u=4$	0.000925	0.000231	0.000116	0.000087
$u=5$	0.000274	0.000069	0.000034	0.000026

Table 2: Shifted negative binomial interclaim times

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